A Procedure for Designing Vector-valued Compactly Supported Wavelets

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Abstract

Wavelet analysis has been applied to many aspects in science and technology. The notion of vector-valued wavelets with four-scale dilation factor associated with an orthogonal vector-valued scaling function is introduced. The existence of orthogonal vector-valued wavelets with quaternary-scale is discussed. A necessary and sufficient condition is presented by means of vector-valued multiresolution analysis and paraunitary vector filter bank theory. An algorithm for constructing a sort of orthogonal vector-valued wavelets with compact support is proposed, and their orthogonal properties are discussed.

I. Introduction

Wavelet analysis has been a widely applied tool in engineering field. The main advantage of wavelets is their time-frequency localization property. Already they have led to exciting applications in signal analysis [1], fractals [2], image processing [3] and so on. Sampling theorems play a basic role in digital signal processing. They ensure that continuous signals can be processed by their discrete samples. Vector-valued wavelets are a class of generalized multiwavelets [4]. Chen and Cheng [5] introduced the notion of vector-valued wavelets and showed that multiwavelets can be degenerated from the component functions in vector-valued wavelets. Vector-valued wavelets and multiwavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms but not necessary for discrete vector-valued wavelet transforms [5]. In real life, Video images are vector-valued signals. Vector-valued wavelet transforms have been recently studied for image coding by W. Li. Chen
and Cheng studied orthogonal compactly supported vector-valued wavelets with 2-scale. Inspired by [5-7], we are about to investigate the construction of a class of orthogonal compactly supported vector-valued wavelets with three-scale. Similar to uni-wavelets, it is more complicated and meaningful to investigate vector-valued wavelets with 4-scale. Based on an observation in [5,8], another purpose of this article is to introduce the notion of orthogonal vector-valued wavelet packets with three-scale and investigate their properties.

2. MULTIRESOLUTION ANALYSIS

By \( R \) and \( C \), we denote the collection of all real and all complex numbers, respectively. \( Z \) and \( Z_+ \) denote all integers and all nonnegative integers, respectively. Set \( u \) be a constant and \( 2 \leq u \in Z \). By \( L^2(R, C^u) \), we denote the aggregate of arbitrary vector-valued functions \( F(t) \), i.e.,

\[
L^2(R, C^u) := \{ F(t) = (f_1(t), f_2(t), \ldots, f_u(t))^T : f_i(t) \in L^2(R), \ t = 1, 2, \cdots, u \},
\]

where \( T \) means the transpose of a vector. For example, video images and digital films are examples of vector-valued functions where \( h_i(t) \) denotes the pixel on the \( i \)-th column at time \( t \). For \( F(t) \in L^2(R, C^u) \), \( \| F \| \) denotes the norm of vector-valued function \( F(t) \), i.e.,

\[
\| F \| := \left( \sum_{i=1}^{u} \int_{R} |f_i(t)|^2 dt \right)^{1/2},
\]

and its integration is defined as

\[
\int_{R} F(t)dt := \left\{ \int_{R} f_1(t)dt, \int_{R} f_2(t)dt, \cdots, \int_{R} f_u(t)dt \right\}^T.
\]

The Fourier transform of \( F(t) \) is defined by

\[
\hat{F}(\omega) := \int_{R} F(t) \cdot e^{-i\omega t} dt.
\]

For two vector-valued functions \( F, G \in L^2(R, C^u) \), their symbol inner product is defined by

\[
\langle F(\cdot), G(\cdot) \rangle := \int_{R} F(t)G(t)^* dt,
\]

where \( * \) means the transpose and the complex conjugate, and \( I_u \) denotes the \( u \times u \) identity matrix.

A sequence \( \{G_i(t)\}_{i \in Z} \subset U \subseteq L^2(R, C^u) \) is called an orthonormal set of the subspace \( U \), if the following condition is satisfied

\[
\langle G_j(\cdot), G_k(\cdot) \rangle = \delta_{j,k} I_u, \ j, k \in Z,
\]

where \( \delta_{j,k} = 1 \) as \( j = k \) and \( \delta_{j,k} = 0 \) otherwise.

**Definition 1.** We say that \( H(t) \in U \subseteq L^2(R, C^u) \) is an orthogonal vector-valued function of the subspace \( U \) if its translations \( \{H(t-v)\}_{v \in Z} \) is an orthonormal collection of the subspace \( Y \), i.e.,

\[
\langle H(\cdot-n), H(\cdot-v) \rangle = \delta_{n,v} I_u, \ n, v \in Z.
\]

**Definition 2** [5]. A sequence \( \{F_v(t)\}_{v \in Z} \subset U \subseteq L^2(R, C^u) \) is called an orthonormal basis of \( Y \), if it satisfies (2), and for any \( G(t) \in U \), there exists a unique sequence of \( u \times u \) constant matrices \( \{Q_k\}_{k \in Z} \) such that...
## Definition 3 \(^{[5]}\) A vector-valued multiresolution analysis of \(L^2(R, C^n)\) is a nested sequence of closed subspaces \(\{Y_l\}_{l \in \mathbb{Z}}\) such that (i) \(Y_l \subseteq Y_{l+1}, \forall l \in \mathbb{Z}\); (ii) \(\bigcap_{l \in \mathbb{Z}} Y_l = \{0\}\); \(\bigcup_{l \in \mathbb{Z}} Y_l\) is dense in \(L^2(R, C^n)\), where \(0\) is the zero vector of \(L^2(R, C^n)\); (iii) \(F(t) \in Y_0\) if and only if \(F(4^l t) \in Y_l\); (iv) there is \(h(t) \in Y_0\) such that the sequence \(\{h(t - v), v \in \mathbb{Z}\}\) is an orthonormal basis of subspace \(Y_0\).

On the basis of Definition 2 and Definition 3, we obtain \(h(t)\) satisfies the following equation

\[
G(t) = \sum_{v \in \mathbb{Z}} Q_v F_v(t). \quad (4)
\]

where \(\{P_v\}_{v \in \mathbb{Z}}\) is a finite supported sequence of \(\mu \times \mu\) constant matrices, i.e., \(\{P_v\}_{v \in \mathbb{Z}}\) has only finite non-zero terms, and the others are zero matrices. By taking the Fouries transform for the both sides of (5), and assuming \(h(\xi)\) is continuous at zero, we have

\[
\hat{h}(4\omega) = \mathcal{P}(\xi) \hat{h}(\xi), \quad \xi \in R, \quad \omega \in R. \quad (6)
\]

\[
4 \mathcal{P}(\xi) = \sum_{v} P_v \cdot \exp(-iv\xi). \quad (7)
\]

Let \(W_j (j \in \mathbb{Z})\) denote the orthocomplement subspace of \(Y_j\) in \(Y_{j+1}\) and there exist three vector-valued functions \(G_s(t) \in L^2(R, C^n), s = 1, 2\) such that their translations and dilations form a Riesz basis of \(W_j\), i.e.,

\[
W_j = \text{clus}_{\xi \in R} \left( \text{span}\{G_s(4^j t - v) : s = 1, 2, 3; v \in \mathbb{Z}\} \right), j \in \mathbb{Z}. \quad (8)
\]

Since \(G_s(t) \in W_0 \subseteq Y_j\), \(s = 1, 2\), there exist three finitely supported sequences \(\{B_v^{(s)}\}_{v \in \mathbb{Z}}\) of \(\mu \times \mu\) constant matrices such that

\[
G_s(t) = \sum_{v} B_v^{(s)} h(4^j t - v), \quad s = 1, 2. \quad (9)
\]

\[
4 B^{(s)}(\xi) = \sum_{v} B_v^{(s)} \exp(-iv\xi). \quad (10)
\]

Then, the refinement equation (10) becomes the following

\[
\hat{G}_s(4\xi) = \mathcal{B}^{(s)}(\xi) \hat{h}(\xi), \quad s = 1, 2, \quad \xi \in R. \quad (11)
\]

If \(h(t) \in L^2(R, C^n)\) is an orthogonal vector-valued scaling function, then it follows from (3) that

\[
\langle h(\cdot), h(\cdot - v) \rangle = \delta_{0,v} I_v, \quad v \in \mathbb{Z}. \quad (12)
\]

We say that \(G_s(t) \in L^2(R, C^n), s = 1, 2, 3\) are orthogonal vector-valued wavelet functions associated with the
vector-valued scaling function $\mathbf{H}(t)$, if they satisfy

$$\langle h(\cdot-n), G_s(\cdot-v) \rangle = O, \quad s = 1,2, \quad n,v \in Z, \quad (13)$$

and the family $\{G_s(t-v), s = 1,2,3, \quad v \in Z\}$ is an orthonormal basis of $W_0$. Thus we have

$$\langle G_s(\cdot), G_s(\cdot-n) \rangle = \delta_{r,n}, \quad r,s = 1,2,3; \quad n \in Z. \quad (14)$$

**Lemma 1**

Let $F(t) \in L^2(R,C^n)$. Then $F(t)$ is an orthogonal vector-valued function if and only if

$$\sum_{k \in Z} \hat{F}(\xi + 2k\pi)\hat{F}(\xi + 2k\pi)^* = I_u, \quad \omega \in R. \quad (15)$$

**Lemma 2.** Let $h(t) \in L^2(R,C^n)$, defined by (5), is an orthogonal vector-valued scaling function, then for $\forall v \in Z$, we have the following equality,

$$\sum_{\sigma \in \mathbb{Z}} P_{\sigma} (P_{\sigma + v})^* = 4\delta_{0,v} I_u. \quad (16)$$

$$\sum_{\sigma \in 0} P(\xi + \sigma \pi / 2)P(\xi + \sigma \pi / 2)^* = I_u, \quad \xi \in R. \quad (17)$$

**Proof.** By substituting equation (5) into the relation (12), for $\forall k \in Z$, we obtain that

$$\delta_{0,k} I_u = \langle h(\cdot-k), h(\cdot) \rangle = \sum_{l \geq 0} \sum_{v \in Z} \int_{R} P_l h(4t-4k-l) h(4t-v)^* (P_v)^* \, dt$$

$$= \frac{1}{4} \cdot \sum_{l \geq 0} \sum_{v \in Z} P_l \langle \mathbf{H}(\cdot-4k-l), \mathbf{H}(\cdot-v) \rangle (P_v)^* = \frac{1}{4} \sum_{l \geq 0} P_l (P_{l+k})^*. \quad (18)$$

Thus, both Theorem 1 and formulas (16), (23) and (24) provide an approach to design a class of compactly supported orthogonal vector-valued wavelets.

### 3. CONSTRUCTION OF WAVELETS

In the following, we begin with considering the existence of a class of compactly supported orthogonal vector-valued wavelets.

**Theorem 1.** Let $h(t) \in L^2(R,C^n)$ defined by (5), be an orthogonal vector-valued scaling function. Assume $G_s(t) \in L^2(R,C^n)$, $s = 1,2$, and $P(\omega)$ and $B^{(s)}(\omega)$ are defined by (7) and (10), respectively.

Then $G_s(t)$ are orthogonal vector-valued wavelet functions associated with $\mathbf{H}(t)$ if and only if

$$\sum_{\sigma = 0}^3 P(\omega + \sigma \pi / 2)B^{(s)}(\omega + \sigma \pi / 2)^* = O, \quad (18)$$

$$\sum_{\sigma = 0}^3 B^{(s)}(\omega + \sigma \pi / 2)B^{(s)}(\omega + \sigma \pi / 2)^* = \delta_{s,s} I_u, \quad (19)$$
where \( r, s \in 1, 2, \omega \in R \). or equivalently,

\[
\sum_{i \in z} P_i (B_{i+4})^* = O, \quad s = 1, 2, 3, \quad v \in Z; \quad (20)
\]

\[
\sum_{i \in z} B_i^*(B_{i+4})^* = 4 \delta_{r,s} \delta_{0,v} I_u, \quad r, s = 1, 2, 3, \quad v \in Z. \quad (21)
\]

**Proof.** Firstly, we prove the necessity. By Lemma 1 and (6), (11) and (13), we have

\[
O = \sum_{v \in z} \hat{h}(4\omega + 2v\pi)\hat{G}_{s}(4\omega + 2v\pi)^*
\]

\[
= \sum_{v \in z} P_{\lambda} (\omega + v\pi / 2) \hat{h}(\omega + v\pi / 2)^* \cdot \hat{h}(\omega + v\pi / 2)^* B^*(\omega + v\pi / 2)^*
\]

\[
= \sum_{\sigma = 0} B^*(\omega + \sigma\pi / 2)B^*(\omega + \sigma\pi / 2)^*. \]

It follows from formula (14) and Lemma 1 that

\[
\delta_{r,s} I_v = \sum_{v \in z} \hat{G}_{s}(4\omega + 2v\pi)\hat{G}_{s}(4\omega + 2v\pi)^*
\]

\[
= \sum_{v \in z} B^*(\omega + v\pi / 2) \hat{h}(\omega + v\pi / 2)^* \cdot \hat{h}(\omega + v\pi / 2)^* B^*(\omega + v\pi / 2)^*
\]

\[
= \sum_{\sigma = 0} (\omega + \sigma\pi / 2)B^*(\omega + \sigma\pi / 2)^*. \]

Next, the sufficiency of the theorem will be proven. From the above calculation, we have

\[
\sum_{v \in z} \hat{h}(4\omega + 2v\pi)\hat{G}_{s}(4\omega + 2v\pi)^*
\]

\[
= \sum_{\sigma = 0} B^*(\omega + \sigma\pi / 2)B^*(\omega + \sigma\pi / 2)^* = \delta_{r,s} I_v. \]

Furthermore

\[
\langle \hat{h}(\cdot), G_{s}(-k) \rangle = \frac{2}{\pi} \int_{0}^{\pi/2} \hat{h}(4\omega + 2v\pi)
\]

\[
\cdot \hat{G}_{s}(4\omega + 2v\pi)^* e^{4ik\omega} d\omega = O, \quad s = 1, 2, 3, \quad k \in Z
\]

\[
\langle G_{s}(\cdot), G_{s}(-k) \rangle = \frac{2}{\pi} \int_{0}^{\pi/2} \hat{G}_{s}(4\omega + 2v\pi)
\]

\[
\cdot \hat{G}_{s}(4\omega + 2v\pi)^* e^{4ik\omega} d\omega = \delta_{s,s} \delta_{r,s} I_u, \quad k \in Z
\]

Thus, \( \hat{h}(t) \) and \( G_{s}(t), s = 1, 2 \) are mutually orthogonal, and \( \{ G_{s}(t), s = 1, 2, 3 \} \) are a family of orthogonal vector-valued functions. This shows the orthogonality of \( \{ G_{s}(\cdot - v), s = 1, 2, 3 \} \). Similar to [7, Proposition 1], we can prove its completeness in \( W_0 \).

**Theorem 2.** Let \( \hat{h}(t) \in L^2 (R, C^m) \) be a 5-coefficient compactly supported orthogonal vector-valued scaling functions satisfying the following refinement equation:
\[
\hat{h}(t) = P_0 \hat{h}(4t) + P_1 \hat{h}(4t-1) + \cdots + P_4 \hat{h}(4t-4).
\]

Assume there exists an integer \( \ell \), such that \( (4I_u - P_1(P_1)^*)^{-1} P_1(P_1)^* \) is a positive definite matrix. Define \( Q_s (s = 1, 2, 3) \) to be two essentially distinct Hermitian matrices, which are all invertible and satisfy

\[
(Q_s)^2 = [4I_u - P_1(P_1)^*]^{-1} P_1(P_1)^*.
\]

Define

\[
\begin{cases}
B_j^{(s)} = Q_j P_j, & j \neq \ell, \\
B_j^{(s)} = -(Q_j)^{-1} P_j, & j = \ell,
\end{cases}
\]

Then \( G_s(t) (s = 1, 2, 3) \), defined by (24), are orthogonal vector-valued wavelets associated with \( \chi(t) \):

\[
G_s(t) = B_0^{(s)} \hat{h}(4t) + B_1^{(s)} \hat{h}(4t-1) + \cdots + B_4^{(s)} \hat{h}(4t-4)
\]

**Proof.** For convenience, let \( \ell = 1 \). By Lemma 2, (20) and (21), it suffices to show that \( \{B_0^{(s)}, B_1^{(s)}, B_2^{(s)}, B_3^{(s)}, B_4^{(s)} ; s = 1, 2, 3\} \) satisfy the following equations:

\[
P_0(B_4^{(s)})^* = O, \quad s = 1, 2, 3,
\]

\[
P_1(B_0^{(s)})^* = O, \quad s = 1, 2, 3,
\]

\[
P_0(B_0^{(s)})^* + P_1(B_1^{(s)})^* + \cdots + P_4(B_4^{(s)})^* = O,
\]

\[
B_0^{(r)}(B_4^{(s)})^* = O, \quad r, s \in \{1, 2, 3\},
\]

\[
B_0^{(s)}(B_0^{(s)})^* + B_1^{(s)}(B_1^{(s)})^* + \cdots + B_4^{(s)}(B_4^{(s)})^* = 4I_u.
\]

If \( \{B_0^{(s)}, B_1^{(s)}, B_2^{(s)}, B_3^{(s)}, B_4^{(s)} ; s = 1, 2\} \) are given by (23), then equations (26), (27) and (29) follow from (16). For the proof of (28) and (30), it follows from (16) and (27) that

\[
P_0(B_0^{(s)})^* + P_1(B_1^{(s)})^* + P_2(B_2^{(s)})^* + P_3(B_3^{(s)})^*
\]

\[
= [P_0(P_1)^* + P_2(P_2)^* + P_3(P_3)^*] Q_s - P_1(P_1)^* (Q_s)^{-1}
\]

\[
= (P_1(P_1)^* - P_1(P_1)^*)(B_j)^{-1} = O.
\]

\[
B_0^{(s)}(B_0^{(s)})^* + B_1^{(s)}(B_1^{(s)})^* + B_2^{(s)}(B_2^{(s)})^* + B_3^{(s)}(B_3^{(s)})^*
\]

\[
= Q_s \{P_1(P_1)^* + [P_1(P_1)^*]^{-1}[4I_u - P_1(P_1)^*]P_1(P_1)^*\} (Q_s)^{-1}
\]

\[
= Q_s \{P_1(P_1)^* + [P_1(P_1)^*]^{-1}[4I_u - P_1(P_1)^*]P_1(P_1)^*\} (Q_s)^{-1}
\]
= Q_x \{ P_1 (P_1)^* + 4I_u - P_1 (P_1)^* (Q_x)^{-1} \} = 4I_u.

So, (28), (30) follow. This completes the proof of Thm 2.

**Example 1.** Let $\lambda(t) \in L^2(R, C^3)$ be a 5-coefficient orthogonal vector-valued scaling function satisfy the following equation:

$$h(t) = P_0 h(4t) + P_1 h(4t - 1) + \cdots + P_4 h(4t - 4).$$

where $P_3 = P_4 = O$, $P_0 (P_1)^* = O$,

$$P_0 (P_0)^* + P_1 (P_1)^* + P_2 (P_2)^* + P_3 (P_3)^* + P_4 (P_4)^* = 4I_3.$$

$$P_0 = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
-\frac{1}{2} & \frac{\sqrt{2}}{3} & 1 \\
0 & 0 & \frac{2\sqrt{3}}{3}
\end{pmatrix}, \quad P_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\sqrt{2}}{6} & 0 \\
0 & 0 & \frac{\sqrt{3}}{3}
\end{pmatrix},$$

$$P_2 = \begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
\frac{1}{2} & \frac{\sqrt{2}}{3} & -1 \\
0 & 0 & \frac{2\sqrt{3}}{3}
\end{pmatrix}.$$

Suppose $\ell = 1$. By using (22), we can choose

$$Q = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & \sqrt{\frac{53}{53}} & 0 \\
0 & 0 & \frac{\sqrt{2}}{4}
\end{pmatrix}, \quad Q_x = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & \sqrt{\frac{53}{53}} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4}
\end{pmatrix}.$$

By applying formula (24), we get that

$$B_0^{(1)} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
-\sqrt{\frac{53}{106}} & \sqrt{\frac{106}{53}} & \frac{\sqrt{53}}{3} \\
0 & 0 & \frac{\sqrt{6}}{6}
\end{pmatrix}, \quad B_0^{(3)} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & \frac{\sqrt{106}}{6} & 0 \\
0 & 0 & 2\sqrt{\frac{6}{3}}
\end{pmatrix}.$$
Applying Theorem 2, we obtain that \( G_1(t) = B_0 \) \( G(t) \)
\[ + B_1 \) \( h(4t - 1) + \cdots + B_4 \) \( h(4t - 4) \), \( t = 1, 2 \) are orthogonal vector-valued wavelet functions associated with the orthogonal vector-valued scaling function.

4. CONCLUSION

A necessary and sufficient condition on the existence of a class of orthogonal vector-valued wavelets is presented.

An algorithm for constructing a class of compactly supported orthogonal vector-valued wavelets is proposed.
References