Linearized Oscillation of Nonlinear Impulsive Delay Differential Equations

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Abstract—The main result of this paper is to show oscillations of nonlinear impulsive delay differential equations are equivalent to those of corresponding linear impulsive delay differential equations. The results of this paper generalize some well-known results in the literature. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—oscillation, impulse, delay differential equation.

1. INTRODUCTION

The impulsive differential equations are adequate mathematical apparatus for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, and industrial robotics (see [1]). Due to this reason, in recent years, they have been an object of active research (see [2-11]). We shall note that, in spite of the great number of investigations of impulsive differential equations, their oscillation theory has not yet been elaborated like that of nonimpulsive delay differential equations. In [12], Györi and Ladas studied the linearized oscillation of nonimpulsive nonlinear delay differential equations. In this paper, we will study the linearized oscillation of the impulsive differential equation

\[ x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad t \neq t_k, \]

\[ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots. \]  

The associated linear equations and inequalities of (1) are

\[ x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad t \neq t_k, \]

\[ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots. \]

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\[
x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) \leq 0, \quad t \neq t_k, \quad (3)
\]
\[
x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots
\]
\[
x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) \geq 0, \quad t \neq t_k, \quad (4)
\]
\[
x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots
\]
\[
x'(t) i \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad t \neq t_k, \quad (5)
\]
\[
x(t_k^+) - x(t_k) = b_kx(t_k), \quad k = 1, 2, \ldots
\]

In this paper, we assume the following.

\((A_1)\) \(0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots\) are fixed points with \(\lim_{k \to \infty} t_k = \infty\).

\((A_2)\) \(p_i \in C([t_0, \infty), R^+), \quad \tau_i \in C([t_0, \infty), R^+), \quad \text{and} \quad \lim_{t \to \infty} \{t - \tau_i(t)\} = \infty, \quad i = 1, 2, \ldots, n.\)

\((A_3)\) \(I_k(x) \in C(R, R), \quad I_k(0) = 0, \quad k = 1, 2, \ldots\)

\((A_4)\) \(|I_k(x)| \leq b_k|x|\).

\((A_5)\) \(b_k \geq 0, \quad k = 1, 2, \ldots, \quad \text{and} \quad \prod_{1 \leq k < \infty} (1 + b_k) < \infty.\)

\((A_6)\) \(\int_{t}^{\infty} \sum_{i=1}^{n} p_i(s) \, ds = \infty.\)

\((A_7)\) \(f_i(\mu) > 0 \quad \text{for} \quad \mu \neq 0.\)

\((A_8)\) \(I_k(x)\) is not decreasing, \(k = 1, 2, \ldots.\)

For any \(\sigma \geq t_0\), let
\[
r_{\sigma} = \min_{1 \leq i \leq n} \inf_{t \geq \sigma} \{t - \tau_i(t)\},
\]
and let \(PC_\sigma\) denote the set of functions \(\phi : [r_{\sigma}, \sigma] \to R\) which are real-valued absolutely continuous in \((r_{\sigma}, \sigma) \cap (t_k, t_{k+1})\) and at \(t_k\) situated in \((r_{\sigma}, \sigma)\) may have discontinuity of the first kind.

DEFINITION 1. For any \(\sigma \geq t_0\) and \(\phi \in PC_\sigma\), a function \(y \in ([r_{\sigma}, \infty) \to R)\) denoted by \(y(t, \sigma, \phi)\) is said to be a solution of \((1)\) on \([\sigma, \infty)\) satisfying the initial value condition
\[
y(t) = \phi(t), \quad t \in [r_{\sigma}, \sigma],
\]
if the following conditions are satisfied.

(i) \(y(t)\) is absolutely continuous on each interval \((t_k, t_{k+1}) \subset [r_{\sigma}, \infty)\).

(ii) For any \(t_k \in [\sigma, \infty)\), \(y(t_k^+), y(t_k^-)\) exist and satisfy \(y(t_k^+) = y(t_k^-) = y(t_k)\), for \(k = 1, 2, \ldots.\)

(iii) \(y(t)\) satisfies \((1)\) almost everywhere in \([\sigma, \infty)\) and at impulsive points \(t_k\) situated in \([\sigma, \infty)\) may have discontinuity of the first kind.

DEFINITION 2. A solution of \((1)\) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Similarly, we have definitions of solutions of \((2)-(5)\).

2. MAIN RESULTS

In this section, we establish theorems which enable us to reduce oscillation of \((1)\) to that of a linear impulsive equation. At the same time, an equivalent condition for oscillations of \((1)\) and \((5)\) is established.

We first give some lemmas which are useful in proving theorems.

LEMMA 1. Assume that \((A_1)-(A_3)\) and \((A_8)\) hold, if \((3)\) has an eventually positive solution \(x(t)\), \((2)\) and \((4)\) have eventually positive solutions \(y(t)\) and \(z(t)\), which satisfy \(x(t) \leq y(t) \leq z(t)\) for sufficiently large \(t\).
PROOF. We assume that \( z(t) \) is an eventually positive solution of (3), then there exists \( k \) such that \( x(t) > 0 \) and \( t - \tau_i(t) \geq t_0 \), for \( t \geq t_k - r_\sigma \), \( i = 1, 2, \ldots, n \). Set
\[
z(t) = y(t) = x(t), \quad t_k - r_\sigma \leq t \leq t_k,
\]
then
\[
y(t^+_k) = y(t_k) + I_k(y(t_k)) = x(t_k) + I_k(x(t_k)) = x(t^+_k),
\]
\[
z(t^+_k) = z(t_k) + I_k(z(t_k)) = x(t_k) + I_k(x(t_k)) = x(t^+_k).
\]
According to Theorem 3.2.1 in [12], we have
\[
z(t) \geq y(t) \geq x(t), \quad t_k < t \leq t_{k+1},
\]
and
\[
y(t^+_{k+1}) = y(t_{k+1}) + I_{k+1}(y(t_{k+1})) \geq x(t^+_{k+1}).
\]
Similarly, we have
\[
z(t^+_{k+1}) \geq y(t^+_{k+1}).
\]
By induction,
\[
z(t) \geq y(t) \geq x(t), \quad t \in (t_m, t_{m+1}] \quad \text{and} \quad z(t^+_{m}) \geq y(t^+_{m}) \geq x(t^+_{m}), \quad m \geq k,
\]
hence,
\[
z(t) \geq y(t) \geq x(t), \quad \text{for} \ t \geq t_k.
\]
The proof of Lemma 1 is complete.

**Lemma 2.** Assume that (A\(_1\))–(A\(_4\)) and (A\(_5\)) hold. if (3) has an eventually positive solution \( x(t) \), (5) has an eventually positive solution \( y(t) \), which satisfies \( y(t) \geq x(t) \).

**Proof.** It is easy to see that there exists \( k \) such that \( x(t) > 0 \) and \( t - \tau_i(t) \geq t_0 \) for \( t \geq t_k - r_\sigma \), \( i = 1, 2, \ldots, n \). Set \( y(t) = x(t) \), \( t_k - r_\sigma \leq t \leq t_k \), then we have
\[
y(t^+_k) = y(t_k) + b_k y(t_k) \geq x(t_k) + I_k(x(t_k)) = x(t^+_k).
\]
According to Theorem 3.2.1 in [12],
\[
y(t) \geq x(t), \quad t_k < t \leq t_{k+1}
\]
holds and
\[
y(t^+_{k+1}) = y(t_{k+1}) + b_{k+1} y(t_{k+1}) \geq x(t_{k+1}) + I_{k+1}(x(t_{k+1})) = x(t^+_{k+1}).
\]
Similarly, by induction,
\[
y(t) \geq x(t), \quad t \in (t_m, t_{m+1}] \quad \text{and} \quad y(t^+_{m}) \geq x(t^+_{m}), \quad m \geq k.
\]
Therefore, \( y(t) \geq x(t) \) for \( t \geq t_k \). The proof of Lemma 2 is complete.

**Lemma 3.** If (A\(_1\))–(A\(_6\)) hold, then every nonoscillatory solution of (1) tends to zero as \( t \to \infty \).

**Proof.** Without loss of generality, we assume that \( z(t) \) is an eventually positive solution of (1). Take a sequence \( \{t^+_k\} \) from \( \{t_j\}_1^\infty \) such that \( I_k(z(t^+_k)) > 0 \) and choose corresponding sequence \( \{b^+_k\} \) from \( \{b_j\}_1^\infty \). Therefore, there exists a sufficiently large \( T \geq t_0 \) such that
\[
z(t) > 0 \quad \text{and} \quad z'(t) \leq 0, \quad \text{for} \ t \geq T, \quad t \neq t_k.
\]
That is, $z(t)$ is decreasing in $(t_j, t_{j+1})$ for $t_j \geq T$, $j = m, m + 1, \ldots$, with $m \in \mathbb{N}$. It is easy to see that $z(t)$ is also decreasing in $(t_k^*, t_{k+1}^*)$ for $t_k^* \geq T$, $k \geq m$. Hence, for $t \geq t_k^*$,

$$z(t) \leq z\left(t_k^{*+}\right) \leq (1 + b_k^*) z\left(t_k^*\right) \leq (1 + b_{k-1}^*) z\left(t_{k-1}^*\right) \leq \cdots \leq (1 + b_{m+1}^*) z\left(t_m^*\right).$$

In view of $(A_5)$ and (8), there exists a constant $M > 0$ such that $z(t) < M$ for $t \geq T$. Now we claim that

$$\liminf_{t \to \infty} z(t) = 0.$$ 

Otherwise, set

$$\liminf_{t \to \infty} z(t) = l > 0,$$

then there exists $T_2 > T_1 > T$ such that $t - \tau_i(t) > T_2$, for $t \geq T_2$, $i = 1, 2, \ldots, n$, and $z(t) > l/2$ for $t \geq T_2$. In view of $l/2 \leq z(t) \leq l$ and the continuity of $f_i(\mu)$, there exists $b > 0$ such that $f_i(z(t - \tau_i(t))) \geq b$ for $t \geq T_2$, then from (1), we have

$$0 = z'(t) + \sum_{i=1}^{n} p_i(t) f_i(z(t - \tau_i(t))) \geq z'(t) + b \sum_{i=1}^{n} p_i(t).$$

Integrating from $t$ to $\infty$ with $t \geq T_2$ yields

$$l - M \sum_{k \geq m} b_k^* - z(t) + b \left(\frac{l}{2}\right) \int_{t}^{\infty} \sum_{i=1}^{n} p_i(s) ds \leq 0,$$

which in view of $(A_5)$ and $(A_7)$ implies a contradiction that completes our claim.

Next, we prove $\lim_{t \to \infty} \sup z(t) = 0$. In view of (7) and the fact that $\lim_{t \to \infty} \inf z(t) = 0$, we can take subsequence $\{\xi_k\}$ from $\{t_k^*\}$ such that

$$\lim_{k \to \infty} z(\xi_k) = 0. \quad (9)$$

Similarly, take another subsequence $\{\eta_k^+\}$ from $\{t_k^*\}$ between $\xi_k$ and $\xi_{k+1}$ such that $\lim_{k \to \infty} z(\eta_k^+) = \lim_{t \to \infty} \sup z(t)$. Assume that $b_k^*$ and $\tilde{b}_k^*$ correspond to the moments $\xi_k$, $\eta_k$ of impulsive effects, respectively. According to (1) and (7), it follows from:

$$0 < z(\eta_k^+) \leq (1 + \tilde{b}_k^*) z(\eta_k) \leq (1 + \tilde{b}_k^*) z(\eta_{k-1}^+) \leq (1 + \tilde{b}_k^*) (1 + \tilde{b}_{k-1}^*) \cdots (1 + b_{m+1}^*) z(\xi_k)$$

and (9) that $\lim_{k \to \infty} z(\eta_k^+) = 0$. Therefore, we have $\lim_{t \to \infty} z(t) = 0$, which completes the proof of Lemma 3.

**Lemma 4.** Assume $(A_2)$ holds, then the following two statements are equivalent.

(i) 

$$x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0 \quad (10)$$

has an eventually positive solution.

(ii) There exists $\epsilon_0 > 0$ such that for every $\epsilon \in [0, \epsilon_0]$,

$$x'(t) + (1 - \epsilon) \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0 \quad (11)$$

has an eventually positive solution.
Proof. 

(ii) \( \Rightarrow \) (i) is obvious.

Proof. 

(i) \( \Rightarrow \) (ii). If (10) has an eventually positive solution such that \( x(t) \geq 0 \) for \( t \geq T \geq t_0 \).

By Theorem 3.1.1 in [12], there exist functions \( \beta, \gamma \in C([t_0, \infty), \mathbb{R}) \) such that for any function \( \delta \in C([t_0, \infty), \mathbb{R}) \) between \( \beta \) and \( \gamma \), the following inequality holds:

\[ \beta(t) \leq S(\delta(t)) \leq \gamma(t), \]  

where

\[ S(\delta(t)) = -\sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \delta(s) \, ds \right), \]  

(13)

\( h_i(t) = \min\{t_0, t - \tau_i(t)\}, \quad H_i(t) = \max\{t_0, t - \tau_i(t)\} \). Without loss of generality, we can assume \( \phi(t) > 0 \) for \( t \in [t_0, t_0] \).

Set

\[ \beta_1(t) = \beta(t), \quad \gamma_1(t) = \gamma(t) - 2\beta(t), \]

and for \( \delta(t) \) between \( \beta_1(t) \) and \( \gamma_1(t) \), set

\[ S_1(\delta(t)) = -(1 - \epsilon) \sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \delta(s) \, ds \right). \]  

(14)

It is easy to see \( S_1(\delta(t)) \geq \beta_1(t) \).

Next, we prove

\[ S_1(\delta(t)) \leq \gamma_1(t). \]

(15)

First, we have

\[ S_1(\delta(t)) = -(1 - \epsilon) \sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \delta(s) \, ds \right) \]

\[ \leq -\sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \delta(s) \, ds \right) \]

\[ + \epsilon \sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \delta(s) \, ds \right) \]

\[ \leq -\sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} (\gamma(s) - 2\beta(s)) \, ds \right) - \epsilon \beta(t) \]

\[ \leq -\sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t))}{\phi(t_0)} \exp \left( -\int_{H_i(t)}^{t} \gamma(s) \, ds \right) \exp \left( \int_{H_i(t)}^{t} 2\beta(s) \, ds \right) - \epsilon \beta(t) \]

\[ \leq \gamma(t) \exp \left( \int_{H(t)}^{t} 2\beta(s) \, ds \right) - \epsilon \beta(t), \]

where \( H(t) = \max_{1 \leq i \leq n} \{t - \tau_i(t)\} \).

If we want to prove (15), we only need to prove

\[ \gamma(t) \exp \left( \int_{H(t)}^{t} 2\beta(s) \, ds \right) - \epsilon \beta(t) \leq \gamma(t) - 2\beta(t). \]

But it is equivalent to

$$-\gamma(t) \left(1 - \exp\left(\int_{H(t)}^{t} 2\beta(s) \, ds\right)\right) \leq -\beta(t)(2 - \epsilon). \quad (16)$$

In view of $-\beta(t) > -\gamma(t)$, we only need to prove

$$1 - \exp\left(\int_{H(t)}^{t} 2\beta(s) \, ds\right) \leq 2 - \epsilon.$$  

It is obvious that this inequality holds. Therefore, (15) holds. By Theorem 3.1.1 in [12], equation (11) has an eventually positive solution. This completes the proof of Lemma 4.

Next, we give the main results.

**THEOREM 1.** Assume $(A_1)$--$(A_8)$ hold and there exists $\delta > 0$ such that

$$|f_i(\mu)| \leq |u|, \quad \text{for } 0 < |\mu| < \delta,$$  

(17)

if (1) is oscillatory, then (2) is also oscillatory.

**PROOF.** Otherwise, if (2) has an eventually positive solution $y(t)$, then there exists $t_k$ such that

$$y(t) > 0, \quad \text{for } t \geq t_k - r_\sigma.$$  

According to Lemma 3, we have $\lim_{t \to \infty} y(t) = 0$.

Set

$$x(t) = y(t), \quad t_k - r_\sigma \leq t \leq t_k,$$

then there exists a neighbourhood of $t_k$ such that $x(t) > 0$ on it, and

$$x'(t) = -\sum_{i=1}^{n} p_i(t)f_i(x(t - \tau_i(t))) \geq -\sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)), \quad x(t_k^+) = x(t_k) + I_k(x(t_k)) = y(t_k) + I_k(y(t_k)) = y(t_k^+).$$

By Lemma 1, $x(t) \geq y(t) > 0$ for sufficiently large $t$, which contradicts the fact that (1) is oscillatory. The proof of Theorem 1 is complete.

**THEOREM 2.** Assume $(A_1)$--$(A_8)$ hold and

$$\lim_{\mu \to 0} \inf \frac{f_i(\mu)}{\mu} \geq 1, \quad i = 1, 2, \ldots, n,$$  

if (5) is oscillatory, then (1) is also oscillatory.

**PROOF.** If (1) has an eventually positive solution $x(t)$, then for every $\epsilon > 0$, there exists $T \geq t_0$ such that

$$f_i(x(t - \tau_i(t))) \geq (1 - \epsilon)x(t - \tau_i(t)), \quad t \geq T, \quad i = 1, 2, \ldots, n.$$  

(19)

Then from (1),

$$x'(t) + \sum_{i=1}^{n} (1 - \epsilon)p_i(t)x(t - \tau_i(t)) \leq 0, \quad t \neq t_k,$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots,$$  

(20)
has an eventually positive solution. By Lemma 2,
\[ x'(t) + \sum_{i=1}^{n} (1 - \epsilon) p_i(t) x(t - \tau_i(t)) = 0, \quad t \neq t_k, \]  
(21)
has an eventually positive solution. According to Theorem 2 in [2],
\[ x'(t) + \sum_{i=1}^{n} (1 - \epsilon) p_i(t) \prod_{\sigma \leq t_k < t - \tau_i(t)} (1 + b_k)^{-1} x(t - \tau_i(t)) = 0 \]
(22)
has an eventually positive solution. So, by Lemma 4,
\[ x'(t) + \sum_{i=1}^{n} p_i(t) \prod_{\sigma \leq t_k < t - \tau_i(t)} (1 + b_k)^{-1} x(t - \tau_i(t)) = 0 \]
(23)
has an eventually positive solution. Therefore, (5) has an eventually positive solution, which is a contradiction. The proof of Theorem 2 is complete.

As a consequence of Theorems 1 and 2, we have the following results.

**COROLLARY 1.** Assume (A1)-(A8) hold, and
\[ \lim_{\mu \to 0} \frac{f_i(\mu)}{\mu} = 1, \quad i = 1, 2, \ldots, n, \]
(24)
then (1) is oscillatory if and only if (5) is oscillatory.

Next, according to Theorem 2 and some of the results in [12], we give some explicit conditions for (1) to be oscillatory or nonoscillatory.

**COROLLARY 2.** Assume (A1)-(A8) and (18) hold, let
\[ \tau(t) = \min_{1 \leq i \leq n} \{ \tau_i(t) \}, \]
if either
\[ \lim_{t \to \infty} \inf \int_{t - \tau(t)}^{t} \sum_{i=1}^{n} \prod_{\sigma \leq t_k < t} (1 + b_k)^{-1} p_i(s) ds > \frac{1}{e} \]
(25)
or
\[ \lim_{t \to \infty} \sup \int_{t - \tau(t)}^{t} \sum_{i=1}^{n} \prod_{\sigma \leq t_k < t} (1 + b_k)^{-1} p_i(s) ds > 1 \]
(26)
holds, then (1) is oscillatory.

**COROLLARY 3.** Assume (A1)-(A8) and (18) are satisfied, let
\[ r(t) = \max_{1 \leq i \leq n} \{ \tau_i(t) \}, \]
if
\[ \lim_{t \to \infty} \inf \int_{t - r(t)}^{t} \sum_{i=1}^{n} \prod_{\sigma \leq t_k < t} (1 + b_k)^{-1} p_i(s) ds \leq \frac{1}{e}, \]
(27)
then (1) has an eventually positive solution.

**EXAMPLE.** We consider the equation
\[ x'(t) + tx(t - 2) e^{x(t-2)} = 0, \quad t \neq t_k, \quad t \geq 2, \]
(28)
\[ x(t^+_k) - x(t_k) = \frac{1}{2k} x(t_k), \quad k = 1, 2, \ldots. \]

The associated linear differential equation is
\[ x'(t) + tx(t - 2) = 0, \quad t \neq t_k, \quad t \geq 2, \]
(29)
\[ x(t^+_k) - x(t_k) = \frac{1}{2k} x(t_k), \quad k = 1, 2, \ldots. \]

It is easy to prove that (28) satisfies the conditions of Corollary 2, so (28) is oscillatory.
REFERENCES