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Removable edges in a k-connected graph and a construction method for k-connected graphs[☆]

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ABSTRACT

An edge e of a k-connected graph G is said to be a *removable edge* if $G \ominus e$ is still k-connected, where $G \ominus e$ denotes the graph obtained from G by deleting e to get $G - e$ and, for any end vertex of e with degree $k - 1$ in $G - e$, say x, deleting x and then adding edges between any pair of non-adjacent vertices in $N_{G-e}(x)$. Xu and Guo [Liqiong Xu, Xiaofeng Guo, Removable edges in a 5-connected graph and a construction method of 5-connected graphs, Discrete Math. 308 (2008) 1726–1731] proved that a 5-connected graph G has no removable edge if and only if $G \cong K_6$, using this result, they gave a construction method for 5-connected graphs. A k-connected graph G is said to be a quasi $(k+1)$-connected if G has no nontrivial k-vertex cut. Jiang and Su [Hongxing Jiang, Jianji Su, Minimum degree of minimally quasi $(k+1)$-connected graphs, J. Math. Study 35 (2002) 187–193] conjectured that for $k \geq 4$ the minimum degree of a minimally quasi k-connected graph is equal to $k - 1$. In the present paper, we prove this conjecture and prove for $k \geq 3$ that a k-connected graph G has no removable edge if and only if G is isomorphic to either K_{k+1} or (when k is even) the graph obtained from K_{k+2} by removing a 1-factor. Based on this result, a construction method for k-connected graphs is given.

1. Introduction

Graph theoretic terminology used here generally follows that of Bondy [5]. We consider only finite and simple graphs.

Let k be an integer such that $k \geq 2$ and G be a k-connected graph. An edge e of G is said to be *k-contractible* if the contraction of the edge results in a k-connected graph. Tutte [26] proved that every 3-connected graph with order at least 5 contains a 3-contractible edge, using this result, he gave a construction methods for 3-connected graphs. A construction methods of 4-connected graphs was given by Slater [22]. A non-complete k-connected graph G is called *contraction-critical k-connected* if every edge of G is not k-contractible. Contractible edges in k-connected graphs and properties of contraction-critical k-connected graphs are investigated by Mader, Egawa, Enomoto, Ando, Kriesell, Kawarabayashi, Su Jianji, and Yuan Xudong et al. [1–3,7,8,12–17,19,24,25].

For removable edges of k-connected graphs, Holton et al. [10] first defined removable edges in a 3-connected graph. Later, Yin Jianhua [29] defined removable edges in a 4-connected graph.The distribution of removable edges in 3-connected and 4-connected graphs has been studied (see [23,27]). Recently, Xu and Guo [28] generalized the concept of removable edges in a 3-connected graph and a 4-connected graph to k-connected graphs.

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Definition 1 ([28]). Let G be a k-connected graph, and let e be an edge of G. Let $G \ominus e$ denote the graph obtained from G by the following operation: (1) delete e from G to get $G - e$; (2) for any end vertex of e with degree $k-1$ in $G-e$, say x, delete x, and then add edges between any pair of non-adjacent vertices in $N_{G-e}(x)$. If $G \ominus e$ is k-connected, then e is said to be a *removable edge* of G, otherwise e is said to be a *non-removable edge* of G.

Barnette and Grunbaum [4] proved that a 3-connected graph of order at least five has a removable edge. Based on the above graph operation and fact, a constructive characterization of minimally 3-connected graphs was given by Dawes [6], which differs from the characterization provided by Tutte [26].

The graph C_n^2, for an integer $n \geq 4$, is defined as follows. Let $C_n = v_1 v_2 \cdots v_n v_1$ be an n-cycle. Then C_n^2 is obtained from C_n by adding edges $v_i v_j$ satisfying that $j \equiv i \pm 2 \bmod n$, for each $1 \leq i \leq n$.

In [29], Yin Jianhua proved that the 4-connected graph without removable edges is either C_5^2 or C_6^2. Based on this result, he provided a constructive characterization of 4-connected graphs, which is simpler than Slater's method [22].

On the other hand, Politof and Satyanarayana [20,21] introduced the concept of quasi 4-connected graphs and investigated their structure and properties, Mader [18] introduced the concept of n^+-connected graph analogous to the quasi $(n+1)$-connected graph. Jiang Hongxing and Su Jianji [11] further investigated some properties of quasi k-connected graphs.

Let S be a vertex cut set of a graph G with $|S| = k$ $(k \geq 2)$. The vertex cut set S is said to be a *nontrivial k-vertex cut* of G, if the set of the components of $G - S$ can be partitioned into two sets, each of which has to contain at least two vertices. A $(k-1)$-connected graph is *quasi k-connected* if it has no nontrivial $(k-1)$-vertex cut. Clearly, every k-connected graph is quasi k-connected. A quasi k-connected graph G is *minimally quasi k-connected* if $G - uv$ is not quasi k-connected for all $uv \in E(G)$. Suppose that G is a quasi k-connected, $uv \in E(G)$, and $G - uv$ is not quasi k-connected. Then either $G - uv$ is not $(k-1)$-connected, or $G - uv$ is $(k-1)$-connected. If $G - uv$ is $(k-1)$-connected, then in $G - uv$ there is a nontrivial $(k-1)$-vertex cut, hence $|V(G)| \geq (k-1) + 2 \times 2 = k+3$.

For the removable edges, non-removable edges, and quasi connectivity of a graph G, the following results are given in Refs. [28,11].

Theorem 1 ([28]). *Let G be a k-connected graph of order at least $k+3$ $(k \geq 3)$, and $xy \in E(G)$. Then xy is non-removable if and only if there exists $S \subseteq V(G - xy)$ with $|S| = k-1$ such that $G - xy - S$ has exactly two components A, B with $|A| \geq 2$ and $|B| \geq 2$, moreover $x \in A$, $y \in B$.*

Theorem 2 ([28]). *Let G be a k-connected graph of order at least $k+3$ $(k \geq 3)$. Then G has no removable edge if and only if G is minimally quasi k-connected.*

Theorem 3 ([11]). *If G is minimally quasi 5-connected, then $\delta(G) = 4$.*

For minimally quasi k-connected graphs, the following conjecture was posed by Jiang and Su in [11].

Conjecture 1 ([11]). *If G is a minimally quasi k-connected graph with $k \geq 4$, then $\delta(G) = k-1$.*

Using Theorem 3, Xu and Guo [28] gave a construction method for 5-connected graphs, and pointed out that if Conjecture 1 is true then the conclusion of the following conjecture would hold.

Conjecture 2 ([28]). *Let G be a k-connected $(k \geq 3)$. G has no removable edge if and only if either $G \cong K_{k+1}$ for k being odd, or G is isomorphic to either K_{k+1} or $H_{(k+2)/2}$ for k being even (here $H_{(k+2)/2}$ denotes the graph obtained from K_{k+2} by deleting a 1-factor).*

Let G be a minimally quasi k-connected graph. Let xy be an edge in G such that $\kappa(G - xy) \geq k-1$. Then in $G - xy$ there is a nontrivial $(k-1)$-vertex cut, say S. A connected component of $G - xy - S$ is called a *(xy, S)-fragment* of G. A (xy, S)-fragment A of G is called a *(xy, S)-atom* of G if A has the minimum number of vertices in all (xy, S)-fragments of G for every edge xy in G with $\kappa(G - xy) \geq k-1$ and every nontrivial $(k-1)$-vertex cut S of $G - xy$. For a (xy, S)-fragment of G, the following property is obvious.

Property 1. *Let G be a minimally quasi k-connected graph, xy an edge in G such that $\kappa(G - xy) \geq k-1$, and S a nontrivial $(k-1)$-vertex cut of $G - xy$. Then* (i) *$G - xy - S$ has at most three connected components;* (ii) *if $G - xy - S$ has exactly three components, then the component containing neither x nor y is a trivial component that is a (xy, S)-atom of G;* (iii) *if $G - xy - S$ has exactly two connected components, then every component has at least two vertices;* (iv) *if G is k-connected, then, for any (xy, S)-fragment of G, $G - xy - S$ has exactly two connected components, and a (xy, S)-atom of G has at least two vertices.*

For a subgraph C of G, when there is no ambiguity, we write simply C for $V(C)$ (resp. $|C|$ for $|V(C)|$).

Theorem 4 ([11]). *Let k be an integer such that $k \geq 3$. If G be a minimally quasi k-connected graph with $\delta(G) = k$, and let A be a (xy, S)-atom of G. Then $|A| = 2$. Let $x \in A$ and $A = \{x, z\}$, then $xz \in E(G)$, $d(x) = d(z) = k$, and $|N(x) \cap N(z)| = k-2$.*

In this paper, we prove that Conjecture 1 holds, and using this result prove for $k \geq 3$ that a k-connected graph G has no removable edge if and only if G is isomorphic to either K_{k+1} or (when k is even) the graph obtained from K_{k+2} by deleting a 1-factor. Based on this result, we give a construction method for k-connected graphs.

2. Minimally quasi *k*-connected graphs

We now prove Conjecture 1.

Theorem 5. *If G is a minimally quasi k-connected graph with $k \geq 3$, then $\delta(G) = k - 1$.*

Proof. Assume that G is a minimally quasi k-connected graph. Since quasi k-connected graph is $(k-1)$-connected, $\delta(G) \geq k - 1$.

If $\delta(G) \geq k$, then G is k-connected. Otherwise there would be a $(k-1)$-vertex cut T of G. Since G is quasi k-connected graph, T is a trivial $(k-1)$-vertex cut of G, and in $G - T$ there is a component with only one vertex, implying $\delta(G) = k - 1$, a contradiction.

It is easy to see that if G is both k-connected and minimally quasi k-connected, then G is a minimally k-connected graph, and so $\delta(G) = k$ by a result on minimally k-connected graph of Halin [9]. Hence for a minimally quasi k-connected G we have that $k - 1 \leq \delta(G) \leq k$, moreover, if $\delta(G) = k$, then G is k-connected.

If $\delta(G) = k$, then G is both k-connected as well as minimally quasi-k-connected. For $uv \in E(G)$, since $G - uv$ is $(k-1)$-connected and is not quasi k-connected, by a discussion in introduction, we have $|V(G)| \geq k + 3$. Then, by Theorem 2, each edge of G is not removable.

By Theorem 4, we can choose a (xz, S)-atom A of G, where $A = \{x, y\},\ xy \in E(G),\ d(x) = d(y) = k,\ N(x) \cap N(y) = W = \{w_1, w_2, \ldots, w_{k-2}\},\ z \in V(G) - A - S$. Let $B = G - A - S$, then $z \in B$. Since xy is non-removable, by Theorem 1, take a (xy, T)-fragment C of G such that $x \in C,\ y \in D = G - T - C$. It is easy to see that $A \cap C = \{x\},\ A \cap D = \{y\},\ z \in B \cap (C \cup T)$, and $W \subseteq S \cap T$. Noting that $|W| = k - 2$ and $|S| = |T| = k - 1$, if $W \neq S \cap T$, then $S \cap T = S = T$, and so $B \cap D = D - y \neq \emptyset$ and T is a $(k-1)$-separator of G, which contradicts that G is k-connected. Hence $W = S \cap T$. Similarly, we have $S \cap D \neq \emptyset$, so $|S \cap D| = 1$. Let $S \cap D = \{s\}$, then s is a unique vertex in S not adjacent to x.

For every $w_i \in W,\ i = 1, 2, \ldots, k-2$, by Theorem 1, take a (yw_i, T_i)-fragment C_i of G such that $y \in C_i,\ w_i \in D_i = G - T_i - C_i$. Let $M_i = C_i \cap S$. In the following, $i, j \in \{1, 2, \ldots, k-2\},\ i \neq j$.

Claim 1. *$A \cap C_i = \{y\},\ A \cap T_i = \{x\},\ S \cap D_i = \{w_i\},\ M_i \neq \emptyset$. Moreover, if $|M_i| = 1$, then $B \cap C_i = \emptyset$, hence $|C_i| = 2$.*

Observe (xz, S)-atom A and (yw_i, T_i)-fragment C_i of G. Clearly, $y \in A \cap C_i,\ w_i \in S \cap D_i$. Since $xy, xw_i \in E(G),\ x \in A \cap T_i$, and so $A \cap C_i = \{y\},\ A \cap T_i = \{x\}, A \cap D_i = \emptyset$. From $N(y) = S \cup \{x\}$, we have $S \cap D_i = \{w_i\}$, hence $B \cap D_i \neq \emptyset$. Note that $|(T_i - \{x\}) \cup \{w_i\}| = |T_i| = k - 1$ and G is k-connected, so $G - ((T_i - \{x\}) \cup \{w_i\})$ is connected, and $z \in B \cap D_i$. In this case, if $M_i = \emptyset$, then $B \cap C_i \neq \emptyset,\ T_i - \{x\}$ is a vertex cut of G. This implies that $|T_i - \{x\}| \geq k$, contradicting $|T_i - \{x\}| = k - 2$. Hence $M_i \neq \emptyset$. If $B \cap C_i \neq \emptyset$, then $(T_i - \{x\}) \cup M_i$ is a vertex cut of G, so $|(T_i - \{x\}) \cup M_i| \geq k$, implying $|M_i| \geq 2$. Therefore if $|M_i| = 1$, then $B \cap C_i = \emptyset, C_i = \{y\} \cup M_i,\ |C_i| = 2$.

By Claim 1, if $k = 3$, then $C_1 \cap A = \{y\},\ C_1 \cap S = M_1 = \{s\},\ C_1 \cap B = \emptyset,\ D_1 \cap S = \{w_1\},\ T_1 \cap S = \emptyset,\ |T_1 \cap B| = 1$. Then $C_1 = \{y, s\},\ |T_1| = 2,\ x \in T_1, d_G(s) \geq k = 3$, and so s must be adjacent to both x and y, $W = \{w_1, s\}$, contradicting that $|W| = k - 2 = 1$.

Hence suppose $k \geq 4$.

Claim 2. *$C_i \cap C_j = \{y\},\ M_i \cap M_j = \emptyset$.*

Observe (yw_i, T_i)-fragment C_i and (yw_j, T_j)-fragment C_j of G. By Claim 1, it is easy to see that $y \in C_i \cap C_j,\ x \in T_i \cap T_j$, $z \in D_i \cap D_j,\ w_i \in D_i \cap (C_j \cup T_j),\ w_j \in D_j \cap (C_i \cup T_i)$. Then $(T_i - C_j) \cup (T_j - C_i)$ is a separator of G, $|(T_i - C_j) \cup (T_j - C_i)| \geq k$, implying that $|(T_i - D_j) \cup (T_j - D_i)| = |T_i| + |T_j| - |(T_i - C_j) \cup (T_j - C_i)| \leq k - 2$. In this case, if $|C_i \cap C_j| \geq 2$, then $(T_i - D_j) \cup (T_j - D_i) \cup \{y\}$ is a $(k-1)$-separator of G, which contradicts that G is k-connected. Hence $C_i \cap C_j = \{y\}$. Since $M_i \cap M_j \subseteq C_i \cap C_j = \{y\}$ and $y \notin M_i \cap M_j, M_i \cap M_j = \emptyset$.

Claim 3. *$|M_i| \geq 2$.*

First observe (xz, S)-fragment A and (yw_i, T_i)-fragment C_i of G. By Claim 1, $|M_i| \geq 1, A \cap C_i = \{y\}$; moreover, if $|M_i| = 1$ and let $M_i = \{t\}$, then $B \cap C_i = \emptyset$ and $C_i = \{y, t\}$. Since $N(t) \subseteq T_i \cup \{y\}$ and G is k connected, $d(t) = k,\ N(t) = T_i \cup \{y\}$. Note that $t \in C_i$ and $w_i \in D_i$, then $t \neq w_i$. Since $ty, tx \in E(G), t \in W$. Let $t = w_j$, then $C_i = \{y, w_j\},\ N(w_j) = T_i \cup \{y\}$, w_j is not adjacent to w_i and is adjacent to every vertex in $S - \{w_i, w_j\}$.

Next observe (xz, S)-fragment A and (yw_j, T_j)-fragment C_j of G. By Claim 1, $S \cap D_j = \{w_j\}$ and $M_j \neq \emptyset$. From the fact that w_i is the unique vertex in $S - \{w_j\}$ not adjacent to w_j, we have $M_j = \{w_i\}$. Replacing M_i with M_j, a similar argument shows that $B \cap C_j = \emptyset,\ C_j = \{w_i, y\},\ N(w_i) = T_j \cup \{y\},\ w_i$ is not adjacent to w_j and is adjacent to every vertex in $S - \{w_i, w_j\}$.

Take a (xw_i, T')-fragment C' of G such that $x \in C',\ w_i \in D'$, where $D' = G - T' - C'$. For (xz, S)-fragment A and (xw_i, T')-fragment C' of G, it is easy to see that $x \in A \cap C',\ w_i \in S \cap D',\ z \in B \cap (C' \cup T')$. Note that $yx, yw_i \in E(G)$, then $A \cap T' = \{y\},\ A \cap C' = \{x\}$. We assert $|S \cap D'| \geq 2$. Otherwise, $S \cap D' = \{w_i\},\ B \cap D' \neq \emptyset$, hence $(T' - \{y\}) \cup \{w_i\}$ would be a $(k-1)$-separator of G, contrary to G is k-connected. Note that s is a unique vertex in S not adjacent to x, so $S \cap D' = \{w_i, s\}$. Then we have $S \cap C' \neq \emptyset$. Otherwise, $B \cap C' \neq \emptyset,\ (T' - \{y\}) \cup \{x\}$ would be a $(k-1)$-separator of G, a contradiction. Since w_j is a unique vertex in $S - \{w_i\}$ not adjacent to w_i, $S \cap C' = \{w_j\}$. This implies that both w_i and s are not adjacent to w_j, again a contradiction.

Hence Claim 3 holds.

Now we complete the proof of Theorem 5.

By Claims 2 and 3, $M_i \cap M_j = \emptyset$, $|M_i| \geq 2$, $|M_j| \geq 2$. From $\bigcup_{i=1}^{k-2} M_i \subseteq S$, we have that

$$2(k-2) \leq \sum_{i=1}^{k-2} |M_i| \leq |S| = k-1.$$

This implies $k \leq 3$, contrary to the assumption $k \geq 4$. □

3. A recursive construction method for k-connected graphs

By the definition of a removable edge of k-connected graphs, Xu and Guo [28] defined the following operations.

Definition 2 (*[28]*). Let G be a k-connected graph with $k \geq 3$, let e be a removable edge of G, and let $H = G \ominus e$. Then H is said to be obtained from G by a θ^--*operation*, denoted by $H = \theta^-(G)$, and G is said to be obtained from H by a θ^+-*operation*, denoted by $G = \theta^+(H)$. A θ^+-operation is said to be the inverse operation of θ^--operation, and vice versa.

Let G be a k-connected graph with $k \geq 3$, and let $e = xy$ be a removable edge of G. Let $E_x = \{x_i x_j | x_i, x_j \in N_{G-e}(x), x_i x_j \notin E(G)\}$, and Let $E_y = \{y_i y_j | y_i, y_j \in N_{G-e}(y), y_i y_j \notin E(G)\}$.

A θ^--operation for G is one of the following three operations:

(1) if $d_G(x) \geq k+1$ and $d_G(y) \geq k+1$, $H = G \ominus e = \theta^-(G) = G - e$;

(2) if $d_G(x) = k$ and $d_G(y) \geq k+1$, $H = G \ominus e = \theta^-(G) = G - x + E_x$;

(3) if $d_G(x) = d_G(y) = k$, $H = G \ominus e = \theta^-(G) = G - x - y + E_x + E_y$.

In order to give an exact definition of a θ^+-operation, we need the following theorem.

For a k-connected graph G and a minimum vertex cut T of G, the vertex set of a connected component of $G - T$ is called a T-fragment of G. A subset S of $V(G)$ is called a fragment of G if there is a minimum vertex cut T of G such that S is a T-fragment. A fragment of G is called an end fragment of G if any of its proper subsets is not a fragment of G.

Theorem 6. *Let H be a k-connected graph with $k \geq 3$, let $X = \{x_1, x_2, \ldots, x_{k-1}\} \subset V(H)$ and $Y = \{y_1, y_2, \ldots, y_{k-1}\} \subset V(H)$.*

(i) If $H[X] \cong K_{k-1}$, then $G_X = (H - E_X) + x + \{xx_i | i = 1, 2, \ldots, k-1\} + xy$ is k-connected if and only if $\kappa(H - E_X) = \kappa(G_X - x) \geq k-1$, where $E_X \subseteq E(H[X])$, $x \notin V(H)$, $y \in V(H) - X$;

(ii) If $H[X] \cong K_{k-1}$ and $H[Y] \cong K_{k-1}$, then $G_{XY} = (H - E_X - E_Y) + x + y + xy + \{xx_i | i = 1, 2, \ldots, k-1\} + \{yy_i | i = 1, 2, \ldots, k-1\}$ is k-connected if and only if $|X \cap Y| \leq k-2$, $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) \geq k-2$, and, if $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) = k-2$, any end fragment of $H - E_X - E_Y$ contains both a vertex in X and a vertex in Y, where $E_X \subseteq E(H[X])$, $E_Y \subseteq E(H[Y])$, $x, y \notin V(H)$.

Proof. The necessity is obvious. We need only prove the sufficiency.

(i) If $\kappa(H - E_X) = \kappa(G_X - x) \geq k$, then G_X clearly is k-connected. Now suppose $\kappa(H - E_X) = \kappa(G_X - x) = k-1$. Let T be any minimum vertex cut of $H - E_X$. Since H is k-connected, any fragment of $H - E_X$ contains a vertex in X, and so T will not be a vertex cut in G_X. Hence G_X is k-connected.

(ii) If $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) \geq k-1$, then by reasoning similar to the proof of (i), G_{XY} is k-connected. Suppose $\kappa(H - E_X - E_Y) = \kappa(G_{XY} - x - y) = k-2$. For any minimum vertex cut T of $H - E_X - E_Y$, since any end fragment of $H - E_X - E_Y$ contains both a vertex in X and a vertex in Y, any connected component of $H - E_X - E_Y - T$ contains both a vertex in X and a vertex in Y, and so any one of T, $T \cup \{x\}$, and $T \cup \{y\}$ will not be a vertex cut of G_{XY}. For a vertex cut S of $H - E_X - E_Y$ with $|S| = k-1$, any connected component of $H - E_X - E_Y - S$ contains either a vertex in X or a vertex in Y, since H is k-connected. Therefore, S is also not a vertex cut of G_{XY}. Now it follows that G_{XY} is k-connected. □

Definition 3. Let H be a k-connected graph with $k \geq 3$, and let $X = \{x_1, x_2, \ldots, x_{k-1}\} \subset V(H)$ and $Y = \{y_1, y_2, \ldots, y_{k-1}\} \subset V(H)$. Let G be a k-connected graph obtained from H by a θ^+-operation. The θ^+-operation is one of the following three operations:

(1) $G = \theta^+(H) = H + xy$, where $x, y \in V(H)$, and $xy \notin E(H)$;

(2) $H[X] \cong K_{k-1}$, $G = \theta^+(H) = H - E_X + x + \{xx_i | i = 1, 2, \ldots, k-1\} + xy$, where $x \notin V(H)$, $y \in V(H) - X$, and $E_X \subseteq E(H[X])$ such that $\kappa(H - E_X) = \kappa(G - x) \geq k-1$;

(3) $H[X] \cong K_{k-1}$ and $H[Y] \cong K_{k-1}$, $G = \theta^+(H) = H - E_X - E_Y + x + y + xy + \{xx_i | i = 1, 2, \ldots, k-1\} + \{yy_i | i = 1, 2, \ldots, k-1\}$, where $x, y \notin V(H)$, $|X \cap Y| \leq k-2$, and $E_X \subseteq E(H[X])$ and $E_Y \subseteq E(H[Y])$ such that $\kappa(H - E_X - E_Y) = \kappa(G - x - y) \geq k-2$, and, if $\kappa(H - E_X - E_Y) = \kappa(G - x - y) = k-2$, any end fragment of $H - E_X - E_Y$ contains both a vertex in X and a vertex in Y.

Theorem 7. *Let G be a k-connected graph with $k \geq 3$. Then G has no removable edge if and only if G is isomorphic to either K_{k+1} or (when k is even) the graph obtained from K_{k+2} by deleting a 1-factor.*

Proof. The sufficiency is obvious. We need only prove the necessity.

Suppose that G has no removable edge.

If $|V(G)| \geq k+3$, then, by Theorems 2 and 5, G is minimally quasi k-connected and $\delta(G) = k-1$, contradicting that G is k-connected. Hence $k+1 \leq |V(G)| \leq k+2$.

If $|V(G)| = k+1$, then $G \cong K_{k+1}$.

If $|V(G)| = k+2$ and k is even, then G can only be the graph obtained from K_{k+2} by removing a 1-*factor*.

If $|V(G)| = k+2$ and k is odd, G is a spanning subgraph of K_{k+2} with $\delta(G) = k$. So G can be obtained from K_{k+2} by removing $(k+1)/2$ independent edges. Then G has a vertex of degree $k+1$ whose any incident edge would be a removable edge of G, a contradiction.

The proof is thus completed. □

By Theorem 7, we can give a recursive construction method of k-connected graphs.

Theorem 8. *Let G be a k-connected graph with $k \geq 3$. Then* (i) *G can be transformed by a number of θ^--operations into either K_{k+1} or (when k is even) the graph $H_{(k+2)/2}$ obtained from K_{k+2} by deleting a* 1*-factor;* (ii) *G can be obtained from either K_{k+1} or $H_{(k+2)/2}$ by a number of θ^+-operations.*

Proof. (i) Let G be a k-connected graph with $k \geq 3$, and suppose that G is not K_{k+1} or (when k is even) $H_{(k+2)/2}$. Then, by Theorem 7, G has a removable edge, say e_1, and $G_1 = \theta^-(G) = G \ominus e_1$ is a k-connected graph with less edges or less vertices than G. Repeating the above discuss, by the finiteness of G, we can obtain a series of k-connected graphs $G_1, G_2, \ldots, G_t$ so that $G_{i+1} = \theta^-(G_i),\ i = 1, 2, \ldots, t-1$, and G_t is isomorphic to either K_{k+1} or (when k is even) $H_{(k+2)/2}$.

(ii) By using θ^+-operations, G can be obtained from either K_{k+1} or (when k is even) $H_{(k+2)/2}$. □

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