The general Steiner problem in Boolean space and application

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Abstract


The Steiner problem is to find the minimum point \( P \) of sum of distances \( \sum_{i=1}^{n} P_iP = \Phi(P) \) where \( P_i, i = 1, \ldots, n, \) are given points in Euclidean plane. It is solved for \( n = 3, \) [1]. When \( n > 3 \) even approximation is unsatisfactory [3]. In Boolean space the distance of Boolean points \( A, B \) is defined as the symmetric difference:

\[
D(A, B) = A \Delta B = AB' + A'B.
\]

Let \( x_1, \ldots, x_n \) be given Boolean points, \( c_i \) the Boolean weight of \( x_i, \) the general Steiner problem is to find the point \( X \) such that \( \Phi(X) = \sum_{i=1}^{n} c_i D(x_i, X) \) be minimum. This paper shows the minimum points fill up the sub-space \( (\alpha', \beta') \) where \( \alpha = \sum_{i=1}^{n} c_i x_i', \beta = \sum_{i=1}^{n} c_i x_i. \) This is a Boolean optimization problem. The paper gives an application to genuine (not pseudo) Boolean optimization to the real world by regarding Boolean elements as sets, \( X \) as target set in consideration, \( x_i \) as sets of basic factors effecting \( X, \) \( \Phi(X) \) gives the total contradiction of \( X \) to the \( x_i's. \) \( |\Phi(X)| \) (the number of individuals in the set \( \Phi(X) \)) gives a numerical measure of the contradiction. The smaller \( |\Phi(X)| \) indicates the better. 

The treatment of the general Steiner problem is reduced to a question of solving the Boolean equation. It can be applied to a wider realm such as genuine (not pseudo) Boolean programming.

1. The general Steiner problem in Boolean space

The Boolean space is defined by a set of geometrical axioms, and was proved equivalent to Boolean algebra. The Boolean points are also Boolean elements so

* This paper was initiated by the late Ting about thirty years ago and proceeded with by Zhao.
can be operated by Boolean operations. Some geometrical facts are mentioned briefly here:

(i) Suppose $A$, $B$ are Boolean points. There exists one and only one least Boolean linear subspace $\langle A, B \rangle$ containing $A$, $B$. The lower and upper end points of $\langle A, B \rangle$ are denoted by $AB$ and $A + B$ respectively, we write also $\langle A, B \rangle = [AB, A + B]$. Thus for any point $X$ of $\langle A, B \rangle$ it holds $AB \leq X \leq A + B$.

(ii) The distance $C$ of Boolean points $A$, $B$ is defined as their symmetric difference, so $D(A, B) = A \oplus B = AB' + A'B$.

(iii) The diameter of the linear subspace $\langle A, B \rangle$, denoted by $\rho(A, B)$, is defined as the maximum distance of points on $\langle A, B \rangle$, and $\rho(A, B) = A \oplus B - AB \oplus (A + B)$.

(iv) Two subspaces $\langle A, B \rangle$ and $\langle C, D \rangle$ are perpendicular if and only if $\langle A \oplus B \rangle \cap \langle C \oplus D \rangle = \emptyset$. And perpendicular subspaces have at most one point in common.

The general Steiner problem (gSp) in Boolean space means: Given $n$ Boolean points $x_1, \ldots, x_n$ and Boolean constants $c_1, \ldots, c_n$, find point $X$ such that

$$\Phi(X) = \sum_{i=1}^{n} c_i D(x_i, X)$$

be minimum.

Put $\Phi(X)$ into disjunctive form, we have

$$\Phi(X) = \left( \sum_{i=1}^{n} c_i x_i \right) X + \left( \sum_{i=1}^{n} c_i x_i \right) X'$$

or

$$\Phi(X) = \alpha X + \beta X', \quad (2)$$

where

$$\alpha = \sum_{i=1}^{n} c_i x_i', \quad \beta = \sum_{i=1}^{n} c_i x_i.$$

**Theorem 1.** The solutions of the gSp fill up the linear subspace $\langle \alpha', \beta \rangle$.

**Comment 1.** $\langle \alpha', \beta \rangle$ is called the solution space of the gSp, and its diameter is $\rho(\alpha', \beta) = \alpha' \oplus \beta$.  

**Proof.**
Proof. By Schröder theorem [17, p. 66, Th. 2.4.] the value of Boolean function ranges over a Boolean interval whose lower end point is the product of the coefficients of the disjunctive form of the function and the upper end point is the sum of the same set of coefficients. Thus we have the minimum of \( \Phi(X) \) being \( \alpha \beta \). The problem now is to find \( X \) making \( \Phi(X) = \alpha \beta \), i.e. to solve \( X \) from the equation \( \alpha X + \beta X' = \alpha \beta \). Put into disjunctive form, we get

\[
\alpha \beta' X + \alpha' \beta X' = \emptyset.
\]

(3)

\( X \) satisfying (3) is equivalent to that \( X \) makes \( \Phi \) minimum. So the remaining is to solve (3), (3) is soluble since the product of its coefficients \( \alpha \beta' \) and \( \alpha' \beta \) is zero. Its solution is easily found to be all the points on \( [\alpha' \beta, \alpha' + \beta] \), which is the linear subspace \( \langle \alpha', \beta \rangle \). □

Comment 2. The values of \( \Phi(X) \) ranges over \( [\alpha \beta, \alpha + \beta] \), i.e. the subspace \( \langle \alpha, \beta \rangle \) determined by \( \alpha \) and \( \beta \). It is easily seen that the solution space \( \langle \alpha', \beta \rangle \) and the range space \( \langle \alpha, \beta \rangle \) of \( \Phi(X) \) are perpendicular, since their diameters \( \alpha \oplus \beta \) and \( \alpha' \oplus \beta \) have zero product. Further, they cut each other at the unique point \( \beta \). To prove this we need to remind the relation between space and its equation. The solution space of \( ax + bx' = 0 \) is known as \( [b, a'] \), so the equation of \( [\alpha \beta, \alpha + \beta] \) should be \( \alpha' \beta' X + \alpha \beta X' = \emptyset \), the equation of \( [\alpha' \beta, \alpha' + \beta] \) should be \( \beta' X + \alpha' \beta X' = \emptyset \). Solve the equations simultaneously, i.e. add the equations and solve the resulting one, we get \( X \oplus \beta = \emptyset \), i.e. \( X = \beta \). This shows the range space and the solution space have just one point \( \beta \) in common.

Theorem 2. Given \( n \) points \( x_1, \ldots, x_n \), not all coincident, and a Boolean value \( \rho \), there exists a set of weights \( c_1, \ldots, c_n \) not all zero such that the solution space of \( gSp \) just has diameter \( \rho : \rho = \rho \langle \alpha', \beta \rangle \).

Comment 3. The theorem tells that the diameter of the solution space can be controled to any magnitude by suitably choosen \( c \)'s.

Proof. By Comment 1,

\[
\rho \langle \alpha', \beta \rangle = \alpha' \oplus \beta = \left( \sum_{i=1}^{n} c_i x_i' \right)' \oplus \left( \sum_{i=1}^{n} c_i x_i \right) = \sum_{i,j=1}^{n} c_i c_j (x_i \oplus x_j) + c_1' \cdots c_n'.
\]

Since \( x_1, \ldots, x_n \) are given points, they may be regarded as constants. Hence \( \rho (\alpha', \beta) \) is a function of \( c_1, \ldots, c_n \). We denote \( \rho (\alpha', \beta) \) by \( \psi(c_1, \ldots, c_n) \). The normal disjunctive form of \( \psi(c_1, \ldots, c_n) \) is

\[
\psi(c_1, \ldots, c_n) = \sum_{c_i \neq \emptyset}^{1} \psi(e_1, \ldots, e_n) x_i^{e_1}, \ldots, x_n^{e_n}.
\]

where \( e^i = \emptyset \) or 1 and \( x^i = x, x'^i = x' \) by [17, p. 22 Cor. 1, and Cor. 3]
\[
\prod_{e_i=0}^{1} \psi(e_1, \ldots, e_n) \leq \psi(c_1, \ldots, c_n) \leq \sum_{e_i=0}^{1} \psi(e_1, \ldots, e_n).
\]

Now \(\psi(0, 0, \ldots, 0) = 1\) and \(\psi(1, 0, \ldots, 0) = 0\), so
\[
\prod_{e_i=0}^{1} \psi(e_1, \ldots, e_n) = 0 \quad \text{and} \quad \sum_{e_i=0}^{1} \psi(e_1, \ldots, e_n) = 1.
\]

Thus the value of \(\psi(c_1, \ldots, c_n)\) ranges throughout the whole space \([0, 1]\).
Conversely when any value of \([0, 1]\) is assigned to \(\rho\), there must be a set \(c_1, \ldots, c_n\) such that \(\psi(c_1, \ldots, c_n) = \rho\).

In order to exclude the trivial case that all the \(c\)'s be zero, we need only to show that the solutions of the equation
\[
F(c_1, \ldots, c_n) = \psi(c_1, \ldots, c_n) \oplus \rho = \emptyset
\]
never be unique. Conditions for the unique solution ([12–13], [17, Th. 6.7. (ii)]) are:
(i) the equation must be consistent,
(ii) the sum of each pair of coefficients of its disjunctive form must be 1.
Condition (ii) can not be fulfilled since at least the sum of one pair of coefficients is not 1, e.g. take \(S = F(1, \ldots, 1) + F(0, \ldots, 0)\), we can show \(S \neq 1\).
Now \(S = [\psi(1, \ldots, 0) \oplus \rho] + [\psi(0, \ldots, 0) \oplus \rho]\). When given \(\rho = 1\),
\[
S = \left[\left(\prod_{i=1}^{n} x_i\right) \oplus \left(\prod_{i=1}^{n} x_i\right) \oplus 1\right]
\]
\[
+ [\emptyset \oplus 0 \oplus 1] = \left(\prod_{i=1}^{n} x_i\right) \oplus \left(\prod_{i=1}^{n} x_i\right) = \left[\sum_{i,j=1}^{n} (x_i \oplus x_j)\right],
\]
which can not be 1 because not all the \(x\)'s coincide. When given \(\rho \neq 1\), let \(S = F(1, 0, \ldots, 0) + F(0, 1, \ldots, 0) = \rho \neq 1\). Hence in any case \(S \neq 1\).
Whenever we get a trivial solution \(c_1 = \cdots = c_n = 0\), there exists another solution different from the trivial one. \(\square\)

**Theorem 3.** The unique solution of the gSp in Boolean space exists when the only when
\[
\sum_{i=1}^{n} c_i x_i = \left(\sum_{i=1}^{n} c_i x_i\right)'.
\]

**Proof.** (4) means \(\alpha' = \beta\), i.e. \(\rho(\alpha, \beta) = \rho(\alpha', \beta') = 0\). \(\rho\) be zero, implies the solution be unique. Conversely, if (4) does not hold, \(\alpha' \neq \beta\), then at least two different solutions exist.

**Corollary.** The unique solution of the gSp is \(X = \sum_{i=1}^{n} c_i x_i\).
Theorem 4. A necessary and sufficient condition for a set of weights $c_1, \ldots, c_n$ to make any set of points $x_1, \ldots, x_n$, not all coincident, to have the unique solution of gSp is that the c’s be unitary orthogonal, i.e.

$$c_i c_j = \emptyset, \quad i \neq j, \quad i, j = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} c_i = 1. \quad (5)$$

Proof. By Theorem 1, the solution space has diameter $\rho(\alpha', \beta) = \alpha' \oplus \beta$. And in the proof of Theorem 2, we see

$$\rho(\alpha', \beta) = \sum_{i,j=1}^{n} c_i c_j (x_i \oplus x_j) + \prod_{i=1}^{n} c'_i.$$

$$\rho(\alpha', \beta) = \emptyset \iff \sum_{i,j=1}^{n} c_i c_j (x_i \oplus x_j) = \emptyset \quad \text{and} \quad \prod_{i=1}^{n} c'_i = \emptyset.$$

This implies that $c_i c_j = \emptyset$ for any $i, j, (i \neq j)$ and $\sum_{i=1}^{n} c_i = 1$. □

Comment 4. In case $\sum_{i=1}^{n} c_i = 1, \sum_{i=1}^{n} c_i x_i$ may be regarded as the arithmetical mean of the given points. Thus the unique solution is the Boolean arithmetical mean of the given points: $X = \sum_{i=1}^{n} c_i x_i$.

Theorem 5. To a given set of points $x_1, \ldots, x_n$, not all coincident, there exist sets of Boolean weights $c_1, \ldots, c_n$ making the solution of gSp unique.

Proof. Regard condition (4) as a Boolean equation in unknowns $c_1, \ldots, c_n$. It is easy to verify that the product of the $2^n$ coefficients of the equation in disjunctive form is zero. The solubility of the equation shows the existence of the wanted weights. □

Theorem 6. Let $x_1, \ldots, x_n$ be a set of given points not all coincident, and $c_1, \ldots, c_n$ a set of weights making the solution of gSp unique. The locus of the unique solution given by all possible sets of weights fill up a hypersphere circumscribed to a simplex having $x_1, \ldots, x_n$ as vertices.

Proof. By Theorem 3 and its corollary the unique solution is given by $X = \sum_{i=1}^{n} c_i x_i$ and $c_i$ should satisfy the condition $\sum_{i=1}^{n} c_i x_i = (\sum_{i=1}^{n} c_i x_i)'$. In other words the locus of $X$ is given by the solution set of the simultaneous equations $X = \sum_{i=1}^{n} c_i x_i$ and (4). It only needs to solve the equation

$$G(c_1, \ldots, c_n) = \left\{ \left( \sum_{i=1}^{n} c_i x_i \right) \oplus \left( \sum_{i=1}^{n} c_i x_i \right)' \right\} + \left\{ X \oplus \sum_{i=1}^{n} c_i x_i \right\} = \emptyset.$$
The consistent condition is that the product of all its coefficients should be zero, i.e.
\[
\prod_{e_i=0,\ldots,e_n=0}^1 G(e_1, \ldots, e_n) = f(X) = 0
\]
Thru the relation \(X \oplus \sum_{i=1}^n c_i x_i = 0\), \(G(c_1, \ldots, c_n)\) can be written as
\[
G(c_1, \ldots, c_n) = \left[\left(\sum_{i=1}^n c_i x_i\right) \oplus X'\right] + \left[ X \oplus \sum_{i=1}^n c_i x_i \right]
\]
\[
= \left[\left(\prod_{i=1}^n (c_i' + x_i)\right) \oplus X\right] + \left[ X \oplus \sum_{i=1}^n c_i x_i \right] = 0.
\]
Now
\[
G^0 = G(0, \ldots, 0) - (X \oplus 0) - 1,
\]
\[
G^1 = G(1, 0, \ldots, 0) = x_1 \oplus X,
\]
\[
G^2 = G(0, 1, 0, \ldots, 0) = x_2 \oplus X,
\]
\[
G^3 = G(0, \ldots, 0, 1) = x_n \oplus X.
\]
And the product of the \(G^i\)'s is \(G' = \prod_{i=1}^n (x_i \oplus X)\). Again
\[
(i \neq j) \quad G^2_{i,j} = G(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) = x_i \oplus X + (x_j \oplus X).
\]
So
\[
G^2_{1,2} = G(1, 1, 0, \ldots, 0) = (x_1 \oplus X) + (x_2 \oplus X),
\]
\[
G^2_{n-1,n} = G(0, \ldots, 1, 1) = (x_{n-1} \oplus X) + (x_n + X).
\]
The product of the \(G^2_{i,j}\)'s is
\[
G^2 = G^2_{1,2} \cdot G^2_{1,3} \cdots G^2_{1,n} \cdot G^2_{2,3} \cdots G^2_{2,n} \cdots G^2_{n,1,n} = (i \neq j) \prod_{i,j=1}^n [(x_i \oplus X) + (x_j \oplus X)].
\]
Finally we get \(G^2 = \prod_{i=1}^n (x_i \oplus X)\). Since \(G^1 \leq G^2 \leq \cdots \leq G^n\), thus
\[
f(X) = \prod_{e_i=0,\ldots,e_n=0}^1 G(e_1, \ldots, e_n) = G^0 \cdot G^1 \cdot G^2 \cdots \cdot G^n = G^1
\]
\[
= \prod_{i=1}^n (x_i \oplus X) = \left(\prod_{i=1}^n x_i\right) X + \left(\prod_{i=1}^n x_i\right) X' = \sigma X + \nu X' = 0,
\]
where \(\sigma = \sum_{i=1}^n x_i\), \(\nu = \prod_{i=1}^n x_i\).

By the Schröder theorem the solution space of \(f(X) = 0\) is \([\nu, \sigma]\). From \(f(X) = \prod_{i=1}^n (x_i \oplus X) = 0\), each of the points \(x_1, \ldots, x_n\) satisfies the equation hence lies on the locus. Further
\[
f(X) = \sigma' X + \nu X' = \sigma' \nu X + \sigma \nu X' = (\sigma \oplus X)(\nu \oplus X) = 0.
\]
indicates each point on the locus subtends a right angle to the couple of points \( \alpha, \nu \). It is a hypersphere and is just the circumsphere of the simplex with vertices \( x_1, \ldots, x_n \). \( \Box \)

**Remark.** Along Theorem 6 we find a way to solve genuine (not pseudo) Boolean programming problems. If we regard \( X = \sum_{i=1}^{n} c_i x_i \) as objective function whose minimum (or maximum) points under the constraint (4) is seeking for, then the method used in Theorem 6 can be applied. Generally when a Boolean function \( \Phi(X) \) is given, by the Schröder Theorem the minimum (maximum) \( m \) can be found easily. Suppose the minimum points of \( \Phi(X) \) under the restraint \( F = \Psi \) is seeking for, we can solve the problem by finding the solution set of the equation \( \Phi(X) \oplus m + F \oplus \Psi = \emptyset \). Finally it reduces to the solution of the Boolean equation. In this facet the algebraic theory of Boolean equation is inherited and developed during the middle of this centary [17]. At the same time another branch of geometrical investigation produced many results but not yet published.

As a whole system some twigs still need complement.

The unweighted case, i.e. \( c_1 = \cdots = c_n = 1 \), is interesting geometrically.

**Theorem 7.** In the unweighted case the solution space of \( gSp \) is a hypersphere circumscribed to a simplex with vertices \( x_1, \ldots, x_n \).

**Proof.** Now the values of \( \alpha, \beta \) are reduced to \( \alpha = \sum_{i=1}^{n} x_i', \beta = \sum_{i=1}^{n} x_i \), thence \( \alpha' \beta = \prod_{i=1}^{n} x_i \). \( \alpha' + \beta = \sum_{i=1}^{n} x_i \). So the solution space should be \( [\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i] \), which is the same hypersphere appeared in the above theorem.

Now we proceed to consider the unweighted case when \( n = 3 \).

When \( x_1, x_2, x_3 \) are given distinct points, \( gSp \) is multi-solutioned. Ellis [2] takes \( h = x_2x_3 + x_3x_1 + x_1x_2 \) as the only solution which is obviously incomplete. In fact \( h \) is the orthocenter of the Boolean triangle \( \Delta(x_1, x_2, x_3) \). Each of the subspaces \( \langle h, x_i \rangle \) (\( i = 1, 2, 3 \)) is a perpendicular from the vertex \( x_i \) to its opposite side determined by the remaining vertices. And \( h \) is the unique point in common of the three perpendiculars. Further points of this kind are listed below.

1. The esccenters \( h_1, h_2 \) and \( h_3 \) of the triangle \( \Delta(x_1, x_2, x_3) \), i.e. the orthocenters of \( \Delta(A', B, C), \Delta(A, B', C) \), and \( \Delta(A, B, C') \), where \( h_1 = x_1x_2 + x_2x_3 + x_3x_1', \ h_2 = x_1x_2' + x_2x_3' + x_3x_1, \ h_3 = x_1x_2 + x_2x_3' + x_3x_1' \) are also solutions of the \( gSp \). The proof is easy. Substituting \( x = h_i \) into \( \Phi(X) = X \sum_{i=1}^{n} x_i' + X' \sum_{i=1}^{n} x_i \), we see that \( \Phi(h_i) \) gives the minimum \( \alpha \cdot \beta \) of \( \Phi(X) \).

2. The vertices \( x_1, x_2, x_3 \) of the triangle \( \Delta(x_1, x_2, x_3) \) and its centre of fixity \( p = x_1 \oplus x_2 \oplus x_3 \) are as well solutions. Since each of these points gives the minimum value of \( \Phi \) when substituted in.

3. The diametrically opposite points \( \nu \) and \( \sigma \) on the circumscribed hypersphere are solution points, since all points on this hypersphere are solutions, so \( \sigma \) and \( \nu \) will do. \( \Box \)
2. Application of gSp as a mathematical model for actual optimization

The rough idea of the application is to conceive Boolean points as subsets of a certain set. Boolean operations then become set calculus. \( \Phi(X) \) being a result of set operations is again a set. Sets are rather concrete and easier to connect with reality than Boolean points or Boolean elements. The detail of the application is introduced through the following examples.

A college makes a preliminary selection from middle schools before the entrance examination of the college. The list of the student’s records of certain courses are available. It is intended to make a rule of selection by applying gSp.

Let \( X \) be the group of students being selected, \( x_1, \ldots, x_n \) be groups of students outstanding in a special course, say \( x_1 \) be outstanding in math, \( x_2 \) be those in literature, \( \ldots, x_n \) be in athletics. Thus \( X' \) be the set eliminated, \( x_1', \ldots, x_n' \) be those non-outstanding in the respective course. The best selecting rule should enlist all the outstanding students and cast out all those of lower level. This happens if the rule makes all \( x_i, i = 1, \ldots, n, \) go into \( X' \), i.e. \( Xx_i = \emptyset \) and all \( x_i' \) go to \( X' \). Otherwise (either \( Xx_i' > \emptyset \) or \( X'x_i > \emptyset \)) indicates the defect of the selecting rule. The larger \( x_i \cap X' \) be, the worse the rule is. The rule making \( x_i \cap X' \) least shows \( x_i \) suits \( X \) best. When all the \( n \) courses are taken into consideration, \( \sum_{i=1}^{n} (x_i \cap X) = \Phi(X) \) may be used as a measure for the goodness of the rule. The rule making \( \Phi(X) \) minimum will be most reasonable. By Theorem 7 the solution set is \( [\bigcap_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i] \).

What is the actual meaning of the solution? Is it practicable? Now \( \prod_{i=1}^{n} x_i \) comprises students outstanding in all aspects, \( \bigcup_{i=1}^{n} x_i \) comprises students outstanding at least in one course. The intermediate represent those outstanding in more than one but not all aspects. We see the rule is good and applicable.

In order to figure out the meaning of the weight \( c_i \), a new condition is bring in the above example. Suppose the selection should care of that some students will be sent abroad and students of \( x_i \) are required to learn foreign language \( c_i \), e.g. \( c_1 \) be English, \( c_2 \) be French, \( \ldots \). By Theorem 1 the solution space is \( [\alpha' \beta, \alpha' + \beta] \).

What is the physical meaning of the solution? Is it practicable?

Now \( c_i x_i \) is the set of students, learning the \( i \)th language and outstanding in the \( i \)th course, this set should be enlisted. And \( \beta = \sum_{i=1}^{n} c_i x_i \) as well as each of its terms \( c_i x_i \) should be in the enlisted set. \( c_i x_i' \) is obviously out of the scope of selection, it should be eliminated. And so does the set \( \alpha = \sum_{i=1}^{n} c_i x_i' \). \( \alpha' \beta = \beta - \alpha \) means the set excluding \( \alpha \) from \( \beta \), i.e. excluding all those not outstanding in any course although possibly attended some language. Moreover by expansion we get

\[
\alpha' = \left( \sum_{i=1}^{n} c_i x_i' \right)' = (c_1' + x_1) \cdots (c_n' + x_n)
\]

\[
= c_1' \cdots c_n' + c_1' \cdots c_{n-1}' x_n + \cdots + c_n' x_1 \cdots x_{n-1} + x_1 \cdots x_n
\]

We see no \( x_i' \) appeared in \( \alpha' \), so when multiplied or added by \( \beta = \sum_{i=1}^{n} c_i x_i \), no \( x_i' \) will occur, and each term is a set of students high leveled in at least one course.
Thus the reasonable selecting rule should cover students of any of these sets (i.e. Boolean points) of \([\alpha'\beta', \alpha' + \beta]\).

The term \(c'_1 \cdots c'_n\) contains \(c'_1 \cdots c'_n x_1 x_2 \cdots x_n\) as well as \(c'_1 \cdots c'_n x'_1 x'_2 \cdots x'_n\). As a selecting rule this is a leakage. An additional interview over this part may be considerable. Thus let \([\alpha'\beta', \alpha' + \beta]\) be the selection set with an additional interview to the set \(c'_1 \cdots c'_n\) may be a reasonable selecting rule.

Another example is using the gSp model to appraise some qualitative problem or to make decision.

A factory is capable to produce \(m\) kinds of products. Each product comprises \(n\) component parts made in \(n\) workshops of the factory in an assigned level as standard. Suppose trial production is made for each product. How to pick the beneficial product based on the trial production.

We try the model of gSp again.

Let \(X_i\) be the set of good pieces (i.e. up to standard ones) of \(i\)th product. \(x_i\) be the set of good (up to standard) pieces of \(i\)th component part. High quality of \(X\) requires high quality of \(x_i\), but overrefinement is a waste. Let us first focus on \(x_i\) and \(X_i\). If some individuals of the set \(x'_i\) are found in \(X_i\), i.e. \(X_i x'_i > 0\), it implies \(x_i\) is over refine for \(X_i\). Conversely if some individuals of \(x_i\) go into \(X'_i\), i.e. \(X'_i x_i > 0\), it implies \(x_i\) is not good enough to make \(X_i\) fine. Thus the magnitude of \(X_i x'_i + X'_i x_i = X'_i \oplus x_i\) is an anti-indicator of goodness of fit for \(x_i\) to \(X_i\). The larger \(X'_i \oplus x_i\) is, the worse the fitness be. Now we take sight of all the \(x_i\), \(i = 1, \ldots, n\), as a whole to \(X_i\). If \(X_i\) is a point in the solution space \(\langle \alpha', \beta \rangle\), \(X_i\) is a minimum point of the gSp. The product \(X_i\) is beneficial. Any \(X_i \in \langle \alpha', \beta \rangle, j = 1, \ldots, m\) will indicate a beneficial product. If \(X_j \in \langle \alpha', \beta \rangle\) for all \(j = 1, \ldots, m\) we may calculate \(\Phi(X_j)\) for each \(j\) and compare their magnitude, the larger be the worse. In this case \(|\Phi(X)|\) is more convenient. If \(|\Phi(X_1)| < |\Phi(X_2)| < \cdots < |\Phi(X_m)|\), it concludes that \(X_1\) is better than \(X_2, \ldots,\) better than \(X_m\). Here \(\Phi(X)\) played the part of decision function.

In order to improve the product the manager may think of rearrangement of the standards \(x_i\) of the workshop. Whether the new standards being an improvement may be tested by the decision function \(|\Phi(X)|\).

**Summary.** The gSp is an optimization problem of genuine (not pseudo) Boolean function. The main idea of solution is reducing the problem to solution of Boolean equations. Other type of optimization such as Boolean programming can be solved in similar manner, e.g. when the objective function \(\Phi(X)\) is given, its extrema \(M\) is easily obtained by the Schröder Theorem. An equation \(\Phi(X) \oplus M = 0\) is obtained. The constraints when given in form of equations or subsumptions can be reduced to equations. Adding these equations, the solution of the resulting equation is the final solution of the programming.

Secondly the gSp gives a mathematical model of application of genuine Boolean method to deal with problems in actual reality by conceiving Boolean
points or elements as sets, so that the abstract mathematics find stage to play a part in the real world. The proceeding examples show that it can solve problems in making regulations, testing the goodness of fit of some objectives to its basic factors and sometimes $|\Phi(X)|$ can be used as a decision function. All these problems are qualitative problems.

The author expects further applications either in mathematical theory and practice.

References