On the Solution of a Class of Integral Equations II

JOHN F. AHNER

Department of Mathematics, Vanderbilt University,
Nashville, Tennessee 37235

AND

JOHN S. LOWNDES

Department of Mathematics, University of Strathclyde,
Glasgow G1 1XH, Scotland

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A class of integral equations is investigated, particular examples of which occur in the consideration of certain three- and four-part mixed boundary-value problems in applied mathematics. A constructive method is given for reformulating the integral equations as Fredholm integral equations of the second kind and three examples are examined in detail to illustrate the general methods developed in the paper. © 1985 Academic Press, Inc.

1. INTRODUCTION

In a previous paper [1] we have examined a class of Fredholm integral equations of the first kind which arise in the study of some dual and triple integral and series equations occurring in several types of mixed boundary-value problems in applied mathematics. In that paper we gave a constructive method for reformulating the integral equations as Fredholm integral equations of the second kind from which the solutions of the original equations can be determined.

In this paper we extend the discussion to another class of integral equations of the type

\[
\left\{ \int_{\beta}^{\gamma} f_1(y) + \int_{\gamma}^{\delta} f_2(y) \right\} K(x, y) \, dy = g(x), \quad 0 \leq x < \beta, \quad 0 \leq \alpha < x < \beta,
\]

\[
= h(x), \quad \gamma < x < \delta, \quad 1.2
\]

where \( \beta \leq \gamma \) and the kernel function can take either of the forms

\[
K(x, y) = L_\sigma(x, y; \alpha), \quad \sigma = \min(x, y), \quad 1.3
\]

\[
K(x, y) = M_\lambda(x, y; \delta), \quad \lambda = \max(x, y). \quad 1.4
\]
The precise definitions of \( L_a(x, y; \alpha) \) and \( M_a(x, y; \beta) \) will be given in Sections 2 and 3, respectively, where it is shown that the problem of determining the solution functions \( f_1(y) \) and \( f_2(y) \) in terms of the known functions \( g(x) \) and \( h(x) \) can be reduced to that of solving a Fredholm integral equation of the second kind.

To illustrate the general methods three specific examples are then considered. In Section 4 a set of quadruple series equations involving Jacobi polynomials is investigated and using the algorithm developed in Section 2, it is readily shown that the solution of the equations can be found from the solution of a Fredholm integral equation of the second kind. Section 5 is devoted to the solution of some quadruple integral equations involving modified Bessel functions by means of the algorithm described in Section 3. As a check on the methods it is shown that limiting cases of our solutions are identical with those found by Lowndes \([2, 3]\) for the solutions of some analogous sets of triple series and triple integral equations.

In the final section of the paper we discuss the solution of some triple integral equations that arise in certain diffraction problems. This set of equations will be seen to be complementary to the triple integral equations investigated in \([1]\); but they are not amenable to solution by the method described in that paper.

2. Integral Equations with the Kernel \( L_a(x, y; 0) \)

We shall first of all consider the problem of determining the solution functions \( f_1(x) \) and \( f_2(x) \) of the integral equations

\[
\begin{align*}
&\left\{ \int_0^a f_1(y) + \int_b^c f_2(y) \right\} L_a(x, y; 0) \, dy = g(x), \quad 0 < x < a, \\
&\quad = h(x), \quad b < x < c,
\end{align*}
\]

(2.1)

where \( a \leq b, \sigma = \min(x, y), g(x) \) and \( h(x) \) are prescribed functions and in general the kernel function has the form

\[
L_\beta(x, y; \alpha) = \int_\alpha^\beta \varphi(t) \, p_1(t, x) \, p_2(t, y) \, dt, \quad 0 \leq \alpha < \beta \leq c.
\]

(2.2)

In the above definition \( \varphi(t) \) is a given nonzero function and the functions \( p_i(t, x), i = 1, 2, \) are the known kernels of the following integral operators

\[
P(A, x) f(x) = \int_\gamma^x p_i(t, x) \, f(t) \, dt,
\]

(2.3)

\[
P_1^*(x, \delta) f(x) = \int_x^\delta p_i(x, t) \, f(t) \, dt, \quad 0 \leq \gamma < x < \delta \leq c,
\]
which are assumed to have unique inverses denoted by \((P_1(y, x))^{-1}\) and \((P_1^*(x, \delta))^{-1}\), respectively.

The integral equations (2.1) and (2.2) can be written as

\[
\begin{align*}
\int_0^x L_\gamma(x, y; 0) f^*_\gamma(y) \, dy + \int_x^c I_{\gamma}(x, y; 0) f^*_\gamma(y) \, dy &= g(x), \quad 0 < x < a, \\
\int_0^a L_\gamma(x, y; 0) f^*_\gamma(y) \, dy + \left\{ \int_b^c L_\gamma(x, y; 0) + \int_x^c L_x(x, y; 0) f^*_\gamma(y) \, dy \right\} &= h(x), \quad b < x < c.
\end{align*}
\]

On inverting the orders of integration in these equations and using the definitions (2.3) and (2.4), we find that they assume the operational forms

\[
\begin{align*}
P_1(0, x) \phi(x) [F_1(x) + P_1^*(b, c) f_2(x)] &= g(x), \quad 0 < x < a, \\
P_1(0, a) \phi(x) F_1(x) + P_1(0, b) \phi(x) P_1^*(b, c) f_2(x) \\
&\quad + P_1(b, x) \phi(x) F_2(x) = h(x), \quad b < x < c,
\end{align*}
\]

where we have written

\[
F_1(x) = P_1^*(x, a) f_1(x), \quad F_2(x) = P_1^*(x, c) f_2(x).
\]

Applying the inverse operators \((P_1(0, x))^{-1}\) and \((P_1(b, x))^{-1}\) to Eqs. (2.7) and (2.8), respectively, we get

\[
\begin{align*}
\phi(x) [F_1(x) + P_1^*(b, c) f_2(x)] &= G(x), \quad 0 < x < a, \\
\phi(x) F_2(x) + (P_1(b, x))^{-1} [P_1(0, a) \phi(x) F_1(x) \\
&\quad + P_1(0, b) \phi(x) P_1^*(b, c) f_2(x)] = H(x), \quad b < x < c,
\end{align*}
\]

where \(G(x)\) and \(H(x)\) are the known functions

\[
G(x) = (P_1(0, x))^{-1} g(x), \quad H(x) = (P_1(b, x))^{-1} h(x).
\]

On eliminating \(F_1(x)\) between Eqs. (2.10) and (2.11) we find that \(F_2(x)\) satisfies the equation

\[
\begin{align*}
\phi(x) F_2(x) + (P_1(b, x))^{-1} P_1(0, a) \phi(x) [G(x) - \phi(x) P_1^*(b, c) f_2(x)] \\
&\quad + (P_1(b, x))^{-1} P_1(0, b) \phi(x) P_1^*(b, c) f_2(x) = H(x), \quad b < x < c,
\end{align*}
\]
which after some obvious simplification reduces to

$$
\phi(x) F_2(x) + (P_1(b, x))^{-1} P_1(a, b) \phi(x) P_2^*(b, c) f_2(x) - \mathcal{H}(x), \quad b < x < c,
$$

(2.14)

where

$$
\mathcal{H}(x) = H(x) - (P_1(b, x))^{-1} P_1(0, a) G(x)
$$

(2.15)

is known.

Using the definitions (2.9) we can write

$$
f_1(x) = (P_2^*(x, a))^{-1} F_1(x), \quad 0 < x < a,
$$

(2.16)

$$
f_2(x) = (P_2^*(x, c))^{-1} F_2(x), \quad b < x < c,
$$

(2.17)

and from Eqs. (2.14) and (2.17) we find that $F_2(x)$ satisfies the operational integral equation

$$
\phi(x) F_2(x) + (P_1(b, x))^{-1} P_1(a, b) \phi(x) P_2^*(b, c)(P_2^*(x, c))^{-1} F_2(x) = \mathcal{H}(x),
$$

(2.18)

After some manipulations it can be shown that the above equation is the integral equation

$$
\phi(x) F_2(x) + (P_1(b, x))^{-1} \int_b^x L_b(x, y; a)(P_2^*(y, c))^{-1} F_2(y) dy = \mathcal{H}(x),
$$

(2.19)

which can finally be brought to the form

$$
\phi(x) F_2(x) + \int_b^x F_2(y)(P_1(b, x))^{-1}(P_2(b, y))^{-1} L_b(x, y; a) dy = \mathcal{H}(x),
$$

(2.20)

Equation (2.20) is a Fredholm integral equation of the second kind from which $F_2(x)$ can be found. The solution function $f_2(x)$ can then be obtained from Eq. (2.17) and the function $F_1(x)$ then follows from Eq. (2.10). Finally the solution function $f_1(x)$ can be determined from Eq. (2.16).
We next consider the integral equations
\[
\left\{ \int_a^b u_1(y) + \int_c^d u_2(y) \right\} M_\lambda(x, y; d) \, dy = g(x), \quad 0 < a < x < b, \quad (3.1)
\]
\[
= h(x), \quad c < x < d, \quad (3.2)
\]
where \( b \leq c \), \( u_1(x) \) and \( u_2(x) \) are the required solution functions, \( g(x) \) and \( h(x) \) are known functions, \( \lambda = \text{max}(x, y) \) and in general the kernel function is defined by
\[
M_\lambda(x, y; \beta) = \int_a^b \psi(t) p_1(x, t) p_2(y, t) \, dt, \quad 0 < \alpha < \beta < d. \quad (3.3)
\]

In the above integral \( \psi(t) \) is a prescribed nonzero function and the \( p_i(x, t), i = 1, 2 \), are the known kernel functions of the integral operators defined by Eqs. (2.4). It should be noted that the arguments of the functions \( p_i \) in Eq. (3.3) are transposed with those in Eq. (2.3), however, we can still use the same definitions (2.4) for the integral operators \( P_i \) and \( P_i^* \).

On taking \( \beta = d, \alpha = \lambda = \text{max}(x, y) \) in definition (3.3) and substituting the resulting expression for \( M_\lambda(x, y; \beta) \) in Eqs. (3.1) and (3.2) we find, after inverting the orders of the integrations, that the integral equations take on the operational forms
\[
P_1^*(x, b) \psi(x) U_1(x) + P_1^*(b, d) \psi(x) P_2(a, b) u_1(x) + P_1^*(c, d) \psi(x) U_2(x) = g(x), \quad a < x < b, \quad (3.4)
\]
\[
P_1^*(x, d) \psi(x) [P_2(a, b) u_1(x) + U_2(x)] = h(x), \quad c < x < d, \quad (3.5)
\]
where we have written
\[
U_1(x) = P_2(a, x) u_1(x), \quad U_2(x) = P_2(c, x) u_2(x), \quad (3.6)
\]
whence
\[
u_1(x) = (P_2(a, x))^{-1} U_1(x), \quad a < x < b, \quad (3.7)
\]
\[
u_2(x) = (P_2(c, x))^{-1} U_2(x), \quad c < x < d. \quad (3.8)
\]

Applying the operator \( (P_1^*(x, d))^{-1} \) to Eq. (3.5) we see that \( U_2(x) \) and \( u_1(x) \) are related through the equation
\[
\psi(x) U_2(x) = \bar{H}(x) - \bar{G}(x) P_2(a, b) u_1(x), \quad c < x < d, \quad (3.9)
\]
where $\tilde{H}(x)$ is the known function

$$\tilde{H}(x) = (P_1^*(x, d))^{-1} h(x). \quad (3.10)$$

Following a procedure very similar to that used in Section 2 it can then be shown that the function $U_1(x)$ satisfies the following Fredholm integral equation of the second kind

$$\psi(x) U_1(x) + \int_a^b U_1(y)(P_1^*(x, b))^{-1}(P_2^*(y, b))^{-1} M_h(x, y; c) \, dy = G(x), \quad a < x < b, \quad (3.11)$$

where the free term

$$G(x) = (P_1^*(x, b))^{-1} [g(x) - P_1^*(c, d) \tilde{H}(x)] \quad (3.12)$$

is a known function.

Once $U_1(x)$ has been found from this equation the solution function $u_1(x)$ can be obtained from Eq. (3.7). The function $U_2(x)$ then follows from Eq. (3.9) and finally the solution function $u_2(x)$ can be determined from Eq. (3.8).

### 4. The Solution of Some Quadruple Series Equations

To illustrate an application of the algorithm developed in Section 2 we shall now investigate the solution of a set of quadruple series equations involving Jacobi polynomials.

We consider the problem of determining the coefficients $A_n$, $n = 0, 1, 2, \ldots$, which satisfy the quadruple series equations (QSE)

$$\sum_{n=0}^{\infty} A_n q_n(\mu - v, \mu) J_n(\alpha, \mu; x) = g_1(x), \quad 0 \leq x < a, \quad (4.1)$$

$$= h_2(x), \quad b < x < c, \quad (4.2)$$

$$\sum_{n=0}^{\infty} A_n J_n(\alpha, \mu; x) = 0, \quad a < x < b, \quad (4.3)$$

$$= 0, \quad c < x \leq 1, \quad (4.4)$$

where $0 \leq a < b < c \leq 1$, $\alpha + 1 > \mu > v$, $0 < v < 1$, $g_1(x)$ and $h_2(x)$ are prescribed functions,

$$q_n(\mu - v, \mu) = \frac{\Gamma(\mu - v + n) \Gamma(1 + \alpha - \mu + n)}{\Gamma(\mu + n) \Gamma(1 + \alpha + v - \mu + n)} \quad (4.5)$$
and
\[ J_n(x, \mu; x) = {}_2F_1(x + n, -n; \mu; x) \]

(4.6)
is the Jacobi polynomial \([S]\).

The orthogonality relation for Jacobi polynomials is
\[ \int_0^1 t^{1/2 - 1/2} (1 - t)^{x - 1/2} J_n(x, \mu; t) J_n(x, \mu; t) \, dt = \frac{\delta_{mn}}{A_n^2}, \quad x + 1 > \mu > 0, \]

(4.7)
where \(\delta_{mn}\) is the Kronecker-delta and
\[ A_n^2 = \frac{(x + 2n + 1)(x + n + 1)(x + n)}{(x + 1)(x + 2)(x + 3)} \]

(4.8)

From [2] we have the result that
\[ L^{(1)}(x, y; 0) = \{J(y)\}^2 (xy)^{1/2 - 1/2} \sum_{n=0}^{\infty} A_n^2 q_n(\mu - v, \mu) J_n(x, \mu; x) J_n(x, \mu; y) \]
\[ = \int_0^\sigma \phi(t)(x - t)^{1/2 - 1/2} (y - t)^{1/2 - 1/2} \, dt, \]

(4.9)

(4.10)
where \(x + 1 + v > \mu > v > 0\), \(\phi(t) = t^{1/2 - 1/2}(1 - t)^{1/2 - 1/2}\) and \(\sigma = \min(x, y)\).

To reduce the problem of solving the QSE to that of solving a pair of integral equations we introduce two unknown functions \(f_1(x)\) and \(f_2(x)\) which are defined by the equations
\[ \sum_{n=0}^{\infty} A_n J_n(x, \mu; x) = (1 - x)^{1/2 - 1/2} f_1(x), \quad 0 < x < a, \]

(4.11)
\[ = (1 - x)^{1/2 - 1/2} f_2(x), \quad b < x < c. \]

(4.12)

On using the orthogonality relation (4.7) we find that Eqs. (4.3), (4.4), (4.11), and (4.12) yield the following expression for the coefficients \(A_n\) in terms of the introduced functions \(f_1(x)\) and \(f_2(x)\)
\[ A_n = A_n^2 \left\{ \int_0^a f_1(y) + \int_b^c f_2(y) \right\} y^{1/2 - 1} J_n(x, \mu; y) \, dy, \quad x + 1 > \mu > 0. \]

(4.13)

Substituting this result into Eqs. (4.1) and (4.2), interchanging the orders of summation and integration and using the results (4.9) and (4.10), we
ARRIVE AT THE FOLLOWING INTEGRAL EQUATIONS OF THE FIRST KIND SATISFIED BY THE FUNCTIONS \( f_1(x) \) AND \( f_2(x) \),

\[
\left\{ \int_0^a f_1(y) + \int_b^c f_2(y) \right\} L_\sigma^{(1)}(x, y; 0) dy = \{ \Gamma(v) \}^2 x^{\mu-1} g_1(x), \quad 0 < x < a, \tag{4.14}
\]

\[
= \{ \Gamma(v) \}^2 x^{\mu-1} h_2(x), \quad b < x < c, \tag{4.15}
\]

whose kernel \( L_\sigma^{(1)}(x, y; 0) \) is given by Eq. (4.9) when \( \sigma + 1 + \nu > \mu - \nu > 0 \).

The above integral equations can be seen to be a special case of the general equations discussed in Section 2. On comparing Eqs. (2.1), (2.2), and (2.3) with Eqs. (4.14), (4.15), and (4.9), respectively, we see that we must take \( g(x) = \{ \Gamma(v) \}^2 x^{\mu-1} g_1(x) \), \( h(x) = \{ \Gamma(v) \}^2 x^{\mu-1} h_2(x) \), \( \phi(t) = \nu - \nu - 1 - \nu - 1 \), and \( p_1 = p_2 = (x - t)^{\nu-1} \).

Associated with the kernel functions \( p_1 = p_2 \) we can, following Eqs. (2.4), define the Abel-type integral operators

\[
P(\gamma, x)f(x) = \int_\gamma^x (x - t)^{\nu-1} f(t) dt, \tag{4.16}
\]

\[
P^*(x, \delta) f(x) = \int_x^\delta (t - x)^{\nu-1} f(t) dt,
\]

where \( 0 \leq \gamma < x < \delta \), which, by the assumption that \( 0 < \nu < 1 \), have the inverses

\[
(P(\gamma, x))^{-1} f(x) = \frac{\sin \nu \pi}{\pi} \frac{d}{dx} \int_\gamma^x \frac{f(t)}{(x - t)^{\nu}} dt, \tag{4.17}
\]

\[
(P^*(x, \delta))^{-1} f(x) = \frac{-\sin \nu \pi}{\pi} \frac{d}{dx} \int_x^\delta \frac{f(t)}{(t - x)^{\nu}} dt.
\]

Using the above results and the general solution given in Section 2 we can now write down the solutions of Eqs. (4.14) and (4.15) and hence obtain the solution of the QSE.

The solution coefficients \( A_n \), \( n = 0, 1, 2, \ldots \), of the QSE are given in terms of the functions \( f_1(x) \) and \( f_2(x) \) by Eq. (4.13) where, from Eqs. (2.16) and (2.17), we have

\[
f_1(x) = -\frac{\sin \nu \pi}{\pi} \frac{d}{dx} \int_x^a \frac{F_1(t)}{(t - x)^{\nu}} dt, \quad 0 < \nu < 1, 0 < x < a, \tag{4.18}
\]

\[
f_2(x) = -\frac{\sin \nu \pi}{\pi} \frac{d}{dx} \int_x^c \frac{F_2(t)}{(t - x)^{\nu}} dt, \quad 0 < \nu < 1, b < x < c. \tag{4.19}
\]
Using Eq. (2.10) we see that $F_1(x)$ and $f_2(x)$ are related through the equation

$$\phi(x) F_1(x) = G(x) - \phi(x) \int_b^c (t-x)^{\nu-1} f_2(t) \, dt, \quad 0 < x < a, \quad (4.20)$$

where $\phi(x) = x^{\mu-\nu-1}(1-x)^{\mu-\nu-\alpha}$ and

$$G(x) = \left\{ \Gamma(v) \right\}^2 \frac{\sin \nu \pi}{\pi} \frac{d}{dx} \int_0^x \frac{t^{\mu-1} g_1(t)}{(x-t)^{\nu}} \, dt \quad (4.21)$$

is a known function.

From Eq. (2.20), after some simple but lengthy manipulations, it can be shown that the function $F_2(x)$ satisfies the Fredholm integral equation of the second kind

$$\phi(x) F_2(x) + \int_b^c F_2(y) T(x, y) \, dy = \mathcal{H}(x), \quad b < x < c, \quad (4.22)$$

whose kernel is given by

$$T(x, y) = \frac{(\sin \nu \pi)^2}{\pi^2 [ (x-b)(y-b) ]^\nu} \int_a^b \frac{(b-t)^{2\nu} \phi(t)}{(x-t)(y-t)} \, dt, \quad (4.23)$$

where $\phi(t) = t^{\mu-\nu-1}(1-t)^{\mu-\nu-\alpha}$ and the free term $\mathcal{H}(x)$ is the known function

$$\mathcal{H}(x) = \left\{ \Gamma(v) \right\}^2 \frac{\sin \nu \pi}{\pi} \frac{d}{dx} \int_b^x \frac{t^{\mu-1} g_2(t)}{(x-t)^{\nu}} \, dt - \frac{\sin \nu \pi}{\pi(x-b)^\nu} \int_0^a \frac{(b-t)^\nu}{x-t} G(t) \, dt, \quad b < x < c. \quad (4.24)$$

Once the function $F_2(x)$ has been determined from Eq. (4.22), $f_2(x)$ can be found from Eq. (4.19); $F_1(x)$ is then obtained from Eq. (4.20) and $f_1(x)$ calculated from Eq. (4.18). Finally knowing $f_1(x)$ and $f_2(x)$ the solution coefficients $A_n$ which satisfy the QSE can be determined using Eq. (4.13).

It is of interest to note that in the limiting case when $c \to 1$ the QSE reduce to the triple series equations of the second kind solved by Lowndes [2] and the above solution (with $c = 1$) is then identical with his solution.
5. The Solution of Some Quadruple Integral Equations

A set of quadruple integral equations which occur in the solution of some mixed boundary-value problems involving the wave equation is as follows

\[
\frac{2}{\pi^2} \int_0^{\infty} t \sinh(\pi t) A(t) K_{it}(px) dt = 0, \quad 0 < x < a, \quad (5.1)
\]

\[
= 0, \quad b < x < c, \quad (5.2)
\]

\[
\frac{2}{\pi^2} \int_0^{\infty} t \sinh(2\pi t) I(\alpha + it) I(\alpha - it) A(t) K_{it}(px) dt = g_1(x), \quad a < x < b, \quad (5.3)
\]

\[
= h_1(x), \quad c < x < \infty, \quad (5.4)
\]

where \(0 < a < b < c < \infty\), \(0 < \alpha < \frac{1}{2}\), \(A(t)\) is the solution function to be determined, \(g_1(x)\) and \(h_1(x)\) are prescribed functions and \(K_{it}(px)\), \(\text{Re}(p) > 0\), is the modified Bessel (Macdonald) function. These equations will be seen to be an extension of the triple integral equations solved in a previous paper [3].

Two relevant results which we shall require are given in [3] and will be stated below for easy reference.

The Kontorovich–Lebedev transform of the function \(f(y)\), \(0 \leq y < \infty\), is defined by

\[
F(t) = \int_0^{\infty} y^{-1} f(y) K_{it}(py) dy, \quad \text{Re}(p) > 0, \quad (5.5)
\]

with the inversion formula

\[
f(y) = \frac{2}{\pi^2} \int_0^{\infty} t \sinh(\pi t) F(t) K_{it}(py) dt. \quad (5.6)
\]

A transform result is

\[
\frac{2}{\pi^2} \int_0^{\infty} t \sinh(2\pi t) I(\alpha + it) I(\alpha - it) K_{it}(px) K_{it}(py) dt
\]

\[
= \frac{2\pi}{[\Gamma((1/2) - \alpha)]^2} (xy)^{\alpha} e^{\phi(x, y)} M_\alpha^{(1)}(x, y; \infty), \quad 0 \leq \alpha < \frac{1}{2}, \quad \text{Re}(p) > 0, \quad (5.7)
\]
where
\[ M_\lambda^{(1)}(x, y; \infty) = \int_{x}^{\infty} \frac{e^{-\lambda t}}{[(t-x)(t-y)]^{a+(1/2)}} dt, \quad \lambda = \max(x, y). \] (5.9)

To reduce the quadruple integral equations (QIE) given by Eqs. (5.1) to (5.4) to a pair of integral equations of the type discussed in Section 3, we introduce the functions \( u_1(x) \) and \( u_2(x) \) defined by the equations
\[ \int_{-\infty}^{x} t \sinh(\pi t) A(t) K_\nu(\nu x) dt = x^{-\alpha} e^{-\nu x} u_1(x), \quad a < x < b, \] (5.10)
\[ = x^{-\alpha} e^{-\nu x} u_2(x), \quad c < x < \infty. \] (5.11)

Applying the inversion formula (5.6) to Eqs. (5.1), (5.2), (5.10), and (5.11) we find that the solution function of the QIE is given in terms of the unknown functions \( u_1(x) \) and \( u_2(x) \) by the expression
\[ A(t) = \left\{ \int_{a}^{b} u_1(y) + \int_{c}^{\infty} u_2(y) \right\} y^{-\alpha} e^{-\nu y} K_\nu(\nu y) dy. \] (5.12)

On substituting the above form for \( A(t) \) into Eqs. (5.3) and (5.4), interchanging the order of the integrations and making use of the results (5.7) and (5.8) we find that \( u_1(x) \) and \( u_2(x) \) satisfy the integral equations
\[ \left\{ \begin{array}{l}
\int_{a}^{b} u_1(y) + \int_{c}^{\infty} u_2(y) \\
M_\lambda^{(1)}(x, y; \infty)
\end{array} \right\} dy = [\Gamma(\frac{1}{2} - \alpha)]^2 (2\pi x^\alpha)^{-1} e^{-\nu x} g_1(x), \]
\[ a < x < b, \] (5.13)
\[ = [\Gamma(\frac{1}{2} - \alpha)]^2 (2\pi x^\alpha)^{-1} e^{-\nu x} h_1(x), \]
\[ c < x < \infty. \] (5.14)

The above equations are a special case of the integral equations investigated in Section 3 and using the general solution given there we can find the solution functions \( u_1(x) \) and \( u_2(y) \) in the following way.

In the definition (5.9) of the kernel \( M_\lambda^{(1)}(x, y; \infty) \) we see that \( \psi(t) = e^{-2\nu t}, p_1 = p_2 = (t-x)^{-(\alpha+(1/2))} \) and the associated Abel-type integral operators are
\[ P(\gamma, x)f(x) = \int_{\gamma}^{x} \frac{f(t)}{(x-t)^{\alpha+(1/2)}} dt, \quad \gamma \leq x < \delta, \] (5.15)
which, since $0 \leq \alpha < \frac{1}{2}$, have the inverses

$$
(P(y, x))^{-1} f(x) = \frac{\cos \alpha \pi}{\pi} \frac{d}{dx} \int_{x}^{\infty} \frac{f(t)}{(x-t)^{1/2}-\alpha} dt,
$$

$$
(P^*(x, \delta))^{-1} f(x) = \frac{-\cos \alpha \pi}{\pi} \frac{d}{dx} \int_{x}^{\delta} \frac{f(t)}{(t-x)^{1/2}-\alpha} dt.
$$

Using Eqs. (3.7) and (3.8) we find that $u_1(x)$ and $u_2(x)$ can be represented in terms of the functions $U_1(x)$ and $U_2(x)$ by the expressions

$$
u_1(x) = \frac{\cos \alpha \pi}{\pi} \frac{d}{dx} \int_{x}^{\infty} \frac{U_1(t)}{(x-t)^{1/2}-\alpha} dt, \quad 0 \leq \alpha < \frac{1}{2}, \quad a < x < b,
$$

$$
u_2(x) = \frac{\cos \alpha \pi}{\pi} \frac{d}{dx} \int_{c}^{x} \frac{U_2(t)}{(x-t)^{1/2}-\alpha} dt, \quad 0 \leq \alpha < \frac{1}{2}, \quad c < x < \infty.
$$

From Eqs. (3.9) and (3.10) it follows that $V_1(x)$ and $u_1(x)$ are connected by the equation

$$
e^{-2px}U_2(x) = \tilde{H}(x) - e^{-2px} \int_{c}^{b} \frac{u_1(t)}{(x-t)^{1/2}+\alpha} dt, \quad c < x < \infty,
$$

where

$$
\tilde{H}(x) = -\frac{1}{2\pi} \int_{a}^{x} \frac{e^{-s\pi}h_1(t)}{(x-t)^{1/2}+\alpha} dt
$$

is a known function.

Finally, after some manipulations, it can be shown from Eqs. (3.11) and (3.12) that $U_1(x)$ satisfies the following Fredholm integral equation of the second kind

$$
e^{-2px} U_1(x) + \int_{c}^{b} U_1(y) S(x, y) dy = \mathcal{G}(x), \quad a < x < b,
$$

where

$$
S(x, y) = \frac{\cos^2(\alpha \pi)}{\pi^2[(b-x)(b-y)]^{1/2}-\alpha} \int_{b}^{\infty} \frac{(t-b)^{1/2}e^{-2pt}}{(t-x)(t-y)} dt
$$

and $\mathcal{G}(x)$ is the known function

$$
\mathcal{G}(x) = -\frac{1}{2\pi} \int_{a}^{b} \frac{e^{-s\pi}h_1(t)}{(x-t)^{1/2}+\alpha} dt
$$

$$
\frac{\cos \alpha \pi}{\pi(b-x)^{1/2}-\alpha} \int_{x}^{\infty} \frac{(u-b)^{1/2}-\alpha \tilde{H}(u)}{(x-u)} du.
$$
When the functions \( u_1(x) \) and \( u_2(x) \) have been determined from Eqs. (5.17) to (5.23), the solution function \( A(t) \) of the QIE can be found from Eq. (5.12).

As a check on the method we see that as \( c \to \infty \) the QIE reduce to the triple integral equations considered in [3] and the above solution, apart from a difference in notation, becomes identical with the solution given in that paper.

6. The Solution of Some Triple Integral Equations

In this final section of the paper we examine a set of triple integral equations which occur in some three-part mixed boundary-value problems in diffraction theory.

We consider the problem of finding the function \( A(t) \) which satisfies the triple integral equations (TIE)

\[
\int_0^\infty t^{1-\mu-\nu} (t^2 - k^2)^\beta A(t) J_\mu(xt) \, dt = g(x), \quad 0 < x < a,
\]

\[
= h(x), \quad b < x < \infty,
\]

\[
\int_0^a tA(t) J_\nu(xt) \, dt = 0, \quad a < x < b,
\]

where \( k > 0, \mu, \nu > \beta > -1, J_\mu(xt) \) is the Bessel function of the first kind and \( g(x) \) and \( h(x) \) are given functions.

These equations are an extension of the dual integral equations solved in [1] and [4] and are different from the set of triple integral equations discussed in [11.

As a first step in solving the TIE we introduce two functions \( f_1(x) \) and \( f_2(x) \) defined by the equations

\[
\int_0^\infty tA(t) J_\mu(xt) \, dt = f_1(x), \quad 0 < x < a,
\]

\[
= f_2(x), \quad b < x < \infty.
\]

An application of the inverse Hankel transform theorem to Eqs. (6.3), (6.4), and (6.5) yields the following expression for the solution function of the TIE

\[
A(t) = \left\{ \int_0^a f_1(y) + \int_b^\infty f_2(y) \right\} yJ_\nu(yt) \, dy, \quad 0 < t < \infty.
\]
Using this definition of \(A(t)\) in Eqs. (6.1) and (6.2) and inverting the orders of the integrations we arrive at the pair of integral equations

\[
\left\{ \int_{0}^{\infty} f_1(y) + \int_{0}^{2} f_2(y) \right\} L^{(2)}_{\sigma}(x, y; 0) \, dy = g(x), \quad 0 < x < a, \quad (6.7)
\]

\[
= h(x), \quad b < x < \infty, \quad (6.8)
\]

where

\[
L^{(2)}_{\sigma}(x, y; 0) = y \int_{0}^{\infty} t^{1-\mu-v(t^2-k^2)^{\beta} J_{\mu}(xt) J_{v}(yt) \, dt.} \quad (6.9)
\]

From Sonine's second finite integral and the Hankel inversion theorem it is shown in [6] that

\[
L^{(2)}_{\sigma}(x, y; 0) - \int_{0}^{\sigma} \phi(t) p_{1}(t, x) p_{2}(t, y) \, dt, \quad \sigma = \min(x, y), \quad (6.10)
\]

where \(\phi(t) = 2^{2\beta-\mu-vt^{2\beta}}\) and

\[
p_{1}(t, x) = 2^{\mu-\beta} k^{1+\beta-\mu} t^{1+2\beta-\mu} x^{-\mu(x^2-t^2)^{1/2}}(\mu-\beta-1) \left\{ k(x^2-t^2)^{1/2} \right\}, \quad x > t, \quad (6.11)
\]

\[
p_{2}(t, y) = 2^{\nu-\beta} k^{1+\beta-\nu} t^{2\beta-\nu} y^{-\nu(y^2-t^2)^{1/2}}(\nu-\beta-1) \left\{ k(y^2-t^2)^{1/2} \right\}, \quad y > t. \quad (6.12)
\]

Associated with the kernel functions \(p_{i}(t, x), \ i = 1, \ 2\), we can define the integral operators

\[
P_{1}(\gamma, x) f(x) = I_{k}(\beta - \frac{1}{2}\mu, \mu - \beta; \gamma, x) f(x), \quad (6.13)
\]

\[
P_{1}^{\ast}(x, \delta) f(x) = x K_{k}(\beta - \frac{1}{2}\mu, \mu - \beta; \gamma, x) x^{-1} f(x), \quad (6.14)
\]

\[
P_{2}(\gamma, x) f(x) = x I_{k}(\beta - \frac{1}{2}\nu, \nu - \beta; \gamma, x) x^{-1} f(x), \quad (6.15)
\]

\[
P_{2}^{\ast}(x, \delta) f(x) = K_{k}(\beta - \frac{1}{2}\nu, \nu - \beta; \gamma, x) f(x), \quad (6.16)
\]

where \(I_{k}\) and \(K_{k}\) are modifications of the generalised Erdélyi-Kober operators of fractional integration [4].

These operators have the general definitions

\[
I_{\lambda}(\eta, \alpha; \gamma, x) f(x)
\]

\[
= 2^{2\alpha-2(\gamma + \alpha)k^{1-2} \int_{\gamma}^{x} t^{2\eta} (x^2 - t^2)^{1/2} G_{\alpha} \left[ k(x^2-t^2)^{1/2} \right] f(t) \, dt, \quad \alpha > 0
\]

\[
= x^{-1-2\eta-2\alpha} \mathcal{D}_{x}^{m} \left\{ x^{2m+1+2\alpha+2\beta} I_{\lambda}(\eta, \alpha + m; \gamma, x) f(x) \right\}, \quad \alpha < 0 \quad (6.17)
\]
A CLASS OF INTEGRAL EQUATIONS II

\[ K_\lambda (\eta, \alpha; x, \delta) f(x) = 2^2 x^{2\eta} k^{1-\lambda} \int_\delta^\Delta t^{t - 2\eta - 2x(t^2 - x^2)^{1/2}(\alpha - 1)} G_k[k(t^2 - x^2)^{1/2}] f(t) dt, \]
\[ \alpha > 0 \]
\[ = (-1)^m x^{2\eta - 1} \Phi_x \{ x^{2m + 1 - 2\eta} K_\lambda (\eta - m, \alpha + m; x, \delta) f(x) \}, \quad \alpha < 0 \]

(6.18)

where \( 0 \leq \gamma < x < \delta, \lambda = k \) or \( \lambda = ik, k \geq 0, G_k(z) = J_{\lambda - 1}(z) \) is the Bessel function of the first kind, \( G_{ik}(z) = I_{\lambda - 1}(z) \) is the modified Bessel function of the first kind, \( m \) is a positive integer such that \( 0 < \alpha + m < 1 \) when \( \alpha < 0 \) and \( \Phi \) denotes the differential operator \( \Phi = d/dx(1/2x) \).

The general inverse operators are given by

\[ I_k^{-1}(\eta, \alpha; \gamma, x) = I_{ik}(\eta + \alpha, -\alpha; \gamma, x), I_k^{-1}(\eta, \alpha; \gamma, x) = I_k(\eta + \alpha, -\alpha; \gamma, x), \]

(6.19)

\[ K_k^{-1}(\eta, \alpha; x, \delta) = K_{ik}(\eta + \alpha, -\alpha; x, \delta), K_k^{-1}(\eta, \alpha; x, \delta) = K_k(\eta + \alpha, -\alpha; x, \delta), \]

(6.20)

and it is easily shown that the operators satisfy the inner product relation

\[ \int_{\gamma}^{\delta} x f(x) I_{\lambda}(\eta, \alpha; \gamma, x) g(x) dx = \int_{\gamma}^{\delta} x g(x) K_{\lambda}(\eta, \alpha; x, \delta) f(x) dx. \]

(6.21)

From the above definitions it follows that the integral operators defined by Eqs. (6.13)-(6.16) possess the inverses given by

\[ (P_1(\gamma, x))^{-1} f(x) = I_{ik}(\mu, \mu; \gamma, x) f(x), \]

(6.22)

\[ (P_1^*(\delta, x))^{-1} f(x) = xK_{ik}(\mu, \mu; x, \delta) x^{-1} f(x), \]

(6.23)

\[ (P_2(\gamma, x))^{-1} f(x) = xI_{ik}(\nu, \nu; \gamma, x) x^{-1} f(x), \]

(6.24)

\[ (P_2^*(\delta, x))^{-1} f(x) = K_{ik}(\nu, \nu; x, \delta) f(x). \]

(6.25)

We are now in a position to use the general results of Section 2 to write down the solution functions \( f_1(x) \) and \( f_2(x) \) of the pair of integral equations (6.7) and (6.8).

Using Eqs. (2.16) and (2.17) we see that \( f_1(x) \) and \( f_2(x) \) are given by

\[ f_1(x) = K_{ik}(\nu, \nu; x, a) F_1(x), \quad 0 < x < a, \]

(6.26)

\[ f_2(x) = K_{ik}(\nu, \nu; x, \infty) F_2(x), \quad b < x < \infty, \]

(6.27)
and from Eq. (2.10) it follows that $F_1(x)$ and $f_2(x)$ are related through the equation

$$
\phi(x)[F_1(x) + K_\beta(\beta - \frac{1}{2} v, v - \beta; b, \infty) f_2(x)] = G(x), \quad 0 < x < a,
$$

(6.28)

where $\phi(x) = 2^{2\beta - \mu} x^{\mu + \frac{v}{2}}$ and $G(x)$ is the known function

$$
G(x) = I_{ik}(\beta, 0, x) g(x).
$$

(6.29)

Finally, on using Eq. (2.20), it can readily be shown that the function $F_2(x)$ satisfies the Fredholm integral equation of the second kind

$$
\phi(x) F_2(x) + \int_b^\infty y F_2(y) I_{ik}(\beta, \mu; b, x) I_{ik}(\beta, v; y) y^{-1} L_{b}^{(2)}(x, y; a) dy = \mathcal{H}(x), \quad b < x < \infty,
$$

(6.30)

where

$$
L_{b}^{(2)}(x, y; a) = \int_a^b \phi(t) p_1(t, x) p_2(t, y) dt,
$$

(6.31)

$p_1$ and $p_2$ being defined by Eqs. (6.11) and (6.12) and

$$
\mathcal{H}(x) = I_{ik}(\beta, \mu; b, x)[h(x) - I_k(\beta - \frac{1}{2}, \mu - \beta; 0, a) G(x)],
$$

(6.32)

are all known functions.

When the functions $f_1(x)$ and $f_2(x)$ have been determined from Eqs. (6.26) to (6.32) the solution function $A(t)$ of the TIE then follows from Eq. (6.6).

If we make $b \to \infty$ the TIE reduce to the dual integral equations solved in [1] and in this case the above solution can be shown to reduce to the solution in agreement with the one given in that paper.

REFERENCES