J. Math. Anal. Appl. 370 (2010) 716–725

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the dependence of the reflection operator on boundary conditions for biharmonic functions

Tatiana V. Savina [a,b,1]

[a] Department of Mathematics, 321 Morton Hall, Ohio University, Athens, OH 45701, USA
[b] Condensed Matter and Surface Science Program, Ohio University, Athens, OH 45701, USA

ARTICLE INFO

Article history:
Received 12 December 2009
Available online 14 April 2010
Submitted by D. Khavinson

Keywords:
Reflection principle
Biharmonic functions

ABSTRACT

The biharmonic equation arises in areas of continuum mechanics including linear elasticity theory and the Stokes flows, as well as in a radar imaging problem. We discuss the reflection formulas for the biharmonic functions $u(x,y) \in \mathbb{R}^2$ subject to different boundary conditions on a real-analytic curve in the plane. The obtained formulas, generalizing the celebrated Schwarz symmetry principle for harmonic functions, have different structures. In particular, in the special case of the boundary, $\Gamma_0 := \{y = 0\}$, reflections are point-to-point when the given on Γ_0 conditions are $u = \partial_n u = 0$, $u = \Delta u = 0$ or $\partial_n u = \partial_n \Delta u = 0$, and point to a continuous set when $u = \partial_n \Delta u = 0$ or $\partial_n u = \Delta u = 0$ on Γ_0.

1. Introduction

In this paper we study the dependence of the structure of the reflection operator on different boundary conditions for the biharmonic functions, where a function $u(x,y)$ of class $C^4(U)$ is said to be biharmonic if it is a solution to the equation $\Delta^2 u = 0$ [4]. Here U is a domain in $\mathbb{R}^2$, and Δ is the Laplacian.

The obtained reflection operator is generally an integro-differential operator, which reduces in the simplest case to the celebrated local (point-to-point) Schwarz symmetry principle for harmonic functions.

In the case of harmonic functions, there are three basic types of boundary conditions: the Dirichlet, Neumann and Robin, and the Schwarz reflection principle can be stated as follows.

Theorem 1.1. *Let $\Gamma \subset \mathbb{R}^2$ be a non-singular real analytic curve and $P' \in \Gamma$. Then, there exist a neighborhood U of P' and an anti-conformal mapping $R : U \to U$ which is identity on Γ, permutes the components U_1, U_2 of $U \setminus \Gamma$ and relative to which any harmonic function $u(x,y)$ defined near Γ and:*

– *vanishing on Γ (the homogeneous Dirichlet condition) is odd* [16,27],

$$u(x_0, y_0) = -u\big(R(x_0, y_0)\big), \tag{1.1}$$

– *subject to the Neumann condition on Γ, $\frac{\partial u}{\partial n} = 0$, is even,*

$$u(x_0, y_0) = u\big(R(x_0, y_0)\big), \tag{1.2}$$

E-mail address: savin@ohio.edu.
URL: http://www.math.ohiou.edu/~tanya/.
[1] Research of this author was supported in part by OU Research Challenge Program, award #RC-09043.

0022-247X/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2010.04.036

– *subject to the Robin condition on Γ, $\alpha(x,y)\frac{\partial u(x,y)}{\partial n}+\beta(x,y)u(x,y)=0$, $(x,y)\in\Gamma$, can be continued by the following integro-differential operator* [5],

$$u(x_0,y_0)=u\big(R(x_0,y_0)\big)+\frac{1}{2i}\int_{\Gamma}^{R(x_0,y_0)}F(x,y,x_0,y_0)\omega\big(u(x,y)\big)-u(x,y)\omega\big(F(x,y,x_0,y_0)\big), \tag{1.3}$$

for any point (x_0,y_0) sufficiently close to Γ. Here $\omega(\cdot)=\frac{\partial}{\partial y}dx-\frac{\partial}{\partial x}dy$, and the integral is independent on the path joining an arbitrary point on Γ with the point $R(x_0,y_0)$. The function F is defined by the coefficients $\alpha(x,y)$ and $\beta(x,y)$, and the curve Γ [5]*. The mapping R mentioned above is given by*

$$R(x_0,y_0)=R(z_0)=\overline{S(z_0)}, \tag{1.4}$$

where $S(z)$ is the Schwarz function [8].

Note that if the point $(x_0,y_0)\in U_1$, then the "reflected" point $R(x_0,y_0)\in U_2$, and the mapping R depends only on the curve Γ and is defined only near Γ but may have conjugate-analytic continuation to a larger domain.

The Schwarz reflection principle has been studied by several researchers (see [1,5,7–12,14–18,24–27] and references therein). In particular, when Γ is a line, H. Poritsky [25] proved that a biharmonic function $u(x,y)$, a solution to the biharmonic equation $\Delta^2_{x,y}u=0$, defined for $y\geqslant 0$ and subject to conditions

$$u(x,0)=\frac{\partial u}{\partial y}(x,0)=0 \tag{1.5}$$

can be continued across the x-axis using the formula

$$u(x_0,y_0)=-u\big(R(x_0,y_0)\big)-2y_0\frac{\partial u}{\partial y}\big(R(x_0,y_0)\big)-y_0^2\Delta_{x,y}u\big(R(x_0,y_0)\big), \tag{1.6}$$

where $R(x_0,y_0)=(x_0,-y_0)$ and $\Delta_{x,y}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. He also applied this formula to problems of planar elasticity, studying bending of plates, where u is a deflection of a thin plate, clamped along $y=0$. An analogous formula has been obtained by R.J. Duffin [9] for three-dimensional case. Duffin also considered spherical boundaries and applied his result to study viscous flows, among other things. J. Bramble [7] considered continuation of biharmonic functions across the circular arc with the clamped boundary conditions

$$u=\frac{\partial u}{\partial r}=0\quad\text{for } x^2+y^2=a^2. \tag{1.7}$$

He has shown that u can be continued using the formula

$$u(x_0,y_0)=-u\big(R(x_0,y_0)\big)-\frac{r_0^2-a^2}{r_0^2}\left(r_0\frac{\partial u}{\partial r}u\big(R(x_0,y_0)\big)+\frac{1}{4}\big(r_0^2-a^2\big)\Delta_{x,y}u\big(R(x_0,y_0)\big)\right),$$

where $r_0=\sqrt{x_0^2+y_0^2}$ and a is the radius of the circle. J. Bramble also applied it to the elastic medium problems in two- and three-dimensional space. Papers by F. John [14] and L. Nystedt [24] are devoted to further studies of reflection of solutions of linear partial differential equations with various linear conditions on a hyperplane. R. Farwig in [11] considered reflection principle for biharmonic functions subject to different boundary conditions on a hyperplane. Following to H.A. Lorenz and R.J. Duffin [9], R. Farwig [11] applied his results to the Stokes system.

The purpose of this paper is to derive and to compare reflection formulas for biharmonic functions across real analytic curves in $\mathbb{R}^2$ subject to different boundary conditions occurring in a variational approach to Δ^2. The motivation of this study is as follows, most of the results for biharmonic functions mentioned above were obtained for the Dirichlet boundary conditions, see (1.5) and (1.7). However, unlike the Laplace equation, which is typically complemented with either the Dirichlet, Neumann or their linear combination, the Robin conditions, the biharmonic equation has much larger set of possible boundary conditions as it is discussed in details in [13]. For this study we have chosen the conditions: (i) $u=\partial_n u=0$, (ii) $u=\Delta u=0$, (iii) $u=\partial_n\Delta u=0$, (iv) $\partial_n u=\Delta u=0$, (v) $\partial_n u=\partial_n\Delta u=0$ and (vi) $\Delta u=\partial_n\Delta u=0$, appearing when one is considering a bilinear form associated with the biharmonic differential operator for $u,v\in C^4(\bar{\Omega})$ [21,22]

$$a(u,v):=\langle\Delta^2u,v\rangle=\int_{\Omega}v\Delta^2u\,d\omega.$$

Employing Green's formula to $a(u,u)$, we obtain

$$a(u,u)=\int_{\partial\Omega}(u\partial_n\Delta u-\partial_n u\Delta u)\,ds+\int_{\Omega}(\Delta u)^2\,d\omega. \tag{1.8}$$

The integral along the boundary $\partial\Omega$ in (1.8) disappears under conditions (i), (ii), (v) and (vi), and therefore $a(u,u) \geqslant 0$. This immediately imply that the operator is strictly positive for (i) and (ii), and the corresponding boundary value problems in $\bar{\Omega}$ are well posed. In the case (v), an extra condition, $\int_\Omega u\,d\omega = 0$, is usually imposed. This yields the coercivity of the bilinear form in suitable function spaces.

Conditions (i), (ii) and (v) are commonly used in applied problems for the biharmonic equation as well as for other equations with the biharmonic operator in the principal elliptic part. To give some examples, we remark that condition (i), the Dirichlet condition, corresponds to the clamped plate model [7,9,25], while condition (ii), the Navier boundary condition, corresponds to the hinged plate, when the contribution of curvature is neglected [6,13]. In the later case the plate is ideally hinged along all of its edges so that it is free to rotate and does not experience any torque or bending moments about the edges. Both the Dirichlet and Navier conditions are also used to model electrostatic actuation [20] alone with the governing non-linear nonlocal elliptic equation with the bi-Laplacian in the principal part.

Condition (v) for biharmonic equation was considered in [28], moreover, this condition is often used for the famous Cahn–Hilliard equation, $u_t = -\Delta(\Delta u + u - u^3)$, which is a semi-linear parabolic equation describing (among other processes) spinodal decomposition [23,28]. Pattern formation resulting this phase transition (spinodal decomposition) has been observed in alloys as well as polymer solutions and glasses. In this case the physical meaning of the second condition in (v), $\partial_n \Delta u = 0$, is a no-flux condition (none of the mixture can pass through the walls of the container). The first condition in (v), $\partial_n u = 0$, is the most natural way to ensure that the total free energy of the mixture decreases in time (this is a requirement from thermodynamics – the variational condition).

We have to comment on condition (vi), for which we will not be able to derive a new reflection formula. Indeed, consider biharmonic function in U subject to condition (vi) on Γ. Denote $v(x,y) := \Delta u(x,y)$, then function v is a harmonic function in U subject to conditions $v = \partial_n v = 0$ on Γ. Thus, $v \equiv 0$ in U, and therefore $u(x,y)$ is a harmonic function in U. Since any harmonic function is a solution to the problem in question, it means that the space of solution does not have finite dimension. Note also that as it was shown in [13], condition (vi) does not satisfy the Complementing Condition, introduced by S. Agmon, A. Douglis, and L. Nirenberg [2]. This condition is necessary for obtaining estimates up to the boundary for solutions of boundary value problems to the elliptic equations and, therefore, is crucial for the existence and uniqueness results [2].

In the case of conditions (iii) and (iv) only one of the term in the boundary integral (1.8) disappears, so our interest to this conditions is mostly theoretical, however, one of the problems arising in radar imaging may be reformulated as a boundary value problem for the biharmonic functions with condition $\partial_n u = u - \beta\Delta u = 0$ [3], which coincide with (i) if $\beta = 0$ and with (iv) if $\beta = -\infty$. We remark, that other linear combinations, the Robin-type conditions, involving (i)–(vi) are used in applications as well, but the corresponding reflections will be discussed elsewhere.

The reflection formula for the case of condition (i) was obtained in [1]. The only known in the literature result for the conditions (ii) and (v), is in [11], where the boundary conditions are given on x-axis, and the odd point-to-point reflection holds for (ii) and even for (v). The author is not aware of any results for the conditions (iii) and (iv). Our aim is to derive reflection formulas for the conditions (ii)–(v), that is, the formulas expressing the value of a biharmonic function $u(x,y)$ at an arbitrary point $(x_0, y_0) \in U_1$ in terms of its values at points in U_2, when the boundary conditions are given on a real-analytic curve, and to study the properties of these mappings.

We remark that the structure of the reflection formulas attracts attention of many researchers, interested, in particular, to answer the question: when the reflection is point-to-point [15,10], point to final set [18] or point to continuous set [5].

The structure of the paper is as follows. In Section 2 we discuss the so-called reflected fundamental solution for each case, whose properties depend on the boundary condition and, therefore, determine the structure of the reflection operator. In Section 3 we formulate and prove the main theorems.

2. The reflected fundamental solutions

This section is devoted to one of the key steps in deriving the reflection formula, that is, to the construction of the reflected fundamental solutions for each case of the boundary conditions.

Let $\Gamma \subset \mathbb{R}^2$ be a non-singular real analytic curve and point $P' \in \Gamma$. Consider the biharmonic differential operator in a neighborhood $U \subset \mathbb{R}^2$ of the point P'. Let U_1 and U_2 be the components of $U \setminus \Gamma$, and $P(x_0, y_0)$ be a point in U_1 sufficiently close to Γ. The fundamental solution can be written in the form

$$G = -\frac{1}{16\pi}\big((x-x_0)^2 + (y-y_0)^2\big)\ln\big((x-x_0)^2 + (y-y_0)^2\big) + g(x,y,x_0,y_0),$$

where g is a regular biharmonic function. It is obvious that G is a real-analytic function in $\mathbb{R}^2$ except for the point $P(x_0, y_0) \in U_1$.

Let us complexify the problem, that is, consider a complex domain W in the space $\mathbb{C}^2$ to which the function f defining the curve $\Gamma := \{f(x,y) = 0\}$ can be continued analytically such that $W \cap \mathbb{R}^2 = U$. Using the change of variables $z = x + iy$, $w = x - iy$, the equation of the complexified curve $\Gamma_{\mathbb{C}}$ can be rewritten in the form

$$f\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) = 0,$$

and if $\operatorname{grad} f(x,y) \neq 0$ on Γ, can be also rewritten in terms of the Schwarz function and its inverse, $w = S(z)$ and $z = \widetilde{S}(w)$ [8]. In the characteristic variables $G(z,w,z_0,w_0)$ can be rewritten as

$$G = -\frac{1}{16\pi}(z-z_0)(w-w_0)\ln\big[(z-z_0)(w-w_0)\big] + g(z,w,z_0,w_0). \tag{2.9}$$

It is obvious that the continuation of G to the complex space (2.9) has logarithmic singularities on the complex characteristics passing through this point, i.e., on $K_P := \{(x-x_0)^2 + (y-y_0)^2 = 0\}$.

Note, that the specific choice of the regular part g of the fundamental solution G does not affect our final result. Thus, for convenience we choose the fundamental solution in the form

$$\begin{aligned} &G(z,w,z_0,w_0) = -\frac{1}{16\pi}\big(G_1(z,w,z_0,w_0) + G_2(z,w,z_0,w_0)\big),\\ &G_1 = (z-z_0)(w-w_0)\big(\ln(z-z_0) - 1\big), \qquad G_2 = (z-z_0)(w-w_0)\big(\ln(w-w_0) - 1\big). \end{aligned} \tag{2.10}$$

Our goal is to construct (multiple-valued) functions $\widetilde{G}^{(j)}$, $j = \overline{1,5}$, which are biharmonic functions satisfying on $\Gamma_{\mathbb{C}}$ one of the pair of conditions: (i) $\widetilde{G} = G$ and $\partial_n \widetilde{G} = \partial_n G$, (ii) $\widetilde{G} = G$ and $\Delta\widetilde{G} = \Delta G$, (iii) $\partial_n \widetilde{G} = \partial_n G$ and $\Delta\widetilde{G} = \Delta G$, (iv) $\widetilde{G} = G$ and $\partial_n \Delta\widetilde{G} = \partial_n \Delta G$, (v) $\partial_n \widetilde{G} = \partial_n G$ and $\partial_n \Delta\widetilde{G} = \partial_n \Delta G$, and having singularities only on the characteristic lines intersecting the real space at point $Q = R(P)$ (see formula (1.4)) in the domain U_2 and intersecting $\Gamma_{\mathbb{C}}$ at $K_P \cap \Gamma_{\mathbb{C}}$. These functions are called the reflected fundamental solutions. According to (2.10) it is convenient to seek these functions in the form

$$\widetilde{G}^{(j)}(z,w,z_0,w_0) = -\frac{1}{16\pi}\big(\widetilde{G}_1^{(j)}(z,w,z_0,w_0) + \widetilde{G}_2^{(j)}(z,w,z_0,w_0)\big). \tag{2.11}$$

It is easy to check that for the case (i) functions $\widetilde{G}_1^{(1)}$ and $\widetilde{G}_2^{(1)}$ are

$$\begin{aligned} &\widetilde{G}_1^{(1)} = (z-z_0)(w-w_0)\big(\ln\big(\widetilde{S}(w) - z_0\big) - 1\big) + \big(z - \widetilde{S}(w)\big)(w-w_0),\\ &\widetilde{G}_2^{(1)} = (z-z_0)(w-w_0)\big(\ln\big(S(z) - w_0\big) - 1\big) + \big(w - S(z)\big)(z-z_0). \end{aligned}$$

Similar functions (up to the regular part) were used in [1], where function g in the expression for G (2.9) was different from the chosen in (2.10).

To obtain $\widetilde{G}_1^{(j)}$ and $\widetilde{G}_2^{(j)}$ for the cases (ii)–(v), we use the asymptotic expansions introduced by Ludwig [19], seeking these functions in the form

$$\widetilde{G}_1^{(j)} = \sum_{k=1}^{\infty} b_k^{(j)}(z,w,w_0) f_k\big(\widetilde{S}(w) - z_0\big), \tag{2.12}$$

$$\widetilde{G}_2^{(j)} = \sum_{k=1}^{\infty} a_k^{(j)}(z,w,z_0) f_k\big(S(z) - w_0\big), \tag{2.13}$$

where

$$\begin{aligned} &f_k(\xi) = \frac{\xi^k}{k!}(\ln\xi - C_k), \quad k = 0,1,\ldots,\\ &f_k(\xi) = (-1)^{-k-1}(-k-1)!\xi^k, \quad k \leqslant -1,\ C_0 = 0,\\ &C_k = \sum_{l=1}^{k}\frac{1}{l}, \quad k = 1,2,\ldots. \end{aligned} \tag{2.14}$$

This form ensures the location of singularities on the desired characteristics. Substituting (2.12) and (2.13) into the biharmonic equation, we obtain the following differential equations for the coefficients $a_k^{(j)}$ and $b_k^{(j)}$,

$$\frac{\partial^2 a_k^{(j)}}{\partial w^2} = 0, \qquad \frac{\partial^2 b_k^{(j)}}{\partial z^2} = 0, \quad k = 1,2,\ldots. \tag{2.15}$$

To find coefficients $a_k^{(j)}$ and $b_k^{(j)}$ for each case we use the conditions on $\Gamma_{\mathbb{C}}$.

Case (ii), $\widetilde{G}^{(2)} = G$ and $\Delta\widetilde{G}^{(2)} = \Delta G$ on $\Gamma_{\mathbb{C}}$,

$$\begin{aligned} &a_1^{(2)} = z - z_0, \qquad \frac{\partial a_1^{(2)}}{\partial w} S' = 1 \quad \text{on } \Gamma_{\mathbb{C}},\\ &a_k^{(2)} = 0, \qquad \frac{\partial a_k^{(2)}}{\partial w} S' = -\frac{\partial^2 a_{k-1}^{(2)}}{\partial w \partial z}, \quad k = 2,3,\ldots, \end{aligned}$$

$$b_1^{(2)} = w - w_0, \qquad \frac{\partial b_1^{(2)}}{\partial z}\widetilde{S}' = 1 \quad \text{on } \Gamma_{\mathbb{C}},$$

$$b_k^{(2)} = 0, \qquad \frac{\partial b_k^{(2)}}{\partial z}\widetilde{S}' = -\frac{\partial^2 b_{k-1}^{(2)}}{\partial w \partial z}, \quad k = 2, 3, \ldots.$$

Thus,

$$a_1^{(2)} = z - z_0 + \frac{(w - S(z))}{S'}, \qquad a_k^{(2)} = (-1)^{k-1}\frac{(w - S(z))}{S'}\hat{D}_z^{k-1}, \tag{2.16}$$

$$b_1^{(2)} = w - w_0 + \frac{(z - \widetilde{S}(w))}{\widetilde{S}'}, \qquad b_k^{(2)} = (-1)^{k-1}\frac{(z - \widetilde{S}(w))}{\widetilde{S}'}\hat{D}_w^{k-1}, \tag{2.17}$$

where $\hat{D}_z = \frac{\partial}{\partial z}(\frac{1}{S'})$, $\hat{D}_z^2 = \frac{\partial}{\partial z}(\frac{1}{S'}\frac{\partial}{\partial z}(\frac{1}{S'}))$, etc., and $\hat{D}_w = \frac{\partial}{\partial w}(\frac{1}{\widetilde{S}'})$.

Note that for the special case when Γ is a straight line $y = 0$, we have

$$\widetilde{G}_1^{(2)} = (z - w_0)(w - z_0)\big(\ln(w - z_0) - 1\big), \tag{2.18}$$

$$\widetilde{G}_2^{(2)} = (w - z_0)(z - w_0)\big(\ln(z - w_0) - 1\big). \tag{2.19}$$

Case (iii), $\partial_n \widetilde{G}^{(3)} = \partial_n G$ and $\Delta\widetilde{G}^{(3)} = \Delta G$ on $\Gamma_{\mathbb{C}}$,

$$a_1^{(3)} = -(z - z_0), \qquad \frac{\partial a_1^{(3)}}{\partial w}S' = 1 \quad \text{on } \Gamma_{\mathbb{C}},$$

$$a_2^{(3)}S' = 1 - \frac{\partial a_1^{(3)}}{\partial z} + \frac{\partial a_1^{(3)}}{\partial w}S', \qquad \frac{\partial a_2^{(3)}}{\partial w}S' = -\frac{\partial^2 a_1^{(3)}}{\partial w \partial z},$$

$$a_k^{(3)}S' = -\frac{\partial a_{k-1}^{(3)}}{\partial z} + \frac{\partial a_{k-1}^{(3)}}{\partial w}S', \qquad \frac{\partial a_k^{(3)}}{\partial w}S' = -\frac{\partial^2 a_{k-1}^{(3)}}{\partial w \partial z}, \quad k \geqslant 3,$$

$$b_1^{(3)} = -(w - w_0), \qquad \frac{\partial b_1^{(3)}}{\partial z}\widetilde{S}' = 1 \quad \text{on } \Gamma_{\mathbb{C}},$$

$$b_2^{(3)}\widetilde{S}' = 1 - \frac{\partial b_1^{(3)}}{\partial w} + \frac{\partial b_1^{(3)}}{\partial z}\widetilde{S}', \qquad \frac{\partial b_2^{(3)}}{\partial z}\widetilde{S}' = -\frac{\partial^2 b_1^{(3)}}{\partial w \partial z},$$

$$b_k^{(3)}\widetilde{S}' = -\frac{\partial b_{k-1}^{(3)}}{\partial w} + \frac{\partial b_{k-1}^{(3)}}{\partial z}\widetilde{S}', \qquad \frac{\partial b_k^{(3)}}{\partial z}\widetilde{S}' = -\frac{\partial^2 b_{k-1}^{(3)}}{\partial w \partial z}, \quad k \geqslant 3,$$

$$a_1^{(3)} = -(z - z_0) + \frac{(w - S(z))}{S'}, \qquad b_1^{(3)} = -(w - w_0) + \frac{(z - \widetilde{S}(w))}{\widetilde{S}'}, \tag{2.20}$$

$$a_k^{(3)} = (-1)^k\left(\frac{2k}{S'}\hat{D}_z^{k-2} - \frac{(w - S(z))}{S'}\hat{D}_z^{k-1}\right), \quad k \geqslant 2, \tag{2.21}$$

$$b_k^{(3)} = (-1)^k\left(\frac{2k}{\widetilde{S}'}\hat{D}_w^{k-2} - \frac{(z - \widetilde{S}(w))}{\widetilde{S}'}\hat{D}_w^{k-1}\right), \quad k \geqslant 2, \tag{2.22}$$

where $\hat{D}_z^0 = \hat{D}_w^0 = 1$.

For the special case when Γ is a straight line $y = 0$, we have

$$\widetilde{G}_1^{(3)} = -(2w - z - w_0)(w - z_0)\big(\ln(w - z_0) - 1\big) + 2(w - z_0)^2\big(\ln(w - z_0) - 3/2\big), \tag{2.23}$$

$$\widetilde{G}_2^{(3)} = -(2z - w - z_0)(z - w_0)\big(\ln(z - w_0) - 1\big) + 2(z - w_0)^2\big(\ln(z - w_0) - 3/2\big). \tag{2.24}$$

Case (iv), $\widetilde{G}^{(4)} = G$ and $\partial_n\Delta\widetilde{G}^{(4)} = \partial_n\Delta G$ on $\Gamma_{\mathbb{C}}$,

$$a_1^{(4)} = z - z_0, \qquad \frac{\partial a_1^{(4)}}{\partial w}S' = -1 \quad \text{on } \Gamma_{\mathbb{C}},$$

$$a_2^{(4)} = 0, \qquad \frac{\partial a_2^{(4)}}{\partial w}\big(S'\big)^2 = -2\frac{\partial^2 a_1^{(4)}}{\partial w \partial z}S' - \frac{\partial a_1^{(4)}}{\partial w}S'',$$

$$a_k^{(4)} = 0, \qquad \frac{\partial a_k^{(4)}}{\partial w}\big(S'\big)^2 = -2\frac{\partial^2 a_{k-1}^{(4)}}{\partial w \partial z}S' - \frac{\partial a_{k-1}^{(4)}}{\partial w}S'' - \frac{\partial^3 a_{k-2}^{(4)}}{\partial w \partial z^2}, \quad k \geqslant 3,$$

$$b_1^{(4)} = w - w_0, \qquad \frac{\partial b_1^{(4)}}{\partial z}\widetilde{S}' = -1 \quad \text{on } \Gamma_{\mathbb{C}},$$

$$b_2^{(4)}=0,\qquad \frac{\partial b_2^{(4)}}{\partial z}\big(\widetilde{S}'\big)^2=-2\frac{\partial^2 b_1^{(4)}}{\partial w\partial z}\widetilde{S}'-\frac{\partial b_1^{(4)}}{\partial z}\widetilde{S}'',$$

$$b_k^{(4)}=0,\qquad \frac{\partial b_k^{(4)}}{\partial z}\big(\widetilde{S}'\big)^2=-2\frac{\partial^2 b_{k-1}^{(4)}}{\partial w\partial z}\widetilde{S}'-\frac{\partial b_{k-1}^{(4)}}{\partial z}\widetilde{S}''-\frac{\partial^3 b_{k-2}^{(4)}}{\partial z\partial w^2},\quad k\geqslant 3,$$

$$a_1^{(4)}=z-z_0-\frac{(w-S(z))}{S'},\qquad a_k^{(4)}=(-1)^k\frac{(w-S(z))}{S'}\hat{D}_z^{k-1},\tag{2.25}$$

$$b_1^{(4)}=w-w_0-\frac{(z-\widetilde{S}(w))}{\widetilde{S}'},\qquad b_k^{(4)}=(-1)^k\frac{(z-\widetilde{S}(w))}{\widetilde{S}'}\hat{D}_w^{k-1},\quad k\geqslant 2.\tag{2.26}$$

For the special case when Γ is a straight line $y=0$, we obtain

$$\widetilde{G}_1^{(4)}=(2w-z-w_0)(w-z_0)\big(\ln(w-z_0)-1\big),\tag{2.27}$$

$$\widetilde{G}_2^{(4)}=(2z-w-z_0)(z-w_0)\big(\ln(z-w_0)-1\big).\tag{2.28}$$

Case (v), $\partial_n\widetilde{G}^{(5)}=\partial_n G$ and $\partial_n\Delta\widetilde{G}^{(5)}=\partial_n\Delta G$ on $\Gamma_{\mathbb{C}}$,

$$a_1^{(5)}=-(z-z_0),\qquad \frac{\partial a_1^{(5)}}{\partial w}S'=-1\quad \text{on } \Gamma_{\mathbb{C}},$$

$$a_2^{(5)}S'=1-\frac{\partial a_1^{(5)}}{\partial z}+\frac{\partial a_1^{(5)}}{\partial w}S',\qquad \frac{\partial a_2^{(5)}}{\partial w}\big(S'\big)^2=-2\frac{\partial^2 a_1^{(5)}}{\partial w\partial z}S'-\frac{\partial a_1^{(5)}}{\partial w}S'',$$

$$a_k^{(5)}S'=-\frac{\partial a_{k-1}^{(5)}}{\partial z}+\frac{\partial a_{k-1}^{(5)}}{\partial w}S',\qquad \frac{\partial a_k^{(5)}}{\partial w}\big(S'\big)^2=-2\frac{\partial^2 a_{k-1}^{(5)}}{\partial w\partial z}S'-\frac{\partial a_{k-1}^{(5)}}{\partial w}S''-\frac{\partial^3 a_{k-2}^{(5)}}{\partial w\partial z^2},\quad k\geqslant 3,$$

$$b_1^{(5)}=-(w-w_0),\qquad \frac{\partial b_1^{(5)}}{\partial z}\widetilde{S}'=-1\quad \text{on } \Gamma_{\mathbb{C}},$$

$$b_2^{(5)}\widetilde{S}'=1-\frac{\partial b_1^{(5)}}{\partial w}+\frac{\partial b_1^{(5)}}{\partial z}\widetilde{S}',\qquad \frac{\partial b_2^{(5)}}{\partial z}\big(\widetilde{S}'\big)^2=-2\frac{\partial^2 b_1^{(5)}}{\partial w\partial z}\widetilde{S}'-\frac{\partial b_1^{(5)}}{\partial z}\widetilde{S}'',$$

$$b_k^{(5)}\widetilde{S}'=-\frac{\partial b_{k-1}^{(5)}}{\partial w}+\frac{\partial b_{k-1}^{(5)}}{\partial z}\widetilde{S}',\qquad \frac{\partial b_k^{(5)}}{\partial z}\big(\widetilde{S}'\big)^2=-2\frac{\partial^2 b_{k-1}^{(5)}}{\partial w\partial z}\widetilde{S}'-\frac{\partial b_{k-1}^{(5)}}{\partial z}\widetilde{S}''-\frac{\partial^3 b_{k-2}^{(5)}}{\partial z\partial w^2},\quad k\geqslant 3,$$

$$a_1^{(5)}=-(z-z_0)-\frac{(w-S(z))}{S'},\tag{2.29}$$

$$a_k^{(5)}=(-1)^{k-1}\left(\frac{(2k-4)}{S'}\hat{D}_z^{k-2}-\frac{(w-S(z))}{S'}\hat{D}_z^{k-1}\right),\quad k\geqslant 2,\tag{2.30}$$

$$b_1^{(5)}=-(w-w_0)-\frac{(z-\widetilde{S}(w))}{\widetilde{S}'},\tag{2.31}$$

$$b_k^{(5)}=(-1)^{k-1}\left(\frac{(2k-4)}{\widetilde{S}'}\hat{D}_w^{k-2}-\frac{(z-\widetilde{S}(w))}{\widetilde{S}'}\hat{D}_w^{k-1}\right),\quad k\geqslant 2.\tag{2.32}$$

For the special case when Γ is a straight line $y=0$, this imply

$$\widetilde{G}_1^{(5)}=-(z-w_0)(w-z_0)\big(\ln(w-z_0)-1\big),\tag{2.33}$$

$$\widetilde{G}_2^{(5)}=-(w-z_0)(z-w_0)\big(\ln(z-w_0)-1\big).\tag{2.34}$$

Thus, we have constructed the reflected fundamental solutions for each case of the boundary conditions (ii)–(v) as formal series. Convergence of the series follows from convergence of the multipliers of the logarithms in (2.12) and (2.13),

$$V_1^{(j)}=\sum_{k=1}^{\infty}b_k^{(j)}(z,w,w_0)\frac{(\widetilde{S}(w)-z_0)^k}{k!},\tag{2.35}$$

$$V_2^{(j)}=\sum_{k=1}^{\infty}a_k^{(j)}(z,w,z_0)\frac{(S(z)-w_0)^k}{k!}.\tag{2.36}$$

Coefficients a_k in (2.36) do not depend on w_0, therefore, this expression can be interpreted as the Taylor series of $V_2^{(j)}$ as a function of $-w_0$ at the point $-S(z)$. Function $V_2^{(j)}$ can be described as the unique solution to the Cauchy–Goursat problem for biharmonic functions with holomorphic data on $\Gamma_{\mathbb{C}}$ and the characteristic line $S(z)=w_0$. The data is prescribed by the boundary conditions for each $j=\overline{2,5}$. For example, for $j=2$ the corresponding problem is

$$\Delta^2 V_2^{(2)} = 0, \quad \text{in } W,$$
$$V_2^{(2)} = (z - z_0)(w - w_0) \quad \text{on } \Gamma_{\mathbb{C}},$$
$$\Delta V_2^{(2)} = 1 \quad \text{on } \Gamma_{\mathbb{C}},$$
$$V_2^{(2)} = 0 \quad \text{on } S(z) - w_0 = 0,$$
$$\frac{\partial V_2^{(2)}}{\partial z} = (z - z_0)S' + \big(w - S(z)\big) \quad \text{on } S(z) - w_0 = 0. \tag{2.37}$$

Note, that $S(z)$ is an analytic function in the neighborhood of $\Gamma_{\mathbb{C}}$ and its derivative does not vanish on $\Gamma_{\mathbb{C}}$. Existence and uniqueness of holomorphic solutions to the Cauchy and Goursat problems for holomorphic partial differential equations with holomorphic data are discussed in [16].

The reflected fundamental solutions as well as functions $V_1^{(j)}$ and $V_2^{(j)}$ are used in the next section for deriving the corresponding reflection formulas.

3. The main result

First we state the reflection formulas for biharmonic functions given in the upper half plane describing the analytic continuation across the x-axis.

Theorem 3.1. *Let $U \subset \mathbb{R}^2$ be a sufficiently small domain divided by a straight line $\Gamma_0 \subset U := \{y = 0\}$ into two components $U_1 \subset \mathbb{R}^2_+$ and $U_2 \subset \mathbb{R}^2_-$, $(x_0, y_0) \in U_1$ and $(x_0, -y_0) \in U_2$. Then any biharmonic function $u(x, y)$ in the domain U_1 subject to one of the conditions* (i)–(v) *on Γ_0 can be continued to the domain U_2, using the following formulas*:

(i) *If $u = \partial_n u = 0$ on Γ_0, then*

$$u(x_0, y_0) = -u(x_0, -y_0) - 2y_0 \frac{\partial u}{\partial y}(x_0, -y_0) - y_0^2 \Delta_{x,y} u(x_0, -y_0), \tag{3.38}$$

(ii) *if $u = \Delta u = 0$ on Γ_0, then*

$$u(x_0, y_0) = -u(x_0, -y_0), \tag{3.39}$$

(iii) *if $u = \partial_n \Delta u = 0$ on Γ_0, then*

$$u(x_0, y_0) = -u(x_0, -y_0) - y_0 \int_0^{-y_0} \Delta u(x_0, y)\, dy, \tag{3.40}$$

where the integral is computed along the segment parallel to y-axis,

(iv) *if $\partial_n u = \Delta u = 0$ on Γ_0, then*

$$u(x_0, y_0) = u(x_0, -y_0) - \int_0^{-y_0} y \Delta u(x_0, y)\, dy, \tag{3.41}$$

(v) *if $\partial_n u = \partial_n \Delta u = 0$ on Γ_0, then*

$$u(x_0, y_0) = u(x_0, -y_0). \tag{3.42}$$

Remark 3.1. If a biharmonic function $u(x, y)$ is also a harmonic function, then (3.39) and (3.40) coincide with the odd continuation (1.1), while formulas (3.41) and (3.42) with the even continuation (1.2) for harmonic functions.

Theorem 3.2. *Let $U \subset \mathbb{R}^2$ be a sufficiently small domain divided by a non-singular real analytic curve Γ into two parts U_1 and U_2. Let also $P(x_0, y_0)$ be a point in U_1 having its reflected point $Q(R(x_0, y_0))$ in U_2 (see (1.4)). Then any biharmonic function $u(x, y)$ in the domain U subject to one of the conditions* (i)–(v) *on the curve $\Gamma \subset U$ can be continued across Γ, using the following reflection formulas*:

(i) *if* $u = \partial_n u = 0$ *on* Γ, *then*

$$u(P) = -u(Q) - \left(x_0 - \frac{S(x_0+iy_0)+\widetilde{S}(x_0-iy_0)}{2}\right)\frac{\partial u}{\partial x}(Q) - \left(y_0 + \frac{S(x_0+iy_0)-\widetilde{S}(x_0-iy_0)}{2i}\right)\frac{\partial u}{\partial y}(Q)$$
$$- \frac{1}{4}\big(x_0^2 + y_0^2 - S(x_0+iy_0)(x_0+iy_0) - \widetilde{S}(x_0-iy_0)(x_0-iy_0) + S(x_0+iy_0)\widetilde{S}(x_0-iy_0)\big)\Delta_{x,y}u(Q), \tag{3.43}$$

(ii) *if* $u = \Delta u = 0$ *on* Γ, *then*

$$u(P) = -u(Q) + \hat{\mathbb{K}}_2, \tag{3.44}$$

(iii) *if* $u = \partial_n \Delta u = 0$ *on* Γ, *then*

$$u(x_0, y_0) = -u(x_0, -y_0) + \hat{\mathbb{K}}_3, \tag{3.45}$$

(iv) *if* $\partial_n u = \Delta u = 0$ *on* Γ, *then*

$$u(x_0, y_0) = u(x_0, -y_0) + \hat{\mathbb{K}}_4, \tag{3.46}$$

(v) *if* $\partial_n u = \partial_n \Delta u = 0$ *on* Γ, *then*

$$u(x_0, y_0) = u(x_0, -y_0) + \hat{\mathbb{K}}_5, \tag{3.47}$$

where

$$\hat{\mathbb{K}}_j = \frac{1}{8i}\int_{\Gamma}^{Q}\left(V^{(j)}\frac{\partial \Delta u}{\partial y} - \Delta u\frac{\partial V^{(j)}}{\partial y} + \Delta V^{(j)}\frac{\partial u}{\partial y} - u\frac{\partial \Delta V^{(j)}}{\partial y}\right)dx$$
$$- \left(V^{(j)}\frac{\partial \Delta u}{\partial x} - \Delta u\frac{\partial V^{(j)}}{\partial x} + \Delta V^{(j)}\frac{\partial u}{\partial x} - u\frac{\partial \Delta V^{(j)}}{\partial x}\right)dy, \tag{3.48}$$

the integral is computed along an arbitrary path joining the curve Γ *with the reflected point* Q, $V^{(j)} = V_1^{(j)} - V_2^{(j)}$, $j = \overline{2,5}$.

It is obvious that Theorem 3.1 is a special case of Theorem 3.2, thus we will prove only the later for the boundary conditions (ii)–(v) (for the case (i) see [1]).

For simplicity, we assume that Γ is an algebraic curve. Under this assumption, the Schwarz function and its inverse are analytic in the whole plane $\mathbb{C}$ except for finitely many algebraic singularities.

Proof of Theorem 3.2. The main step of the proof is already done by constructing the reflected fundamental solution for each case of boundary conditions (see Section 2). The rest of the proof is based on the contour deformation in Green's formula [12] and is similar to [1].

The Green's formula, expressing the value of biharmonic function at a point P via the values of this function on a contour $\gamma \subset U_1$ surrounding the point P, is

$$u(P) = \int_{\gamma}\left(G\frac{\partial \Delta u}{\partial y} - \Delta u\frac{\partial G}{\partial y} + \Delta G\frac{\partial u}{\partial y} - u\frac{\partial \Delta G}{\partial y}\right)dx - \left(G\frac{\partial \Delta u}{\partial x} - \Delta u\frac{\partial G}{\partial x} + \Delta G\frac{\partial u}{\partial x} - u\frac{\partial \Delta G}{\partial x}\right)dy, \tag{3.49}$$

where $G = G(x, y, x_0, y_0)$ is an arbitrary fundamental solution of the bi-Laplacian. The most suitable one for what follows is (2.10).

Since the integrand in (3.49) is a closed form, the value of the integral does not change while we deform the contour γ homotopically. Thus, the goal is to deform the contour γ from the domain U_1 to the domain U_2 by deforming it first to the complexified curve $\Gamma_{\mathbb{C}}$. This part of the deformation is possible if the point P lies so close to the curve Γ that there exists a connected domain $\Omega \subset \Gamma_{\mathbb{C}}$ such that Ω contains both points of intersections of the characteristic lines passing through the point P and Ω can be univalently projected onto a plane domain (for details, see [26]). Thus, we can replace the contour γ in (3.49) with the contour $\gamma\prime \subset \Omega$, which is homotopic to γ in $\mathbb{C}^2 \setminus \{(x-x_0)^2 + (y-y_0)^2 = 0\} =: \mathbb{C}^2 \setminus K_P$.

Note that due to homogeneous boundary conditions (ii)–(v) a half of the terms (different for each case) in the integrand of (3.49), while integrating along $\gamma\prime$, vanishes. For example, formula (3.49) in the case (ii) can be rewritten in the form

$$u(P) = \int_{\gamma\prime}\left(G\frac{\partial \Delta u}{\partial y} + \Delta G\frac{\partial u}{\partial y}\right)dx - \left(G\frac{\partial \Delta u}{\partial x} + \Delta G\frac{\partial u}{\partial x}\right)dy. \tag{3.50}$$

To deform the contour $\gamma\prime$ from $\Gamma_{\mathbb{C}}$ to the real domain U_2 we replace the fundamental solution with the corresponding reflected fundamental solution $\widetilde{G}^{(j)}$, $j = \overline{1,5}$ (see Section 2). Since functions $\widetilde{G}^{(j)}$ have singularities only on the characteristic lines intersecting the real space at point $Q = R(P)$ in the domain U_2 and intersecting $\Gamma_{\mathbb{C}}$ at $K_P \cap \Gamma_{\mathbb{C}}$, we are able to deform contour $\gamma\prime$ from the complexified curve $\Gamma_{\mathbb{C}}$ to the real domain U_2 without changing the value of the integral [26]. As a result, we obtain

$$u(P) = \int_{\widetilde{\gamma}} \left(\widetilde{G}^{(j)} \frac{\partial \Delta u}{\partial y} - \Delta u \frac{\partial \widetilde{G}^{(j)}}{\partial y} + \Delta \widetilde{G}^{(j)} \frac{\partial u}{\partial y} - u \frac{\partial \Delta \widetilde{G}^{(j)}}{\partial y} \right) dx - \left(\widetilde{G}^{(j)} \frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial \widetilde{G}^{(j)}}{\partial x} + \Delta \widetilde{G}^{(j)} \frac{\partial u}{\partial x} - u \frac{\partial \Delta \widetilde{G}^{(j)}}{\partial x} \right) dy, \tag{3.51}$$

where $\widetilde{\gamma} \subset U_2$ is a contour surrounding the point Q and having endpoints on the curve Γ. Formula (3.51) in the characteristic variables has the form,

$$u(P) = 4i \int_{\widetilde{\gamma}} \left(\widetilde{G}^{(j)} \frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial^2 \widetilde{G}^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial z} - u \frac{\partial^3 \widetilde{G}^{(j)}}{\partial z^2 \partial w} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial \widetilde{G}^{(j)}}{\partial z} \right) dz - \left(\widetilde{G}^{(j)} \frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial^2 \widetilde{G}^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial w} - u \frac{\partial^3 \widetilde{G}^{(j)}}{\partial z \partial w^2} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial \widetilde{G}^{(j)}}{\partial w} \right) dw. \tag{3.52}$$

If we substitute (2.11)–(2.13) into (3.52) and move one endpoint of the contour $\widetilde{\gamma}$ along the curve Γ to the other endpoint, the integral terms containing products of the function u and regular part of the function $\widetilde{G}^{(j)}$ and their derivatives vanish, while the integral terms containing logarithms can be combined as follows,

$$-\frac{i}{4\pi} \int_{\widetilde{\gamma}} \ln(\widetilde{S}(w) - z_0) \left\{ \left(V_1^{(j)} \frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial^2 V_1^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V_1^{(j)}}{\partial z} \right) dz - \left(V_1^{(j)} \frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial^2 V_1^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial w} - u \frac{\partial^3 V_1^{(j)}}{\partial z \partial w^2} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V_1^{(j)}}{\partial w} \right) dw \right\}, \tag{3.53}$$

$$-\frac{i}{4\pi} \int_{\widetilde{\gamma}} \ln(S(z) - w_0) \left\{ \left(V_2^{(j)} \frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial^2 V_2^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial z} - u \frac{\partial^3 V_2^{(j)}}{\partial z^2 \partial w} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V_2^{(j)}}{\partial z} \right) dz - \left(V_2^{(j)} \frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial^2 V_2^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial w} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V_2^{(j)}}{\partial w} \right) dw \right\}, \tag{3.54}$$

where $\widetilde{\gamma}$ is a loop surrounding the point Q and having endpoints on the curve Γ.

The logarithm $\ln(\widetilde{S}(w) - z_0)$ obtains the increment $2\pi i$ along the loop, while the logarithm $\ln(S(z) - w_0)$ gets $(-2\pi i)$. Thus, compressing $\widetilde{\gamma}$ to a segment joining Q to Γ, we find that the integral (3.53) can be rewritten as

$$\hat{\mathbb{K}}_j = \frac{1}{2} \int_{\Gamma}^{Q} \left\{ \left(V^{(j)} \frac{\partial^3 u}{\partial z^2 \partial w} + \frac{\partial^2 V^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial z} + u \frac{\partial^3 V_2^{(j)}}{\partial z^2 \partial w} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V^{(j)}}{\partial z} \right) dz - \left(V^{(j)} \frac{\partial^3 u}{\partial z \partial w^2} + \frac{\partial^2 V^{(j)}}{\partial z \partial w} \frac{\partial u}{\partial w} - u \frac{\partial^3 V_1^{(j)}}{\partial z \partial w^2} - \frac{\partial^2 u}{\partial z \partial w} \frac{\partial V^{(j)}}{\partial w} \right) dw \right\}, \tag{3.55}$$

where $V^{(j)} = V_1^{(j)} - V_2^{(j)}$.

The rest of nonzero terms in (3.52) are terms involving derivatives of the logarithms. Some of these integrals also vanish due to the properties of functions $V_1^{(j)}$ and $V_2^{(j)}$, resulting in

$$-\frac{i}{4\pi} \int_{\widetilde{\gamma}} \left(-\frac{\partial a_1^{(j)}}{\partial w} \frac{(S'(z))^2 u}{S(z) - w_0} dz + \frac{\partial b_1^{(j)}}{\partial z} \frac{(\widetilde{S}'(w))^2 u}{\widetilde{S}(w) - z_0} dw \right) = -\frac{1}{2} u(Q) \left(\frac{\partial a_1^{(j)}}{\partial w}(Q) S'(Q) + \frac{\partial b_1^{(j)}}{\partial z}(Q) \widetilde{S}'(Q) \right). \tag{3.56}$$

Combining (3.55) and (3.56) we finally obtain,

$$u(P) = -\frac{1}{2} u(Q) \left(\frac{\partial a_1^{(j)}}{\partial w}(Q) S'(Q) + \frac{\partial b_1^{(j)}}{\partial z}(Q) \widetilde{S}'(Q) \right) + \hat{\mathbb{K}}_j. \tag{3.57}$$

Here the expression in the parentheses with the appropriate choice of the coefficients $a_1^{(j)}$ and $b_1^{(j)}$ (see Section 2) is equal to either 1 or -1. Thus, formula (3.57) in variables x, y is equivalent to (3.44)–(3.48). □

Remark 3.2. Formula (3.57) gives continuation of a biharmonic function from the domain $U_1 \subset \mathbb{R}^2$ to the domain $U_2 \subset \mathbb{R}^2$ as a multiple-valued function whose singularities coincide with the singularities of the functions S or $\widetilde{S}$.

Remark 3.3. In the special case when Γ is a straight line and boundary conditions (ii) or (v) are applied, $V_1^{(j)} = V_2^{(j)}$, and, therefore, $\hat{\mathbb{K}}_j \equiv 0$, $j = 2, 5$.

Example 3.4. As an example of formula (3.57) consider a biharmonic function $u(x, y)$ subject to the Navier condition (ii), $u = \Delta u = 0$, on a unit circle centered at the origin, $x^2 + y^2 = 1$. In this case series (2.35) and (2.36) with coefficients (2.16), (2.17) can be summed, and $V_2^{(2)} = (z - z_0)(1/z - w_0) + (1/z - w)(1/w_0 - z)$. The reflection formula then has the form

$$u(r_0, \theta_0) = -u\left(\frac{1}{r_0}, \theta_0\right) + \frac{r_0^2 - 1}{4r_0} \int_1^{r_0^{-1}} \frac{1 - r^2}{r^2} \left(\frac{1}{r} \partial_r u(r, \theta_0) + \partial_{rr}^2 u(r, \theta_0)\right) dr, \tag{3.58}$$

where the integral is computed along the straight line $\theta = \theta_0$.

Acknowledgment

I would like to thank Professor Reinhard Farwig for sending me reprints of his interesting papers.

References

[1] D. Aberra, T. Savina, The Schwarz reflection principle for polyharmonic functions in $\mathbb{R}^2$, Complex Var. Theory Appl. 41 (1) (2000) 27–44.
[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, Comm. Pure Appl. Math. 14 (1964) 35–92.
[3] L.-E. Andersson, T. Elfving, G.H. Golub, Solution of biharmonic equations with application to radar imaging, J. Comput. Appl. Math. 94 (2) (1998) 153–180.
[4] N. Aronszajn, T.M. Creese, L.J. Lipkin, Polyharmonic Functions, Oxford University Press, USA, 1983.
[5] B.P. Belinskiy, T.V. Savina, The Schwarz reflection principle for harmonic functions in R^2 subject to the Robin condition, J. Math. Anal. Appl. 348 (2008) 685–691.
[6] E. Berchio, F. Gazzola, E. Mitidieri, Positivity preserving property for a class of biharmonic elliptic problems, J. Differential Equations 229 (1) (2006) 1–23.
[7] J. Bramble, Continuation of biharmonic functions across circular arcs, J. Math. Mech. 7 (6) (1958) 905–924.
[8] Ph. Davis, The Schwarz Function and Its Applications, Carus Math. Monogr., MAA, 1979.
[9] R.J. Duffin, Continuation of biharmonic functions by reflection, Duke Math. J. 22 (2) (1955) 313–324.
[10] P. Ebenfelt, D. Khavinson, On point to point reflection of harmonic functions across real analytic hypersurfaces in $\mathbb{R}^n$, J. Anal. Math. 68 (1996) 145–182.
[11] R. Farwig, A note on a reflection principle for the biharmonic equation and the Stokes system, Acta Appl. Math. 37 (1994) 41–51.
[12] P.R. Garabedian, Partial differential equations with more than two independent variables in the complex domain, J. Math. Mech. 9 (1960) 241–271.
[13] F. Gazzola, H.-Ch. Grunau, G. Sweers, Polyharmonic Boundary Value Problems, Lecture Notes in Math., vol. 1991, Springer, 2010.
[14] F. John, Continuation and reflection of solutions of partial differential equations, Bull. Amer. Math. Soc. 63 (1957) 327–344.
[15] D. Khavinson, H.S. Shapiro, Remarks on the reflection principles for harmonic functions, J. Anal. Math. 54 (1991) 60–76.
[16] D. Khavinson, Holomorphic Partial Differential Equations and Classical Potential Theory, Univ. de La Laguna, 1996.
[17] H. Lewi, On the reflection laws of second order differential equations in two independent variables, Bull. Amer. Math. Soc. 65 (1959) 37–58.
[18] R.R. López, On reflection principles supported on a finite set, J. Math. Anal. Appl. 351 (2009) 556–566.
[19] D. Ludwig, Exact and asymptotic solutions of the Cauchy problem, Comm. Pure Appl. Math. 13 (3) (1960) 473–508.
[20] D. Lin, Y. Yang, Nonlinear non-local elliptic equation modelling electrostatic actuation, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 463 (2081) (2007) 1323–1337.
[21] S.G. Mikhlin, Variational Methods in Mathematical Physics, The MacMillan Company, New York, 1964.
[22] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.
[23] A. Novick-Cohen, The Cahn–Hilliard equation, in: Handbook of Differential Equations: Evolutionary Equations, IV, Elsevier, Amsterdam, 2008, pp. 201–228.
[24] L. Nystedt, On polyharmonic continuation by reflection formulas, Ark. Mat. (1982) 201–247.
[25] H. Poritsky, Application of analytic functions to two-dimensional biharmonic analysis, Trans. Amer. Math. Soc. 59 (2) (1946) 248–279.
[26] T.V. Savina, B.Yu. Sternin, V.E. Shatalov, On a reflection formula for the Helmholtz equation, J. Comm. Techn. Electronics 38 (7) (1993) 132–143.
[27] H.S. Shapiro, The Schwarz Function and Its Generalization to Higher Dimensions, John Wiley and Sons, Inc., 1992.
[28] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer, 1997.