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Smooth polynomial approximation of spiral arcs

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ABSTRACT

Constructing fair curve segments using parametric polynomials is difficult due to the oscillatory nature of polynomials. Even NURBS curves can exhibit unsatisfactory curvature profiles. Curve segments with monotonic curvature profiles, for example spiral arcs, exist but are intrinsically non-polynomial in nature and thus difficult to integrate into existing CAD systems. A method of constructing an approximation to a generalised Cornu spiral (GCS) arc using non-rational quintic Bézier curves matching end points, end slopes and end curvatures is presented. By defining an objective function based on the relative error between the curvature profiles of the GCS and its Bézier approximation, a curve segment is constructed that has a monotonic curvature profile within a specified tolerance.

1. Introduction

Bézier curves are a popular parametric form used in a variety of CAD/CAGD systems. They are simple to use since there is a close relationship between the defining characteristic polygon and the curve. However, the Bézier form, along with all polynomials, suffers from oscillations that are conspicuous in the curvature profile. Much research has been carried out to counter this effect either by controlling the curve directly [1] or by using NURBS [2] which offers a more general solution but is unsatisfactory since the fitting is not automatic, each data set being treated differently. An alternative approach is to use curves that have defined monotonic curvature profiles. Such curves are well known and include spirals, logarithmic spirals and generalised Cornu spirals [3]. These forms clearly guarantee fairness since they have monotonic curvature profiles. However, they are unsuitable for use in CAD/CAGD systems, because they cannot be expressed exactly as polynomials [3], are difficult to construct and manipulate, and may require numerical methods to evaluate. Hence the need to approximate spirals with polynomial forms.

Previous work mainly concentrates on approximating spiral arcs, including circular arcs, by polynomials, such as Cornu spirals [4,2,5–9], generalised Cornu spirals [3] and logarithmic spirals [4]. All these approaches construct the approximation based on minimising the positional error between points on the spiral and corresponding points on the polynomial approximation. As such, they do not control the curvature profile of the resulting approximations. The method adopted in this article is to minimise the error in the curvature profiles between the spiral and the polynomial approximation. Since the curvature profile completely defines a planar curve up to an affine transformation, the curves will be positioned by fixing the end points and end slopes and matching the curvature profile. The Bézier form is adopted due to its popularity in existing CAD systems. The quintic is used as this is the lowest order polynomial that admits independent adjustment of end point curvatures which is desirable for piecewise curve construction.

In outline, the method constructs a Bézier quintic curve that matches the constraint data; end points, end slopes and end curvatures. To simplify the approximation, the Bézier is reformulated in terms of chord lengths and angles between the vertices of the characteristic polygon. Upper and lower bounds on the expected error in the curvature of

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the approximation are determined and are used to derive the anticipated approximation error between the spiral and the polynomial approximation. The result is a family of Bézier quintic curves that match the curvature along the spiral to within a specified tolerance. The final stage is to search for the Bézier curve that gives an almost monotonic curvature profile.

2. Curvature profile

The curvature profile $\kappa(s)$ of a 2D regular curve $\mathbf{r}(s)$ parameterised with respect to (*wrt*) arc length, is given by: $c(s) = \mathbf{r}''(s) = \left(\mathrm{d}^2 r(s)/\mathrm{d}s^2\right)\ s \in [0, S]$, where $\mathbf{r}''(s)$ is the second derivative *wrt* arc length. The magnitude of the radius curvature is given by $|\kappa(s)| = \|\mathbf{c}(s)\| = \left\|\mathbf{r}''(s)\right\|$.

A spiral, $\mathbf{r}(s), s \in [0, S]$, is a 2D curve of arc length s that possesses a monotonic curvature profile. The parametric form of the 2D spiral is given by $\mathbf{r}(t)(t \in [t_0, t_1])$ with curvature $\kappa_r(t)$ given by:

$$\kappa_r(t) = \frac{\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t) \cdot \mathbf{b}}{\|\dot{\mathbf{r}}(t)\|^3} \tag{1}$$

where $\dot{\mathbf{r}}(t)$is the derivative *wrt* t and $\mathbf{b} = (0, 0, 1)$.

The aim is to approximate a spiral segment with a non-rational quintic Bézier curve defined by: $\mathbf{p}(t) = \sum_{i=0}^{5} {}^5C_i(1-t)^{5-i}t^i \cdot \mathbf{p}_i\ t \in [0, 1]$, where $\mathbf{p}_i = (x_i, y_i)$ are the 2D control points. Without loss of generality, it is assumed that the spiral $\mathbf{r}(t)$ is defined by the following constraints:

- $\mathbf{r}(t_0) = \mathbf{0}$ with corresponding end unit tangent $\mathbf{t}(t_0) = (1, 0)$. This is easily satisfied since $\mathbf{r}(t)$ can always be rotated and translated. The resulting approximated curve can be recovered using the inverse transformation.
- The magnitude of the winding angle, θ, between the end unit tangents satisfies: $|\theta| \leq \pi/2$. This constraint has been determined empirically and results in extremely tight tolerances. Spans with $\theta \geqslant \pi/2$ can still be approximated but the error between the curvature profiles increases as θ increases.
- $\kappa_r(t)$ is monotonic for $t \in [t_0, t_1]$, ensuring that $\mathbf{r}(t)$ is a spiral.

3. Reparameterisation of spirals and 2D quintic Bézier curves

In order to simplify the comparison of a spiral with a 2D Bézier curve, it is necessary to reparameterise the two curves in terms of a common parameter. A spiral is an arc length parameterised curve [10]. A Bézier curve is parameterised *wrt* an arbitrary parameter, t, $(t \in [t_0, t_1])$. It is possible to reparameterise the Bézier curve *wrt* arc length. If the total arc lengths of the two curves are the same, they can be compared simply by comparing corresponding points along the spiral arc and its approximation. However, in general, the arc lengths will not be equal, although in most practical cases they will be reasonably close. The approach is to normalise the arc lengths of the two curves by proportional distance, $\bar{s}$ and to compare the curves at corresponding proportional arc length parameter values.

3.1. Reparameterisation of a 2D curve with proportional distance

Given $\mathbf{r}(t)$ with parameter $t, (t \in [t_0, t_1])$, the arc length s of the curve between t_0 and t is given by: $s = \int_{t_0}^{t} \sqrt{(\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t))}\mathrm{d}t\ t \in [t_0, t_1]$ and the total arc length $S = \int_{t_0}^{t_1} \sqrt{(\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t))}\mathrm{d}t$.

The curve $\mathbf{r}(t)$ can be reparameterised *wrt* arc length, giving $\mathbf{r}(t(s)), s(s \in [0, S])$ which is written as $\mathbf{r}(s)$ to simplify the notation and to emphasise that it is the curve and not the function that is of interest in this paper. Now the curvature of $\mathbf{r}(s)$ at s is equal to the curvature of $\mathbf{r}(t)$ at the corresponding parameter value t. Since the curvature $\kappa_r(t)$ of $\mathbf{r}(t)$ is given by (1), $\kappa_r(t) = \kappa(s),\ t \in [t_0, t_1],\ s \in [0, S]$. The arc length s of a curve can be normalised by its proportional distance $\bar{s} = sS^{-1}\ \bar{s} \in [0, 1]$, so $\mathbf{r}(s) \Rightarrow \mathbf{r}(\bar{s})\ (\bar{s} \in [0, 1])$ with curvature $\kappa_{\mathrm{r}}(\bar{s}) = \kappa(s)$. Thus the curvature of $\mathbf{r}(\bar{s})$ at $\bar{s}$ is equal to the curvature of $\mathbf{r}(t)$ at the corresponding t i.e. $\kappa_{\mathrm{r}}(\bar{s}) = \kappa_{\mathrm{r}}(t)$.

3.2. Reparameterisation of a spiral with proportional distance

A spiral $\mathbf{r}(t)$ parameterised *wrt* $t,\ t \in [t_0, t_1]$ can be reparameterised to $\mathbf{r}(\bar{s})\ (\bar{s} \in [0, 1])$ *wrt* proportional distance $\bar{s};\bar{s} \in [0, 1]$, then $\mathbf{r}(\bar{s}) = \mathbf{r}(t);\ t \in [t_0, t_1]$, and since $\frac{\mathrm{d}s}{\mathrm{d}\bar{s}} = S$, $\frac{\mathrm{d}\kappa_r(\bar{s})}{\mathrm{d}\bar{s}} = \frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\bar{s}} = S\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}$, i.e., the first derivative of the curvature $\kappa_{\mathrm{r}}(\bar{s})$ keeps the same sign for $\bar{s} \in [0, 1]$.

4. Determining a tolerance for the approximation

To construct the approximation, a tolerance on $\kappa(s)$ is established using circular arcs, i.e. constant curvature spirals, expressed in terms of a non-rational cubic Bézier curve, $\mathbf{p}(t) = \sum_{i=0}^{3} {}^3C_i(1-t)^{3-i}t^i\mathbf{p}_i$. A GCS $\mathbf{r}(s)$ is bounded by two circular arcs, $\mathbf{c}_0(s)$ and $\mathbf{c}_1(s)$, defined by two end points, $\mathbf{r}(0)$ and $\mathbf{r}(S)$ and the end tangent directions at $\mathbf{r}(0)$ and $\mathbf{r}(S)$ respectively [6]. Since the aim is to construct a Bézier approximation to the GCS, the two circular arcs are approximated by cubic Bézier curves, which are completely defined by the end conditions. The two Bézier circular approximations will then be used to specify a suitable tolerance for the optimisation of the Bézier approximation.

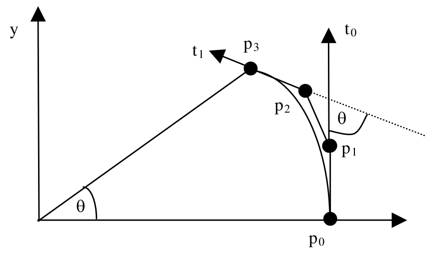

Fig. 1. Representation of a non-rational cubic Bézier curve.

Given a circular arc, $\mathbf{r}(\theta) = \rho(\cos\theta, \sin\theta)$, of radius $\rho = \kappa^{-1} > 0$, where $\theta(0 < \theta \le \pi/2)$ is the angle between the two end unit tangents (Fig. 1). Further, let $\mathbf{p}(t),\ t \in [0, 1]$, be a cubic Bézier curve that approximates $\mathbf{r}(\theta)$. Without loss of generality, suppose that $\mathbf{r}(0) = \rho(1, 0)$, $\mathbf{r}(1) = \rho(\cos\theta, \sin\theta)$ and the end unit tangents are $\mathbf{t}_0 = (0, 1)$, $\mathbf{t}_1 = (-\sin\theta, \cos\theta)$. Then the cubic Bézier curve must satisfy:

- $\mathbf{p}(0) = \mathbf{p}_0 = \mathbf{r}(0) = \rho(1, 0)$ and $\mathbf{p}(1) = \mathbf{p}_3 = \mathbf{r}(\theta) = \rho(\cos\theta, \sin\theta)$.
- $\mathbf{p}_1 - \mathbf{p}_0 = \rho\alpha\mathbf{t}_0$ and $\mathbf{p}_3 - \mathbf{p}_2 = \rho\beta\mathbf{t}_1$, $(\alpha > 0, \beta > 0)$.
- $\rho(0) = \rho(1) = 1/\kappa$.

The control points can be written as:

$$\begin{aligned}
&\mathbf{p}_0 = \rho(1, 0),\\
&\mathbf{p}_1 = \rho(1, \alpha),\\
&\mathbf{p}_2 = \rho[(\cos\theta, \sin\theta) + \beta(\sin\theta, -\cos\theta)](\alpha > 0, \beta > 0),\\
&\mathbf{p}_3 = \rho(\cos\theta, \sin\theta),
\end{aligned}$$

and the approximating cubic Bézier curve is given by:

$$\mathbf{p}(t) = (x(t), y(t)) = \rho\begin{pmatrix}[(1-t)^3 + 3t(1-t)^2 + 3(\cos\theta + \beta\sin\theta)t^2(1-t) + \cos\theta\, t^3],\\ [3\alpha t(1-t)^2 + 3(\sin\theta - \beta\cos\theta)t^2(1-t) + \sin\theta\, t^3]\end{pmatrix},$$

with curvatures at the end points are given by:

$$\kappa(0) = \frac{2\kappa(1 - \cos\theta - \beta\sin\theta)}{3\alpha^2}, \qquad \kappa(1) = \frac{2\kappa(1 - \cos\theta - \alpha\sin\theta)}{3\beta^2}.$$

Now the end curvatures of the circular arc are $\kappa(0) = \kappa(1) = \kappa$, hence:

$$\begin{aligned}
&3\alpha^2 + 2\beta\sin\theta = 2 - 2\cos\theta,\\
&3\beta^2 + 2\alpha\sin\theta = 2 - 2\cos\theta.
\end{aligned}$$

Giving:

$$(\alpha - \beta)\left(\alpha + \beta - \frac{2}{3}\sin\theta\right) = 0$$

with solutions: $\alpha = \beta$ and $\alpha + \beta = \frac{2}{3}\sin\theta$. Solving for the roots determines the four control points of the approximating Bézier cubic curve.

When $\alpha = \beta$: $3\alpha^2 + 2\alpha\sin\theta - (2 - 2\cos\theta) = 0$. Let the solutions be α_1 and α_2, i.e.:

$$\alpha_1 = \frac{2\sin\frac{\theta}{2}\left(-\cos\frac{\theta}{2} + \sqrt{3 + \cos^2\frac{\theta}{2}}\right)}{3}; \qquad \alpha_2 = \frac{2\sin\frac{\theta}{2}\left(-\cos\frac{\theta}{2} - \sqrt{3 + \cos^2\frac{\theta}{2}}\right)}{3}.$$

Since $0 < \theta \le \pi/2$ and $\alpha > 0$, there is only one solution α_1.

When $\alpha + \beta = \frac{2}{3}\sin\theta$: $3\alpha^2 - 2\alpha\sin\theta - (2 - 2\cos\theta - \frac{4}{3}\sin^2\theta) = 0$. Let the solutions be (α_2, β_2) and (α_3, β_3), then:

$$\alpha_i = \frac{2}{3}\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + (-1)^{i-1}\sqrt{3}\sin\frac{\theta}{2}\right), \qquad \beta_i = \frac{2}{3}\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + (-1)^{i}\sqrt{3}\sin\frac{\theta}{2}\right); \quad i = 2, 3. \tag{2}$$

Since $\alpha, \beta > 0$ and $0 < \theta \le \pi/2$, it is necessary that $\cos\frac{\theta}{2} > \sqrt{3}\sin\frac{\theta}{2}$, i.e., $0 \le \theta < \pi/3$.

When $0 \le \theta < \pi/3$, there are three solutions of α and β, i.e., there are three distinct cubic Bézier curves for approximating the circular arc. The tolerance ε_{ci} $i = 1, 2, 3$ representing the relative error of the curvature $\sigma_i(t)$ given by the three cubic curves respectively, to the circular arc is given by:

$$\varepsilon_{ci} = \max_{t\in[0,1]}\{\sigma_i(t)\} = \max_{t\in[0,1]}\left\{\frac{|\kappa_i(t) - \kappa|}{\max\{|\kappa_i(t)|, \kappa\}}\right\}, \quad i = 1, 2, 3.$$

The smallest tolerance is given by $\bar{\varepsilon} = \min\{\varepsilon_{c_1}, \varepsilon_{c_2}, \varepsilon_{c_3}\}$.

Numerical experimentation suggests $\bar{\varepsilon}$ occurs when $\alpha = \beta$. Thus the two bounding circular arc approximations give two tolerances, ε_0 and ε_1, from which a conservative tolerance is given by:

$$\bar{\varepsilon} = 2\max\{\varepsilon_0, \varepsilon_1\}. \tag{3}$$

$\bar{\varepsilon}$ is then used to control the search for an acceptable Bézier approximation to a given GCS. A curve is completely defined by its curvature, $\kappa(s)$, up to an affine transformation. Specifying end conditions fixes the position and orientation. Thus if the relative error between the curvature profiles of the GCS and the current Bézier curve is less than $\bar{\varepsilon}$, the resulting Bézier curve segment possesses a curvature profile that is monotonic within the specified tolerance and is an acceptable approximation to the given GCS.

5. Approximation of a spiral by a quintic Bézier curve

Let $\mathbf{r}(t)$ $t \in [t_0, t_1]$ be a GCS arc of length S with curvature $\kappa_r(\bar{s})$ $\bar{s} \in [0, 1]$, with $\bar{s}$ the proportional distance. Further, let $\theta(0)$ and $\theta(\bar{S})$ be the angles between the end tangents and the x-axis. Then

$$\kappa(\bar{s}) = \frac{(\kappa_1 - \kappa_0 + r\kappa_1)\,\bar{s} + \kappa_0\bar{S}}{r\bar{s} + \bar{S}}; \quad (0 \le \bar{s} \le \bar{S} = 1, r > -1),$$

where κ_0 and κ_1 are the end curvatures of the spiral and r is the shape factor [1]. It is noted that since $\kappa(\bar{s})$ is monotonic the spiral arc can have at most one inflection.

Let the spiral arc be approximated by a quintic Bézier curve $\mathbf{p}(t)$ that matches end points, end slopes and end curvatures. The reparameterisation *wrt* proportional arc length enables the curvatures, $\kappa_p(\bar{s})$ and $\kappa_r(\bar{s})$, to be directly compared. Since the arc lengths will be close, a simple numerical search along the Bézier curve for the corresponding proportional arc length parameter can be carried out.

A quintic Bézier curve $\mathbf{p}(t)$ is sought such that the maximum relative error in curvature is within the specific tolerance i.e.:

$$\mu(\bar{s}) \le \xi(\bar{s}), \quad \bar{s} \in [0, 1], \tag{4}$$

where $\mu(\bar{s}) = \left|\kappa_p(\bar{s}) - \kappa_r(\bar{s})\right|$ and $\xi(\bar{s}) = \bar{\varepsilon}\max\{\kappa_p(\bar{s}), \kappa_r(\bar{s})\}$.

Cases may exist where it may not be possible to guarantee that the error in the curvature $\mu(\bar{s})$ is always less than $\xi(\bar{s})$ for $\bar{s} \in [0, 1]$. In such cases, one method of proceeding is to subdivide the GCS until each span is within the tolerance on the curvature. A further complication occurs when the GCS inflects. Since the spiral is monotonic *wrt* curvature, it can have at most one inflection, which can be located using simple bisection. As the curvature approaches zero, the relative measure becomes unstable and must be adapted. A simple device is to use the relative curvature error away from the zero crossing, and use the absolute curvature error in the region around the inflection. The transition between the two regions can be assumed to be linear and a simple average taken to represent this change as suggested in [11].

5.1. Identification of the degrees of freedom for non-rational approximation

The end points and first and second derivatives of the Bézier curve are given by:

$$\begin{aligned}
&\mathbf{p}(0) = \mathbf{p}_0, \quad \mathbf{p}(1) = \mathbf{p}_5,\\
&\dot{\mathbf{p}}(0) = 5\,(\mathbf{p}_1 - \mathbf{p}_0)\,, \qquad \dot{\mathbf{p}}(1) = 5\,(\mathbf{p}_5 - \mathbf{p}_4)\,,\\
&\ddot{\mathbf{p}}(0) = 20\,(\mathbf{p}_2 - 2\mathbf{p}_1 + \mathbf{p}_0)\,, \qquad \ddot{\mathbf{p}}(1) = 20\,(\mathbf{p}_5 - 2\mathbf{p}_4 + \mathbf{p}_3)\,,
\end{aligned}$$

and the two end unit tangents and end curvatures are given by [2]:

$$\begin{aligned}
&\mathbf{t}_p(0) = d_1^{-1}(\mathbf{p}_1 - \mathbf{p}_0), \qquad \mathbf{t}_p(1) = d_5^{-1}(\mathbf{p}_5 - \mathbf{p}_4),\\
&\left|\kappa_p(0)\right| = 1.6d_1^{-3}\mathit{area}[\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2], \qquad \left|\kappa_p(1)\right| = 1.6d_5^{-3}\mathit{area}[\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5],
\end{aligned}$$

where $d_i = \left\|\mathbf{p}_i - \mathbf{p}_{i-1}\right\|, i = 1, \ldots, 5$; $\mathit{area}[\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2] = (d_1 d_2\,|\sin\varphi_1|)/2$, $\mathit{area}[\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5] = (d_4 d_5\,|\sin\varphi_4|)/2$, φ_i is the angle between $\mathbf{p}_{i+1} - \mathbf{p}_i$ and $\mathbf{p}_i - \mathbf{p}_{i-1}$, $i = 1, 2, 3, 4$.

Thus, the constraints on the approximating Bézier curve to the spiral $\mathbf{r}(t)$, translated and rotated onto the origin with unit tangent vector in the direction of the x-axis, can be expressed as:

$$\begin{aligned}
&\mathbf{p}(0) = \mathbf{p}_0 = \mathbf{r}(t_0) = (0, 0), \qquad \mathbf{p}(1) = \mathbf{p}_5 = \mathbf{r}(t_1),\\
&\mathbf{t}_p(0) = d_1^{-1}(\mathbf{p}_1 - \mathbf{p}_0) = \mathbf{t}_r(t_0) = (1, 0), \qquad \mathbf{t}_p(1) = d_5^{-1}(\mathbf{p}_5 - \mathbf{p}_4) = \mathbf{t}_r(t_1),\\
&1.6d_1^{-3}\mathit{area}[\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2] = |\kappa_r(t_0)|\,, \qquad 1.6d_5^{-3}\mathit{area}[\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5] = |\kappa_r(t_1)|\,.
\end{aligned}$$

The six control points of the Bézier quintic are then uniquely defined by the parameters $\mathbf{p}_0, \mathbf{p}_5, d_1, d_2, d_4, d_5, \theta_0, \theta_5, \varphi_1$ and φ_4 (Fig. 2). Since $\mathbf{p}_0, \mathbf{p}_5, \theta_0$ and θ_5 are specified by the initial conditions, there are 6 scalar degrees of freedom $d_1, d_2, d_4, d_5, \varphi_1$ and φ_4 that can be manipulated.

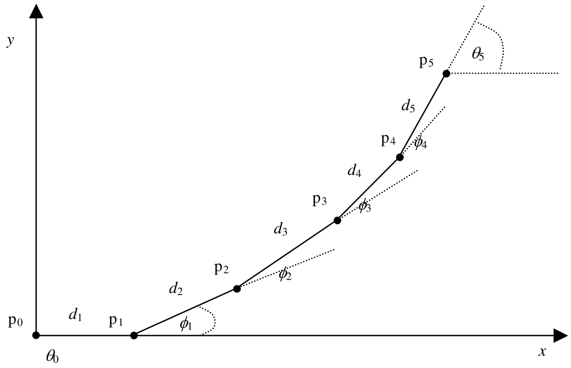

Fig. 2. Representation of a non-rational Bézier quintic curve.

Now $\mathbf{p}_1$ and $\mathbf{p}_4$ are given by:

$$\mathbf{p}_1 = \mathbf{p}_0 + d_1(\cos\theta_0, \sin\theta_0); \qquad \mathbf{p}_4 = \mathbf{p}_5 - d_5(\cos\theta_5, \sin\theta_5),$$

where $\theta(0) = \theta_0 = 0$ and $\theta(S) = \theta_5(0 \le |\theta_5| < \pi/2)$ are the angles between the tangents $\mathbf{t}_p(0)$ and $\mathbf{t}_p(1)$ and the x-axis respectively (Fig. 2).

Acceptable values for the parameters can be found using standard multi-dimensional searching algorithms [12]. For illustrative purposes, suppose that the angles φ_1 and φ_4 can be estimated. For example one could take $\varphi_1 = \theta_0/2$; $\varphi_4 = \theta_5/2$, then $0.8d_1^{-2}d_2\,|\sin\varphi_1| = |\kappa_r(t_0)|$ and $0.8d_4d_5^{-2}\,|\sin\varphi_4| = |\kappa_r(t_1)|$, which results is just 2 degrees of freedom. In the following case studies, a simple numerical search is carried out to numerically optimise the approximation by balancing the remaining parameters. In outline the algorithm implemented has the following steps. Set d_1 and d_5 to be $0.2\,\|\mathbf{p}_5 - \mathbf{p}_0\|$ and $\varphi_1 = \theta_0/2$; $\varphi_4 = \theta_5/2$ then simple step through values of the parameters between 0 and $\|\mathbf{p}_5 - \mathbf{p}_0\|$ for the chords and 0 and $\pi/2$ for the angles with small increments. At each step determine the relative curvature error until within the specified tolerance, then terminate. This clearly will not find the optimum quintic Bézier, merely the first to be within tolerance.

6. Numerical results

To illustrate the performance of the proposed method, a range of generalised spiral arcs are approximated with a Bézier quintic curve segment which minimises the relative error between the curvature profiles.

6.1. Unit semicircle

The simplest GCS arc is the circular arc which has constant curvature. Consider a unit semicircle centred on (0, 1), curvature, $\kappa^{-1} = \rho = 1$. The initial conditions on the GCS are $r = 0$ and $\kappa(0) = \kappa_0 = 1$ and $\kappa(1) = \kappa_1 = 1$. Further, $\mathbf{r}(0) = (0, 0)$, $\mathbf{r}(1) = (0, 2)$ and $\theta(0) = \theta_0 = 0$, $\theta(1) = \theta_5 = \pi$. Using (3), Section 4, the tolerance for the approximation is given by $\varepsilon = 3.5\mathrm{e}^{-2}$. Given the symmetry of the circular arc, it is supposed that $d_1 = d_5$ and $\varphi_1 = \varphi_4$. Iteration on the remaining parameters resulted in a maximum relative error, given by Eq. (4), $\mu = 2.43\mathrm{e}^{-5}$ which is well within the anticipated tolerance, ε and hence a Bézier quintic curve segment satisfying a monotonic curvature profile within the specified tolerance has been constructed. The parameters of the Bézier curve segment are given by: $d_1 = d_5 = 0.667855$ and $\varphi_1 = \varphi_4 = 0.750278$. The control points of the Bézier are given by: $\mathbf{p}_0 = (0.0, 0.0)$, $\mathbf{p}_1 = (0.668, 0.0)$, $\mathbf{p}_2 = (1.266, 0.558)$, $\mathbf{p}_3 = (1.266, 1.442)$, $\mathbf{p}_4 = (0.668, 2.0)$, $\mathbf{p}_5 = (0.0, 2.0)$. The circular arc and resulting Bézier curve are given in Fig. 3, whilst the curvature profiles and relative errors between the circle and the Bézier approximation are given in Figs. 4 and 5 respectively. The curves shown in Fig. 3 appear coincident. Whilst it is clear that the curvature profile of the approximating Bézier quintic oscillates about the required profile, $\kappa(\bar{s}) = 1.0$, the size of the errors are within tolerance.

6.2. Non-inflecting GCS

The simplest non-degenerate GCS is the non-inflecting arc such that the curvature profile, $\kappa(\bar{s})$ does not change sign. A typical non-inflecting GCS is give by $S = 3.0$, $\kappa(0) = 0.1$, $\kappa(S) = 0.5$, $\mathbf{r}(0) = (0, 0)$, $\mathbf{r}(S) = (2.67, 1.076)$, $\theta(0) = \theta_0 = 0$ and $\theta(S) = \theta_5 = 0.967$ and shape factor, $r = 0.4$ (Fig. 6). Again, from Section 4, the tolerance is given by $\varepsilon = 1.0\mathrm{e}^{-2}$ and using the same initial conditions are for the unit circle, i.e. $d_1 = d_5$ and $\varphi_1 = \varphi_4$, iteration on the parameters resulted in a relative error $\mu = 1.0\mathrm{e}^{-3}$ which is within the anticipated tolerance, ε, and hence a Bézier quintic curve segment satisfying a monotonic curvature profile within the specified tolerance has been constructed. The parameters of the Bézier curve segment are given by: $d_1 = d_5 = 0.559024$ and $\varphi_1 = 0.633513$ $\varphi_4 = 0.363078$ with corresponding control points: $\mathbf{p}_0 = (0, 0)$, $\mathbf{p}_1 = (0.559, 0.0)$, $\mathbf{p}_2 = (0.612, 0.039)$, $\mathbf{p}_3 = (1.90, 0.304)$, $\mathbf{p}_4 = (2.353, 0.616)$, $\mathbf{p}_5 = (2.670, 1.076)$. The

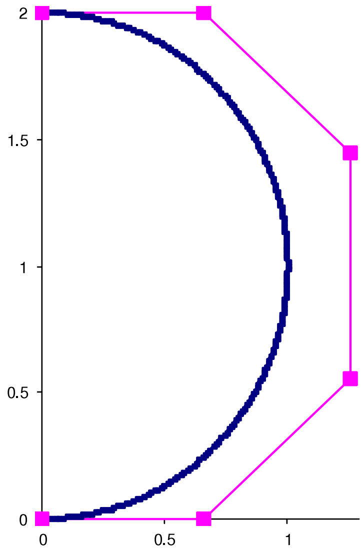

Fig. 3. Unit semicircle and approximating Bézier Quintic.

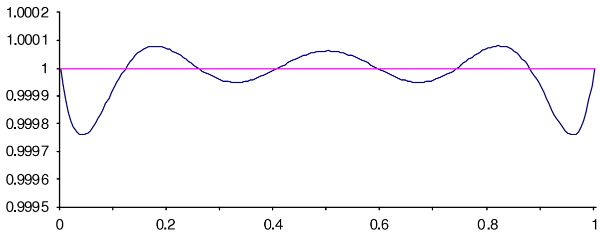

Fig. 4. Curvature profile: Circle vs. Bézier Quintic.

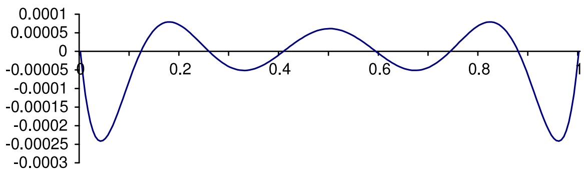

Fig. 5. Relative curvature error of approximating Bézier Quintic.

resulting Bézier quintic is plotted along with the GCS in Fig. 6. Again the curves appear coincident. The curvature profile ($\kappa(\bar{s})$) against proportion distance ($\bar{s}$) of the GCS and the approximating Bézier quintic are given in Fig. 7 which illustrates the close match of the two, which is further shown by the relative curvature error in Fig. 8. The rather naïve implementation of the search algorithm is highlighted by the resulting control polygon of the approximating Bézier quintic (Fig. 7). The algorithm stops as soon as a control polygon configuration satisfies the approximation criteria. The initial condition $d_1 = d_5$ has given rise to the unbalanced control polygon. However, this is only one of the family of Bézier curves which satisfies the relative curvature constraint. Clearly one could improve the algorithm to identify the best configuration based on further criteria, possibly application dependent, for example, most evenly distributed chord lengths, etc.

6.3. Inflecting GCS

An inflecting spiral has the added complication that as the curvature approaches zero, the relative error becomes unstable. This example illustrates how the algorithm is adapted to cope with this situation. An inflecting GCS has initial conditions given by: $S = 20.0$, $\kappa(0) = 0.3$, $\kappa(S) = -0.2$ $\mathbf{r}(0) = (0, 0)$, $\mathbf{r}(S) = (10.245, 15.616)$, $\theta(0) = \theta_0 = 0$ and $\theta(S) = \theta_5 = 0.328$ and shape factor $r = 0.5$ (Fig. 9). Taking the tolerance $\varepsilon = 8.9\mathrm{e}^{-2}$ and assuming $d_1 = d_5$, iteration on the parameters resulted in a relative error $\mu = 4.7\mathrm{e}^{-2}$ which is within the anticipated tolerance, ε and hence a Bézier quintic curve segment satisfying a monotonic curvature profile within the specified tolerance has been constructed. The parameters of the Bézier curve segment are given by: $d_1 = d_5 = 4.0645$ and $\varphi_1 = 1.1871$ $\varphi_4 = -0.8134$. The resulting control points for the Bézier approximation are given by: $\mathbf{p}_0 = (0.0, 0.0)$, $\mathbf{p}_1 = (4.065, 0.0)$, $\mathbf{p}_2 = (6.565, 6.195)$, $\mathbf{p}_3 = (4.000, 9.139)$,

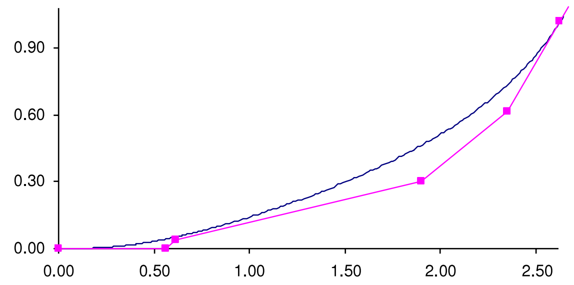

Fig. 6. Non-inflecting GCS and approximating Bézier Quintic.

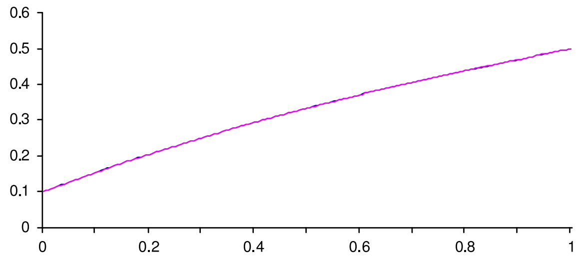

Fig. 7. Curvature profile: Non-inflecting GCS vs. Bézier Quintic.

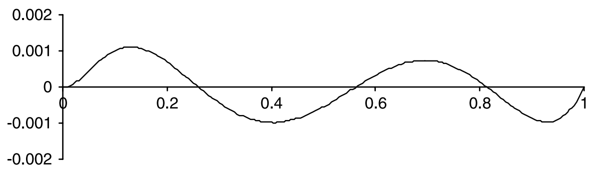

Fig. 8. Relative curvature error of approximating Bézier Quintic.

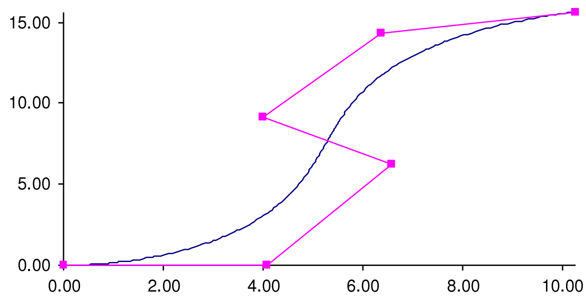

Fig. 9. Inflecting GCS and approximating Bézier Quintic.

$\mathbf{p}_4 = (6.367, 14.307)$, $\mathbf{p}_5 = (10.215, 15.616)$. The GCS and Bézier curves are given in Fig. 9 and illustrates the closeness of fit achieved by the approximations, the two curves being indistinguishable. Fig. 10 gives the curvature profiles of the inflecting GCS and the Bézier quintic approximation. The profiles are closest in the central region of the curve and furthest at the ends. This is due to terminating the search once the relative curvature error is within tolerance. The relative curvature error away from the inflection is given in Fig. 11(a) and the absolute curvature error in Fig. 11(b).

7. Conclusions

The aim of the paper is to give the theoretical background for approximating a spiral by a non-rational quintic Bézier curve based on curvature profiles. Using proportional distance reparameterisation, the curvature profiles of two curves can be compared at corresponding points. A suitable tolerance in the curvature has been derived based on the bounding circular arcs. This tolerance is used to control the approximation. The parameters of the Bézier quintic are found using a simple numerical search such that the resulting relative error in the curvature between the two curves is within the specified tolerance in the curvature. The method of constructing an approximation curve based on matching curvature profiles to within an acceptable tolerance is applicable to other curve forms providing the curvature profile is known.

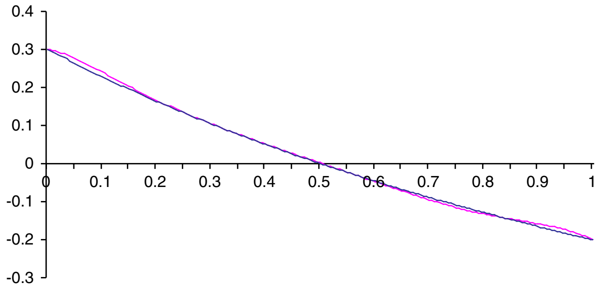

Fig. 10. Curvature profile: Inflecting GCS vs. Bézier Quintic.

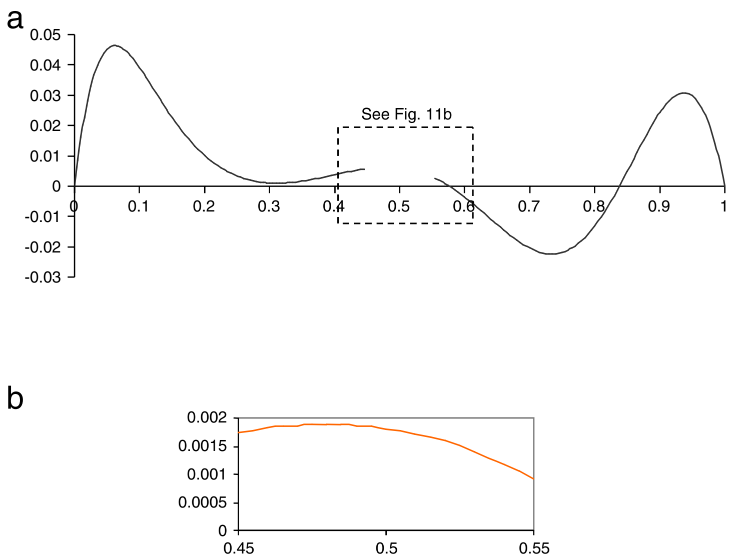

Fig. 11. (a) Relative curvature error of approximating Bézier Quintic. (b) Absolute error in region of small curvature.

The approach outlined in the paper can be extended to approximate spiral arcs with rational Bézier curves. This simply increases the degrees of freedom of the system as there are now 5 extra parameters, the weights associated with the control points. The current implementation demonstrates that such an approximation method is feasible. It is recognised that an entire family of acceptable Bézier curves exists and identification of the optimum member of that family is currently under consideration.

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References

[1] R.J. Cripps, Y. Li, Characterisation of Planar Rational Cubic Curves, Third SIAM Conference of Geometric Design, Arizona, 1993.
[2] G. Farin, Curves and Surfaces for Computer Aided Design: A Practical Guide, 4th ed., Academic Press, San Diago, 1993.
[3] J.M. Ali, R.M. Tooky, J.V. Ball, A.A. Ball, The generalised Cornu spiral and its application to span generation, Journal of Computational and Applied Mathematics 102 (1999) 37–47.
[4] C. Baumgarten, G. Farin, Approximation of logarithmic spirals, Computer Aided Geometric Design 14 (1997) 515–532.
[5] D.S. Meek, D.J. Walton, The use of Cornu spirals in drawing planar curves of controlled curvature, Journal of Computational and Applied Mathematics 25 (1989) 79–92.
[6] D.S. Meek, D.J. Walton, Clothoid spline transition spirals, Mathematics of Computation 59 (199) (1992) 117–133.
[7] D.S. Meek, D.J. Walton, Planar G^1 Hermite interpolation with spirals, Computer Aided Geometric Design 15 (1998) 787–801.
[8] D.S. Meek, D.J. Walton, An arc spline approximation to a clothoid, Journal of Computational and Applied Mathematics 170 (2004) 59–77.
[9] L.Z. Wang, K.T. Miura, E. Nakamae, T. Yamamoto, T.J. Wang, An approximation approach of the clothoid curve defined in the interval $[0, \pi/2]$ and its offset by free-form curves, Computer Aided Design 33 (2001) 1049–1058.
[10] D.J. Struik, Lectures on Classical Differential Geometry, Dover Publication, New York, 1988.
[11] F.E. O'Neill, Geometry Based Constructions for Curves and Surfaces, Ph.D. Thesis, University of Birmingham, 1994.
[12] W.H. Press, S.A. Teukolsk, W.T. Vettering, B.P. Flannery, Numerical Recipes in C, The Art of Scientific Computing, 2nd ed., Cambridge University Press, New York, 1994.