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Sharp estimates for pseudo-differential operators with symbols of limited smoothness and commutators

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Abstract

We consider here pseudo-differential operators whose symbol $\sigma(x, \xi)$ is not infinitely smooth with respect to x. Decomposing such symbols into four—sometimes five—components and using tools of paradifferential calculus, we derive sharp estimates on the action of such pseudo-differential operators on Sobolev spaces and give explicit expressions for their operator norm in terms of the symbol $\sigma(x, \xi)$. We also study commutator estimates involving such operators, and generalize or improve the so-called Kato–Ponce and Calderon–Coifman–Meyer estimates in various ways.

Keywords: Pseudo-differential operators; Paradifferential calculus; Commutator estimates

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1. Introduction

1.1. General setting and description of the results

Among the widely known properties of pseudo-differential operators with symbol in Hormander's class $S^m_{1,0}$, two are discussed in this paper. The first one concerns their action on Sobolev spaces, and the second one deals with the properties of commutators.

It is a classical result that for all $\sigma^1 \in S^{m_1}_{1,0}$, the operator $\mathrm{Op}(\sigma^1)$ maps $H^{s+m_1}(\mathbb{R}^d)$ into $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$. Moreover, the proof shows that

$$\forall u \in H^{s+m_1}(\mathbb{R}^d), \quad |\mathrm{Op}(\sigma^1)u|_{H^s} \leqslant \underline{C}(\sigma^1)|u|_{H^{s+m_1}}. \tag{1.1}$$

Concerning the study of commutators, Taylor (following works of Moser [16] and Kato–Ponce [10]) proved in [17] that for all $\sigma^1 \in S^{m_1}_{1,0}$, with $m_1 > 0$, and all $\sigma^2 \in H^\infty(\mathbb{R}^d)$, one has for all $s \geqslant 0$,

$$\big|[\mathrm{Op}(\sigma^1), \sigma^2]u\big|_{H^s} \leqslant \underline{C}(\sigma^1)\big(|\sigma^2|_{W^{1,\infty}}|u|_{H^{s+m_1-1}} + |\sigma^2|_{H^{s+m_1}}|u|_\infty\big); \tag{1.2}$$

another well-known commutator estimate is the so-called Calderon–Coifman–Meyer estimate: if $m_1 \geqslant 0$ then for all $s \geqslant 0$ and $t_0 > d/2$ such that $s + m_1 \leqslant t_0 + 1$, one has (see [20, Proposition 4.2] for instance):

$$\big|[\mathrm{Op}(\sigma^1), \sigma^2]u\big|_{H^s} \leqslant \underline{C}(\sigma^1)|\sigma^2|_{H^{t_0+1}}|u|_{H^{s+m_1-1}}. \tag{1.3}$$

A drawback of (1.1)–(1.3) is that the dependence of the constant $\underline{C}(\sigma^1)$ on σ^1 is not specified. This may cause these estimates to be inoperative in the study of some nonlinear PDE; indeed, when solving such an equation by an iterative scheme, one is led to study the pseudo-differential operator corresponding to the linearized equations around some reference state. Generally, the symbol of this operator can be written $\sigma(x,\xi) = \Sigma(v(x),\xi)$, where $\Sigma(v,\xi)$ is smooth with respect to v and of order m with respect to ξ, while $v(\cdot)$ belongs to some Sobolev space $H^s(\mathbb{R}^d)$. For instance, in the study of nonlinear water waves, one is led to study the operator associated to the symbol (see [11])

$$\sigma(x,\xi) := \sqrt{(1+|\nabla a|^2)|\xi|^2 - (\nabla a \cdot \xi)^2}, \tag{1.4}$$

which is of the form described above, with $\Sigma(v,\xi) = \sqrt{(1+|v|^2)|\xi|^2 - (v\cdot\xi)^2}$ and $v(\cdot) = \nabla a$. Such symbols $\sigma(x,\xi)$ are not infinitely smooth with respect to x, since their regularity is limited by the regularity of the function v. One must therefore be able to handle symbols of limited smoothness to deal with such situations; moreover, one must be able to say which norms of $v(\cdot)$ are involved in the constant $\underline{C}(\sigma^1)$ of (1.1)–(1.3).

But even knowing precisely the way the constants $\underline{C}(\sigma^1)$ depend on σ^1, estimates (1.1) and (1.2) may not be precise enough in some situations. Indeed, when one has to use, say, a Nash–Moser iterative scheme, *tame* estimates are needed. For instance, in such situations, the product estimate $|uv|_{H^s} \lesssim |u|_{H^s}|v|_{H^s}$ $(s > d/2)$ is inappropriate and must be replaced by Moser's tame product estimate $|uv|_{H^s} \lesssim |u|_\infty |v|_{H^s} + |u|_{H^s}|v|_\infty$ ($s \geqslant 0$). Obviously, (1.1) is not precise enough to contain this latter estimate. Part of this paper is therefore devoted to the derivation of sharper versions of (1.1).

In the works dealing with pseudo-differential operators with nonregular symbols, the focus is generally on the continuity of such operators on Sobolev or Zygmund spaces (see for instance [17,18]) and not on the derivation of precise (and tame) estimates. In [7], Grenier gave some description of the constants $\underline{C}(\sigma^1)$ in (1.1)–(1.2) but his results, though sufficient for his purposes, are far from optimal. In this article, we aim at proving more precise versions of (1.1)–(1.3), and we also give some extensions of these results. Let us describe roughly some of them:

Action of pseudo-differential operators on Sobolev spaces (see Corollary 30): Take a symbol $\sigma \in S^m_{1,0}$ of the form $\sigma(x,\xi) = \Sigma(v(x),\xi)$, with Σ as described above and $v \in H^\infty$. Then Moser's tame product estimate can be generalized to pseudo-differential operators of order $m > 0$: for all $s > 0$,

$$|\mathrm{Op}(\sigma)u|_{H^s} \lesssim C(|v|_\infty)(|v|_{H^{s+m}}|u|_\infty + |u|_{H^{s+m}}).$$

Another estimate which does not assume any restriction on the order m and also holds for negative values of s is the following: for all $t_0 > d/2$, one has

$$\forall -t_0 < s < t_0, \qquad |\mathrm{Op}(\sigma)u|_{H^s} \lesssim C(|v|_\infty)|v|_{H^{t_0}}|u|_{H^{s+m}},$$
$$\forall\, t_0 \leqslant s, \qquad |\mathrm{Op}(\sigma)u|_{H^s} \lesssim C(|v|_\infty)(|v|_{H^s}|u|_{H^{m+t_0}} + |u|_{H^{s+m}}).$$

Commutator estimates: In this paper, we give a precise description of the constant $\underline{C}(\sigma^1)$ which appears in (1.2) and (1.3), and generalize these estimates in three directions:

- We control the symbolic expansion of the commutator in terms of the Poisson brackets. For instance, in the particular case when the symbol $\sigma^1(x,\xi) = \sigma^1(\xi)$ does not depend on x, we derive the following estimate (see Theorem 5): if $m_1 \in \mathbb{R}$ and $n \in \mathbb{N}$ are such that $m_1 > n$, then for all $s \geqslant 0$, one has

$$\left|[\mathrm{Op}(\sigma^1),\sigma^2]u - \mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\right|_{H^s}$$
$$\lesssim C(\sigma^1)\left(|\nabla^{n+1}\sigma^2|_\infty |u|_{H^{s+m_1-n-1}} + |\sigma^2|_{H^{s+m_1}}|u|_\infty\right),$$

  and a precise description of $C(\sigma^1)$ is given; if $\sigma^1(\cdot)$ is regular at the origin, we have a more precise version involving only derivatives of σ^2,

$$\left|[\mathrm{Op}(\sigma^1),\sigma^2]u - \mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\right|_{H^s}$$
$$\lesssim C'(\sigma^1)\left(|\nabla^{n+1}\sigma^2|_\infty |u|_{H^{s+m_1-n-1}} + |\nabla^{n+1}\sigma^2|_{H^{s+m_1-n-1}}|u|_\infty\right).$$

  For a similar generalization of (1.3), see Theorem 6.
- We allow σ^2 to be a pseudo-differential operator and not only a function (Theorems 3, 4, 6, 7 and 8, and Corollaries 39 and 43);

- We give an alternative to (1.2) allowing the cases $m_1 \leqslant 0$ and $s < 0$ (Theorems 3 and 7 and Corollaries 39 and 43); similarly, we show that negative values of s and m_1 are possible in (1.3) (see Theorems 6 and 8). For instance, if σ^1 is a Fourier multiplier of order $m_1 \in \mathbb{R}$ and σ^2 is of order $m_2 \in \mathbb{R}$ with $\sigma^2(x,\xi) = \Sigma^2(v(x),\xi)$ and $v \in H^\infty(\mathbb{R}^d)^p$ then for all $s \in \mathbb{R}$ such that $\max\{-t_0, -t_0 - m_1\} < s$ (with $t_0 > d/2$ arbitrary),

$$\big|\big[\sigma^1(D), \sigma^2(x,D)\big]u - \mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\big|_{H^s} \\ \leqslant C(\sigma^1, |v|_{W^{n+1,\infty}})\big(|u|_{H^{s+m_1+m_2-n-1}} + |v|_{H^{(s+m_1\wedge n)_+}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}\big).$$

The above results admit generalization to L^p-based Sobolev spaces and Besov spaces, but we deliberately chose to work with classical L^2-based Sobolev spaces to ease the readability. We refer the reader interested by this kind of generalizations to [21,22,12,17,18] for instance.

The methods used to prove the above results rely heavily on Bony's paradifferential calculus [3] as well as on the works of Coifman and Meyer [14,15].

In Section 2, we introduce the class of symbols adapted to our study; they consist in all the symbols $\sigma(x,\xi)$ such that $\sigma(\cdot,\xi)$ belongs to some Sobolev space for all ξ. These symbols are decomposed into four components, one of them being the well-known paradifferential symbol associated to σ, and some basic properties are given.

In Section 3, we study the action on Sobolev (and Zygmund) spaces of the four components into which each symbol is decomposed, and give precise estimate on the operator norm. These results generalize classical results of paraproduct theory and are in the spirit of [12] and especially [21] (but the estimate we give here are different from the ones given in this latter reference). Gathering the estimates obtained on each component, we obtain a tame estimate on the action of the operator associated to the full symbol $\sigma(x,\xi)$.

Section 4 is devoted to the study of commutator estimates. We first give in Proposition 31 precise estimates for Meyer's well-known result on the symbolic calculus for paradifferential operators. In Section 4.1, we address the case of commutators between a Fourier multiplier $\sigma^1(D)$ and a pseudo-differential operator $\sigma^2(x,D)$; we study some particular cases, including the case when $\sigma^2(x,\xi) = \sigma^2(x)$ is a function. The case when $\sigma^1(x,D)$ is a pseudo-differential operator (and not only a Fourier multiplier) is then addressed in Section 4.2.

Throughout this paper, we use the following notations.

Notations: (i) For all $a, b \in \mathbb{R}$, we write $a \wedge b := \max\{a,b\}$;

(ii) For all $a \in \mathbb{R}$, we write $a_+ := \max\{a, 0\}$, while $[a]$ denotes the biggest integer smaller than a;

(iii) If $f \in F$ and $g \in G$, F and G being two Banach spaces, the notation $|f|_F \lesssim |g|_G$ means that $|f|_F \leqslant C|g|_G$ for some constant C which does not depend on f nor g.

(iv) Here, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decaying functions, and for any distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, we write respectively $\widehat{f}$ and $\check{f}$ its Fourier and inverse Fourier transform.

(v) We use the classical notation $f(D)$ to denote the Fourier multiplier, namely, $\widehat{f(D)u}(\cdot) = f(\cdot)\widehat{u}(\cdot)$.

1.2. Brief reminder of Littlewood–Paley theory

We recall in this section basic facts in Littlewood–Paley theory.

Throughout this article, $\psi \in C_0^\infty(\mathbb{R}^d)$ denotes a smooth bump function such that

$$\psi(\xi) = 1 \quad \text{if } |\xi| \leqslant 1/2 \qquad \text{and} \qquad \psi(\xi) = 0 \quad \text{if } |\xi| \geqslant 1, \tag{1.5}$$

and we define $\varphi \in C_0^\infty(\mathbb{R}^d)$ as

$$\varphi(\xi) = \psi(\xi/2) - \psi(\xi) \quad \forall \xi \in \mathbb{R}^d, \tag{1.6}$$

so that φ is supported in the annulus $1/2 \leqslant |\xi| \leqslant 2$, and one has

$$1 = \psi(\xi) + \sum_{p \geqslant 0} \varphi(2^{-p}\xi) \quad \forall \xi \in \mathbb{R}^d. \tag{1.7}$$

For all $p \in \mathbb{Z}$, we introduce the functions φ_p, supported in $2^{p-1} \leqslant |\xi| \leqslant 2^{p+1}$, and defined as

$$\varphi_p = 0 \text{ if } p < -1, \quad \varphi_{-1} = \psi, \quad \varphi_p(\cdot) = \varphi(2^{-p}\cdot) \quad \text{if } p \geqslant 0. \tag{1.8}$$

This allows us to give the classical definition of Zygmund spaces:

Definition 1. Let $r \in \mathbb{R}$. Then $C_*^r(\mathbb{R}^d)$ is the set of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$|u|_{C_*^r} := \sup_{p \geqslant -1} 2^{pr} |\varphi_p(D)u|_\infty < \infty.$$

Remark 2. We recall the continuous embeddings $H^s(\mathbb{R}^d) \subset C_*^{s-d/2}(\mathbb{R}^d)$, for all $s \in \mathbb{R}$, and $L^\infty(\mathbb{R}^d) \subset C_*^0(\mathbb{R}^d)$.

We now introduce admissible cut-off functions, which play an important role in paradifferential theory ([3,14]; [13, Appendix B]).

Definition 3. A smooth function $\chi(\eta, \xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ is an admissible cut-off function if and only if:

- There are δ_1 and δ_2 such that $0 < \delta_1 < \delta_2 < 1$ and

$$\forall\, |\xi| \geqslant 1/2, \quad \chi(\eta, \xi) = 1 \quad \text{for } |\eta| \leqslant \delta_1 |\xi|,$$
$$\forall\, |\xi| \geqslant 1/2, \quad \chi(\eta, \xi) = 0 \quad \text{for } |\eta| \geqslant \delta_2 |\xi|;$$

- For all $\alpha, \beta \in \mathbb{N}^d$, there is a constant $C_{\alpha,\beta}$ such that

$$\forall (\eta, \xi) \in \mathbb{R}^{2d}, \quad |\xi| \geqslant 1/2, \quad \left|\partial_\xi^\alpha \partial_\eta^\beta \chi(\eta, \xi)\right| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{-|\alpha|-|\beta|}. \tag{1.9}$$

Example 1. A useful admissible cut-off function is given, for all $N \geqslant 2$, by

$$\chi(\eta, \xi) = \sum_{p \geqslant -1} \psi(2^{-p+N}\eta)\varphi_p(\xi); \tag{1.10}$$

One can check that it satisfies indeed the conditions of Definition 3 with $\delta_1 = 2^{-N-2}$ and $\delta_2 = 2^{1-N}$.

An important property satisfied by admissible cut-off functions is that $\check{\chi}(\cdot, \xi)$ and its derivatives with respect to ξ enjoy good estimates in L^1-norm. The next lemma is a simple consequence of the estimates imposed in Definition 3; we refer for instance to [13, Appendix B] for a proof.

Lemma 4. *Let $\chi(\eta, \xi)$ be an admissible cut-off function. Then for all $\beta \in \mathbb{N}^d$, there exists a constant C_β such that*

$$\forall \xi \in \mathbb{R}^d, \quad \left|\partial_\xi^\beta \check{\chi}(\cdot, \xi)\right|_{L^1(\mathbb{R}^d)} \leqslant C_\beta \langle \xi \rangle^{-|\beta|}.$$

Finally, we end this section with the classical characterization of Sobolev spaces (see for instance [5, Theorem 2.2.1] or [18, Lemma 9.4]).

Lemma 5. *Let $(u_p)_{p \geqslant -1}$ be a sequence of $\mathcal{S}'(\mathbb{R}^d)$ such that for all $p \geqslant 0$, $\widehat{u_p}$ is supported in $A2^{p-1} \leqslant |\xi| \leqslant B2^{p+1}$, for some $A, B > 0$, and such that $\widehat{u_{-1}}$ is compactly supported.*

If, for some $s \in \mathbb{R}$, $\sum_{p \geqslant -1} 2^{2ps}|u_p|_2^2 < \infty$, then

$$\sum_{p \geqslant -1} u_p =: u \in H^s(\mathbb{R}^d) \quad \text{and} \quad |u|_{H^s}^2 \leqslant \text{Cst} \sum_{p \geqslant -1} 2^{2ps}|u_p|_2^2.$$

Conversely, if $u \in H^s(\mathbb{R}^d)$ then

$$\sum_{p \geqslant -1} 2^{2ps}|\varphi_p(D)u|_2^2 \leqslant \text{Cst } |u|_{H^s}^2.$$

2. Symbols

As said in the introduction, we are led to consider nonregular symbols $\sigma(x,\xi)$ such that

$$\sigma(x,\xi) = \Sigma(v(x),\xi), \tag{2.1}$$

where $v \in C^0(\mathbb{R}^d)^p$ for some $p \in \mathbb{N}$, while Σ is a smooth function belonging to the class $C^\infty(\mathbb{R}^p, \mathcal{M}^m)$ defined below.

Definition 6. Let $p \in \mathbb{N}$, $m \in \mathbb{R}$ and let Σ be a function defined over $\mathbb{R}^p_v \times \mathbb{R}^d_\xi$. We say that $\Sigma \in C^\infty(\mathbb{R}^p, \mathcal{M}^m)$ if

- $\Sigma_{|\mathbb{R}^p \times \{|\xi| \leqslant 1\}} \in C^\infty(\mathbb{R}^p; L^\infty(\{|\xi| \leqslant 1\}))$;
- For all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^d$, there exists a nondecreasing function $C_{\alpha,\beta}(\cdot)$ such that

$$\sup_{\xi \in \mathbb{R}^d, |\xi| \geqslant 1/4} \langle \xi \rangle^{|\beta| - m} \left| \partial_v^\alpha \partial_\xi^\beta \Sigma(v,\xi) \right| \leqslant C_{\alpha,\beta}(|v|).$$

Example 2. One can write the symbol $\sigma(x,\xi)$ given in (1.4) under the form $\sigma(x,\xi) = \Sigma(\nabla a, \xi)$, with $\Sigma(v,\xi) = \sqrt{(1+|v|^2)|\xi|^2 - (v \cdot \xi)^2}$. One can check that $\Sigma \in C^\infty(\mathbb{R}^d, \mathcal{M}^1)$.

Remark 7. (i) We do not assume in this article that the symbols are smooth at the origin with respect to ξ (for instance, (1.4) has singular derivatives at the origin). This is the reason why the estimate in Definition 6 is taken over frequencies away from the origin, namely $|\xi| \geqslant 1/4$.

(ii) When $p = 0$, then $C^\infty(\mathbb{R}^p, \mathcal{M}^m)$ coincides with the class $\mathcal{M}^m$ of symbols of Fourier multipliers of order m.

Let us remark now that if $\sigma(x,\xi)$ is as in (2.1), then one can write

$$\sigma(x,\xi) = \left[\sigma(x,\xi) - \Sigma(0,\xi)\right] + \Sigma(0,\xi);$$

the interest of such a decomposition is that the second term is a simple Fourier multiplier while the first one is in $H^s(\mathbb{R}^d)$ if $v \in H^s(\mathbb{R}^d)^p$, $s > d/2$:

Lemma 8. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$, $s_0 > d/2$ and take $v \in H^{s_0}(\mathbb{R}^d)^p$ and $\Sigma \in C^\infty(\mathbb{R}^p, \mathcal{M}^m)$; set also $\sigma(x,\xi) := \Sigma(v(x),\xi)$.*

Defining $\tau(x,\xi) = \sigma(x,\xi) - \Sigma(0,\xi)$, one has $\tau(\cdot,\xi) \in H^{s_0}(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$; moreover:

- *One has $\tau_{|\mathbb{R}^d \times \{|\xi| \leqslant 1\}} \in L^\infty(\{|\xi| \leqslant 1\}; H^{s_0}(\mathbb{R}^d))$;*

- *For all* $\beta \in \mathbb{N}^d$ *and* $0 \leqslant s \leqslant s_0$,

$$\sup_{\xi \in \mathbb{R}^d, |\xi| \geqslant 1/4} \langle \xi \rangle^{|\beta|-m} \left| \partial_\xi^\beta \tau(\cdot, \xi) \right|_{H^s} \leqslant C'_{s,\beta}(|v|_{L^\infty}|)|v|_{H^s},$$

where $C'_{s,\beta}(\cdot)$ *is some nondecreasing function depending only on the* $C_{\alpha,\beta}(\cdot)$, $|\alpha| \leqslant [s] + 2$, *introduced in Definition 6.*

Proof. This is a simple consequence of Moser's inequality (see e.g. [18, Proposition 3.9]) and the properties of Σ set forth in Definition 6. □

The previous lemma motivates the introduction of the following class of symbols (see also [21,12] for similar symbol classes).

Definition 9. Let $m \in \mathbb{R}$ and $s_0 > d/2$. A symbol $\sigma(x, \xi)$ belongs to the class $\Gamma^m_{s_0}$ if and only if

- One has $\sigma_{|_{\mathbb{R}^d \times \{|\xi| \leqslant 1\}}} \in L^\infty(\{|\xi| \leqslant 1\}; H^{s_0}(\mathbb{R}^d))$;
- For all $\beta \in \mathbb{N}^d$, one has

$$\sup_{\xi \in \mathbb{R}^d, |\xi| \geqslant 1/4} \langle \xi \rangle^{|\beta|-m} \left| \partial_\xi^\beta \sigma(\cdot, \xi) \right|_{H^{s_0}} < \infty.$$

We now set some terminology concerning the regularity of the symbols at the origin.

Definition 10. We say that $\Sigma \in C^\infty(\mathbb{R}^p, \mathcal{M}^m)$ is *k-regular* at the origin if $\Sigma_{|_{\mathbb{R}^p \times \{|\xi| \leqslant 1\}}} \in C^\infty(\mathbb{R}^p; W^{k,\infty}(\{|\xi| \leqslant 1\}))$.

Similarly, we say that $\sigma \in \Gamma^m_{s_0}$ is *k-regular* at the origin if $\sigma_{|_{\mathbb{R}^d \times \{|\xi| \leqslant 1\}}} \in W^{k,\infty}(\{|\xi| \leqslant 1\}; H^{s_0}(\mathbb{R}^d))$.

Notation 1. *It is quite natural to introduce the seminorms* $N^m_{k,s}(\cdot)$ *and* $M^m_{k,l}(\cdot)$ *defined for all* $k, l \in \mathbb{N}$, $s \in \mathbb{R}$ *and* $m \in \mathbb{R}$ *as*

$$N^m_{k,s}(\sigma) := \sup_{|\beta| \leqslant k} \sup_{|\xi| \geqslant 1/4} \langle \xi \rangle^{|\beta|-m} \left| \partial_\xi^\beta \sigma(\cdot, \xi) \right|_{H^s} \tag{2.2}$$

and

$$M^m_{k,l}(\sigma) := \sup_{|\beta| \leqslant k} \sup_{|\xi| \geqslant 1/4} \langle \xi \rangle^{|\beta|-m} \left| \partial_\xi^\beta \sigma(\cdot, \xi) \right|_{W^{l,\infty}}. \tag{2.3}$$

To get information on the low frequencies, we also define

$$n_{k,s}(\sigma) := \sup_{|\beta|\leqslant k, |\xi|\leqslant 1} \left|\partial_\xi^\beta \sigma(\cdot,\xi)\right|_{H^s} \quad \text{and} \quad m_k(\sigma) := \sup_{|\beta|\leqslant k, |\xi|\leqslant 1} |\partial_\xi^\beta \sigma(\cdot,\xi)|_\infty. \tag{2.4}$$

Note that $M^m_{k,l}(\sigma)$ and $m_k(\sigma)$ still make sense when σ is the symbol of a Fourier multiplier (i.e. if $\sigma(x,\xi)=\sigma(\xi)$). When $l=0$, we simply write $M^m_k(\sigma)$ instead of $M^m_{k,0}(\sigma)$.

Finally, the notation $N^m_{k,s}(\nabla^l_x\sigma)$, $l\in\mathbb{N}$, stands for $\sup_{|\alpha|=l} N^m_{k,s}(\partial^\alpha_x\sigma)$; we use the same convention for the other seminorms defined above.

Associated to the class Γ^m_s is the subclass of paradifferential symbols Σ^m_s (in the sense of [3,14], see also [15] and [13, Appendix B]). In the definition below, the notation Sp is used to denote the spectrum of a function, that is, the support of its Fourier transform.

Definition 11. Let $m\in\mathbb{R}$ and $s_0>d/2$. A symbol $\sigma(x,\xi)$ belongs to the class $\Sigma^m_{s_0}$ if and only if

- One has $\sigma\in\Gamma^m_{s_0}$,
- There exists $\delta\in(0,1)$ such that

$$\forall\xi\in\mathbb{R}^d,\quad \mathrm{Sp}\ \sigma(\cdot,\xi)\subset\{\eta\in\mathbb{R}^d, |\eta|\leqslant\delta|\xi|\}. \tag{2.5}$$

Remark 12. If $\psi\in C_0^\infty(\mathbb{R}^d)$ is as in (1.5), then the spectral condition (2.5) implies that for all $\sigma\in\Sigma^m_s$, one has

$$\forall\xi\in\mathbb{R}^d,\quad \sigma(\cdot,\xi)=\sigma(\cdot,\xi)*\left[(2\delta\langle\xi\rangle)^d\check{\psi}(2\delta\langle\xi\rangle\cdot)\right]. \tag{2.6}$$

It is classical (Bernstein's lemma) to deduce that for all $\alpha,\beta\in\mathbb{N}^d$, one has

$$\forall\xi\in\mathbb{R}^d, |\xi|\geqslant 1/4,\quad \left|\partial_x^\alpha\partial_\xi^\beta\sigma(\cdot,\xi)\right|_{L^\infty}\leqslant \mathrm{Cst}\ M^m_{|\beta|}(\sigma)\langle\xi\rangle^{m-|\beta|+|\alpha|}, \tag{2.7}$$

where $M^m_{|\beta|}(\cdot)$ is defined in (2.3).

It is well known that symbols of Γ^m_s can be smoothed into paradifferential symbols of Σ^m_s. In order to give a precise description of the difference between these two symbols (and of the operator associated to it), we split every $\sigma\in\Gamma^m_s$ into four components:

$$\sigma(x,\xi)=\sigma_{lf}(x,\xi)+\sigma_I(x,\xi)+\sigma_{II}(x,\xi)+\sigma_R(x,\xi), \tag{2.8}$$

with, for some $N \in \mathbb{N}$, $N \geqslant 4$,

$$\sigma_{lf}(\cdot, \xi) = \psi(\xi)\sigma(\cdot, \xi), \tag{2.9}$$

$$\sigma_I(\cdot, \xi) = \sum_{p \geqslant -1} \psi(2^{-p+N} D_x)\sigma(\cdot, \xi)(1 - \psi(\xi))\varphi_p(\xi), \tag{2.10}$$

$$\sigma_{II}(\cdot, \xi) = \sum_{p \geqslant -1} \varphi_p(D_x)\sigma(\cdot, \xi)(1 - \psi(\xi))\psi(2^{-p+N}\xi), \tag{2.11}$$

$$\sigma_R(\cdot, \xi) = \sum_{p \geqslant -1} \sum_{|p-q| \leqslant N} \varphi_q(D_x)\sigma(\cdot, \xi)(1 - \psi(\xi))\varphi_p(\xi), \tag{2.12}$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$ is a bump function satisfying (1.5).

We also need sometimes a further decomposition of σ_R as $\sigma_R = \sigma_{R,1} + \sigma_{R,2}$, with

$$\sigma_{R,1}(\cdot, \xi) := (1 - \psi(D_x))\sigma_R(\cdot, \xi) \quad \text{and} \quad \sigma_{R,2}(\cdot, \xi) := \psi(D_x)\sigma_R(\cdot, \xi); \tag{2.13}$$

note that $\sigma_{R,2}$ is given by the *finite* sum

$$\sigma_{R,2}(\cdot, \xi) = \sum_{p \leqslant N+1} \sum_{|p-q| \leqslant N} \psi(D_x)\varphi_q(D_x)\sigma(\cdot, \xi)(1 - \psi(\xi))\varphi_p(\xi).$$

Remark 13. (i) The fact that the sum of the four terms given in (2.9)–(2.12) equals $\sigma(x, \xi)$ follows directly from (1.7).

(ii) When $\sigma(x, \xi) = \sigma(x)$ does not depend on ξ, one has $\mathrm{Op}(\sigma_{lf})u = \sigma\psi(D)u$, $\mathrm{Op}(\sigma_I)u = T_\sigma\widetilde{u}$, $\mathrm{Op}(\sigma_{II})u = T_{\widetilde{u}}\sigma$ and $\mathrm{Op}(\sigma_R)u = R(\sigma, \widetilde{u})$, with $\widetilde{u} := (1 - \psi(D))u$ and where T_f denotes the usual paraproduct operator associated to f and $R(f, g) = fg - T_f g - T_g f$ (see [3,14,5]).

(iii) In [21], Yamazaki used a similar decomposition of symbols into three components. We need a fourth one here, namely σ_{lf}, in order to take into account symbols which are not infinitely smooth with respect to ξ at the origin. A fifth component is also introduced in (2.13); it is used in the proof of the second parts of Theorems 3–6.

In the next proposition, we check that σ_I belongs to the class of paradifferential symbols Σ_s^m.

Proposition 14. *Let $m \in \mathbb{R}$, $s_0 > d/2$, and let $\sigma \in \Gamma_{s_0}^m$.*

Then, the symbol σ_I defined in (2.10) belongs to $\Sigma_{s_0}^m$ and, for all $k \in \mathbb{N}$ and $s \leqslant s_0$,

$$N_{k,s}^m(\sigma_I) \leqslant \mathrm{Cst}\ N_{k,s}^m(\sigma) \quad \textit{and} \quad M_k^m(\sigma_I) \leqslant \mathrm{Cst}\ M_k^m(\sigma).$$

Proof. One can write $\sigma_I(\cdot,\xi) = (1-\psi(\xi))\check{\chi}(\cdot,\xi) * \sigma(\cdot,\xi)$, where $\chi(\eta,\xi)$ denotes the admissible cut-off function constructed in (1.10). The spectral property (2.5) is thus obviously satisfied by σ_I and the result follows therefore from simple convolution estimates, together with the bounds on the L^1-norm on the derivatives $\partial_\xi^\alpha \check{\chi}(\cdot,\xi)$ given in Lemma 4. □

Together with the decomposition given in (2.8), we shall also need another kind of decomposition, namely, Coifman and Meyer's decomposition into elementary symbols. The proof of the next proposition is a quite close adaptation of the proof of Proposition II.5 of [6]; it is given in Appendix A.

Proposition 15. *Let $m \in \mathbb{R}$ and $s_0 > d/2$, and let $\sigma \in \Gamma^m_{s_0}$. With $\psi \in C_0^\infty(\mathbb{R}^d)$ as given by* (1.5), *one has*

$$(1-\psi(\xi))\sigma(x,\xi) = \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^{[\frac{d}{2}]+1}} p_k(x,\xi)\langle\xi\rangle^m,$$

with $p_k(x,\xi) = \sum_{q\geqslant -1} c_{k,q}(x)\lambda_k(2^{-q}\xi)$, *and where*:

(i) *The coefficients $c_{k,q}(\cdot)$ are in $H^{s_0}(\mathbb{R}^d)$ and for all $s \leqslant s_0$, one has*

$$|c_{k,q}|_{H^s} \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,s}(\sigma);$$

moreover, for all $p \geqslant -1$, one has $|\varphi_p(D)c_{k,q}|_{H^s} \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,s}(\sigma_{(p)})$, where the symbol $\sigma_{(p)}$ is defined as $\sigma_{(p)}(\cdot,\xi) = \varphi_p(D_x)\sigma(\cdot,\xi)$.

(ii) *For all $k \in \mathbb{Z}^d$, the functions $\lambda_k(\cdot)$ are smooth and supported in $2/5 \leqslant |\xi| \leqslant 12/5$. Moreover, $\left|\check{\lambda}_k\right|_{L^1}$ is bounded from above uniformly in $k \in \mathbb{Z}^d$.*

3. Operators

To any symbol $\sigma(x,\xi) \in C^0(\mathbb{R}^d \times \mathbb{R}^d\backslash\{0\})$, one can associate an operator $\sigma(x,D) = \mathrm{Op}(\sigma)$ acting on functions whose Fourier transform is smooth and compactly supported in $\mathbb{R}^d\backslash\{0\}$:

$$\forall x \in \mathbb{R}^d, \quad \mathrm{Op}(\sigma)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\sigma(x,\xi)\widehat{u}(\xi)\,d\xi.$$

The aim of this section is to study $\mathrm{Op}(\sigma)$ when $\sigma \in \Gamma^m_s$. In order to do so, we study successively $\mathrm{Op}(\sigma_{lf})$, $\mathrm{Op}(\sigma_I)$, $\mathrm{Op}(\sigma_{II})$ and $\mathrm{Op}(\sigma_R)$, where σ_{lf}, σ_I, σ_{II} and σ_R are the four components of the decomposition (2.8).
The operator $\mathrm{Op}(\sigma_{lf})$ is handled as follows:

Proposition 16. *Let $m \in \mathbb{R}$ and $s_0 > d/2$, and let $\sigma \in \Gamma^m_{s_0}$.*

(i) *The operator* $\mathrm{Op}(\sigma_{lf})$ *extends as an operator mapping any Sobolev space into* $H^s(\mathbb{R}^d)$, *for all* $s \leqslant s_0$. *Moreover,*

$$\forall t \in \mathbb{R}, \quad \forall s \leqslant s_0, \quad \forall u \in H^t(\mathbb{R}^d), \quad \big|\mathrm{Op}(\sigma_{lf})u\big|_{H^s} \lesssim n_{0,s}(\sigma)|u|_{H^t},$$

where $n_{0,s}(\sigma)$ *is defined in* (2.4).

(ii) *If* σ *is* $2[\frac{d}{2}]+2$*-regular at the origin, the following estimates also hold, for all* $s \leqslant s_0$,

$$|\mathrm{Op}(\sigma_{lf})u|_{H^s} \lesssim n_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{C^0_*} \lesssim n_{2[\frac{d}{2}]+2,s}(\sigma)|u|_\infty.$$

Proof. By definition, one has

$$\begin{aligned}\mathrm{Op}(\sigma_{lf})u(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(\xi)\sigma(x,\xi)\widehat{u}(\xi)\,d\xi\\ &= (2\pi)^{-d}\int_{|\xi|\leqslant 1} e^{ix\cdot\xi}\psi(\xi)\sigma(x,\xi)\widehat{u}(\xi)\,d\xi.\end{aligned}$$

Since $|x \mapsto e^{ix\cdot\xi}\sigma(x,\xi)|_{H^s} \leqslant \mathrm{Cst}\ \langle\xi\rangle^{|s|}|\sigma(\cdot,\xi)|_{H^s}$, one has

$$|\mathrm{Op}(\sigma_{lf})u|_{H^s} \leqslant \mathrm{Cst}\ n_{0,s}(\sigma)\int_{|\xi|\leqslant 1}\langle\xi\rangle^{|s|}|\widehat{u}(\xi)|\,d\xi.$$

One can then obtain the first estimate by a simple Cauchy–Schwarz argument.

In order to prove the second estimate, remark that a simple expansion in Fourier series shows that

$$\psi(\xi)\sigma(x,\xi) = \mathbf{1}_{\{|\xi|\leqslant 1\}}(\xi)\sum_{k\in\mathbb{Z}^d}\frac{1}{(1+|k|^2)^{1+[\frac{d}{2}]}}c_k(x)e^{i\xi\cdot k},$$

where $\mathbf{1}_{\{|\xi|\leqslant 1\}}$ is the characteristic function of the ball $\{|\xi|\leqslant 1\}$ and

$$c_k(x) = (1+|k|^2)^{1+[d/2]}(2\pi)^{-d}\int_{[-\pi,\pi]^d} e^{-i\xi\cdot k}\psi(\xi)\sigma(x,\xi)\,d\xi.$$

Using methods similar to those used in the proof of Proposition 15, one obtains that $|c_k(\cdot)|_{H^s} \leqslant n_{2[\frac{d}{2}]+2,s}(\sigma)$. Since

$$\mathrm{Op}(c_k(x)e^{i\xi\cdot k}\mathbf{1}_{\{|\xi|\leqslant 1\}}(\xi))u = c_k(\cdot)(\mathbf{1}_{\{|\xi|\leqslant 1\}}(D)u)(\cdot + k),$$

the result follows from the next lemma:

Lemma 17. *Let $u, v \in \mathcal{S}(\mathbb{R}^d)$ and assume that $\widehat{v}$ is supported in the ball $\{|\xi| \leqslant A\}$, for some $A > 0$. Then for all $s \in \mathbb{R}$, one has*

$$|uv|_{H^s} \leqslant \mathrm{Cst}\ |u|_{H^s}|v|_\infty.$$

Proof. Write $uv = \sum_{q\geqslant -1} v\varphi_q(D)u$; except the first ones, each term of this sum has its spectrum included in an annulus of size $\sim 2^q$. Thanks to Lemma 5, the H^s-norm of the product uv can therefore be controlled in terms of $|v\varphi_q(D)u|_{L^2}$, $q \geqslant -1$. Since these quantities are easily bounded from above by $|v|_\infty|\varphi_q(D)u|_{L^2}$, the lemma follows from another application of Lemma 5. $\square$

We now turn to study $\mathrm{Op}(\sigma_I)$. As already said, σ_I is the paradifferential symbol associated to σ so that it is well-known that $\mathrm{Op}(\sigma_I)$ maps H^{s+m} into $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ (see [3,14]; [13, Proposition B.9]). However, since we need a precise estimate on the operator norm of $\mathrm{Op}(\sigma_I)$, we cannot omit the proof.

Proposition 18. *Let $m \in \mathbb{R}$ and $s_0 > d/2$, and let $\sigma \in \Gamma^m_{s_0}$.*

If σ_I is as defined in (2.10), *then* $\mathrm{Op}(\sigma_I)$ *extends as a continuous mapping on $H^{s+m}(\mathbb{R}^d)$ with values in $H^s(\mathbb{R}^d)$, for all $s \in \mathbb{R}$. Moreover,*

$$\forall s \in \mathbb{R}, \quad \forall u \in H^{s+m}(\mathbb{R}^d), \quad |\mathrm{Op}(\sigma_I)u|_{H^s} \lesssim M^m_d(\sigma)|u|_{H^{s+m}},$$

where $M^m_d(\sigma)$ is defined in (2.3).

Proof. Let us first prove the following lemma, which deals with the action of operators whose symbol satisfies the spectral property (2.5).

Lemma 19. *Let $m \in \mathbb{R}$ and $\sigma(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d\backslash\{0\})$ be such that $M^m_d(\sigma) < \infty$, where $M^m_d(\cdot)$ is as defined in* (2.3).

If moreover $\sigma(x,\xi)$ vanishes for $|\xi| \leqslant 1/2$ and satisfies the spectral condition (2.5) *then* $\mathrm{Op}\,(\sigma)$ *extends as a continuous mapping on $H^{s+m}(\mathbb{R}^d)$ with values in $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ and*

$$\forall u \in H^{s+m}(\mathbb{R}^d), \quad |\mathrm{Op}\,(\sigma)\,u|_{H^s} \lesssim \sup_{|\xi|\geqslant 1/2}\ \sup_{|\beta|\leqslant d} \Big(\langle\xi\rangle^{|\beta|-m}|\partial^\beta_\xi \sigma(\cdot,\xi)|_\infty\Big)|u|_{H^{s+m}}.$$

Proof. Using (1.7), we write $\sigma(x,\xi) = \sum_{p\geqslant -1} \sigma_p(x,\xi)$, with $\sigma_p(x,\xi) = \varphi_p(\xi)\sigma(x,\xi)$. For all $u \in \mathcal{S}(\mathbb{R}^d)$, (1.7) and (1.8) yield

$$\mathrm{Op}(\sigma)u = \sum_{p\geqslant -1} \mathrm{Op}(\sigma_p) \sum_{|p-q|\leqslant 1} \varphi_q(D)u. \tag{3.1}$$

Let us now define $\widetilde{\sigma_p}(x,\xi) := \sigma_p(2^{-p}x, 2^p\xi)$ for all $p \in \mathbb{N}$. One obviously has (see e.g. [6, Lemma II.1]) $\|\mathrm{Op}(\sigma_p)\|_{L^2\to L^2} = \|\mathrm{Op}(\widetilde{\sigma_p})\|_{L^2\to L^2}$. Moreover, Hwang proved in [9] that

$$\|\mathrm{Op}(\widetilde{\sigma_p})\|_{L^2\to L^2} \leqslant \mathrm{Cst} \sum_{\alpha,\beta\in\{0,1\}^d} \left|\partial_x^\alpha \partial_\xi^\beta \widetilde{\sigma_p}\right|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)};$$

it follows therefore from (2.7) that

$$\|\mathrm{Op}(\sigma_p)\|_{L^2\to L^2} \leqslant \mathrm{Cst}\ M_d^m(\sigma) 2^{pm}. \tag{3.2}$$

The result follows therefore from (3.1), (3.2) and Lemma 5 if we can prove that for all $p \in \mathbb{N}$, $\mathrm{Op}(\sigma_p)\sum_{|p-q|\leqslant 1} \varphi_q(D)u$ has its spectrum supported in an annulus $A2^{p-1} \leqslant |\eta| \leqslant B2^{p+1}$ for some $A, B > 0$. Since this is an easy consequence of the spectral property (2.5), the proof of the lemma is complete. □

The proof of the proposition is now very simple. One just has to apply Lemma 19 to σ_I, and to use Proposition 14.

The next proposition gives details on the action of $\mathrm{Op}(\sigma_{II})$.

Proposition 20. *Let $m \in \mathbb{R}$ and $s_0 > d/2$, and let $\sigma \in \Gamma_{s_0}^m$.*
If σ_{II} is defined as in (2.11) *then* $\mathrm{Op}(\sigma_{II})$ *extends as an operator on any Sobolev space and one has, for all $s \leqslant s_0$ and $t > 0$,*

$$\forall u \in C_*^{t+m}, \quad |\mathrm{Op}(\sigma_{II})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{C_*^{t+m}},$$

and

$$\forall u \in C_*^{-t+m}, \quad |\mathrm{Op}(\sigma_{II})u|_{H^{s-t}} \lesssim N^m_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{C_*^{-t+m}},$$

where $N^m_{2[\frac{d}{2}]+2,s}(\sigma)$ is defined in (2.2).

Remark 21. (i) One can replace the quantity $N^m_{2[\frac{d}{2}]+2,s}(\sigma)$ by $N^m_{2[\frac{d}{2}]+2,s-k}(\nabla_x^k\sigma)$, $k \in \mathbb{N}$, in the estimates of the proposition. This follows from the fact that $|f|_{H^s} \leqslant \mathrm{Cst}\ |\nabla^k f|_{H^{s-k}}$, $k \in \mathbb{N}$, whenever $\widehat{f}$ vanishes in a neighborhood of the origin, and from the observation that one can replace σ by $(1-\psi(D_x))\sigma$ in the definition of σ_{II}.

(ii) As said previously, when $\sigma(x,\xi) = \sigma(x)$ does not depend on ξ, one has $\mathrm{Op}(\sigma_{II})u = T_{(1-\psi(D))u}\sigma$ and thus $|\mathrm{Op}(\sigma_{II})u|_{H^s} \lesssim |u|_\infty |\sigma|_{H^s}$, that is, the endpoint case $t = 0$ holds in Proposition 20 if one weakens the $|u|_{C_*^0}$-control into a $|u|_\infty$-control. This is no longer true in general when dealing with general symbols.

Proof. Proposition 15 allows us to reduce the study to the case $m = 0$ and to the reduced symbols $p_k(x,\xi)$ given in that proposition.

By definition of σ_{II}, one has, for all $u \in \mathcal{S}(\mathbb{R}^d)$, $\sigma_{II}(x,D)u = \sum_{p\geqslant -1} v_p$, with $v_p = \sigma_{(p)}(x,D)(1-\psi(D))\psi(2^{-p+N}D)u$ and $\sigma_{(p)}(\cdot,\xi) = \varphi_p(D_x)\sigma(\cdot,\xi)$. Since the spectrum of v_p is supported in $(1-2^{1-N})2^{p-1} \leqslant |\xi| \leqslant (1+2^{-1-N})2^{p+1}$, Lemma 5 reduces the control of $|\sigma_{II}(x,D)u|_{H^s}$ to finding an estimate on each $|v_p|_2$, and hence on

$$I = \left| \sum_{q\geqslant -1} \left[\varphi_p(D_x)c_{k,q}\right]\lambda_k(2^{-q}D)\psi(2^{-p+N}D)(1-\psi(D))u \right|_2$$
$$\leqslant \sum_{q\geqslant -1} \left|\varphi_p(D_x)c_{k,q}\right|_2 \left|\lambda_k(2^{-q}D)\psi(2^{-p+N}D)u\right|_\infty .$$

Remarking that $\lambda_k(2^{-q}\xi)\psi(2^{-p+N}\xi) = 0$ when $q \geqslant p-N+2$, and using Proposition 15, one deduces

$$I \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)}) \sum_{q=-1}^{p-N+1} \left|\lambda_k(2^{-q}D)\psi(2^{-p+N}D)u\right|_\infty , \tag{3.3}$$

where we recall that $\sigma_{(p)}(\cdot,\xi) = \varphi_p(D_x)\sigma(\cdot,\xi)$.

We now need the following lemma:

Lemma 22. *Let $A, B > 0$ and $\lambda \in C_0^\infty(\mathbb{R}^d)$ supported in $A \leqslant |\xi| \leqslant B$. Then, for all $t \in \mathbb{R}$ and $q \geqslant -1$, one has,*

$$\forall u \in C^t_*, \quad \left|\lambda(2^{-q}D)u\right|_\infty \leqslant C_t|\check{\lambda}|_{L^1}2^{-qt}|u|_{C^t_*}.$$

Proof. Since λ is supported in $A \leqslant |\xi| \leqslant B$, there exists $n_0 \in \mathbb{N}$ such that for all $\xi \in \mathbb{R}^d$ and $q \geqslant -1$, one has $\lambda(2^{-q}\xi) = \lambda(2^{-q}\xi)\sum_{|r-q|\leqslant n_0} \varphi_r(\xi)$. Therefore, one can write

$$|\lambda(2^{-q}D)u|_\infty = \left|\lambda(2^{-q}D)\sum_{|r-q|\leqslant n_0} \varphi_r(D)u\right|_\infty$$
$$\leqslant |\check{\lambda}|_{L^1} \sum_{|r-q|\leqslant n_0} |\varphi_r(D)u|_\infty,$$

so that the lemma follows from the very definition of Zygmund spaces. □

In order to prove the first part of Proposition 20, take any $t > 0$ and use the lemma to remark that (3.3) yields

$$I \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)}) \sum_{q\geqslant -1} 2^{-qt} \left|\psi(2^{-p+N}D)u\right|_{C^t_*}$$
$$\leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)})\, |u|_{C^t_*}, \tag{3.4}$$

since $\sum_{q\geqslant -1} 2^{-qt} < \infty$. From Lemma 5 and the definition of $N^m_{2[\frac{d}{2}]+2,0}(\cdot)$, it is obvious that

$$\sum_{p\geqslant -1} 2^{2ps} N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)})^2 \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,s}(\sigma)^2, \tag{3.5}$$

so that (3.4) and Lemma 5 give the result.

To prove the second part of the proposition, proceed as above to obtain

$$I \leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)}) \left(\sum_{q=-1}^{p-N+1} 2^{qt}\right) |u|_{C^{-t}_*},$$
$$\leqslant \mathrm{Cst}\ N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)}) 2^{pt}\, |u|_{C^{-t}_*};$$

the end of the proof is done as for the first part of the proposition. □

We finally turn to study $\mathrm{Op}(\sigma_R)$:

Proposition 23. *Let $m \in \mathbb{R}$ and $s_0 > d/2$, and let $\sigma \in \Gamma^m_{s_0}$.*

If σ_R is as given in (2.12) and if $s + t > 0$ and $s \leqslant s_0$ then $\mathrm{Op}(\sigma_R)$ extends as a continuous operator on $H^{m+t}(\mathbb{R}^d)$ with values in $H^{s+t-\frac{d}{2}}(\mathbb{R}^d)$. Moreover,

$$\forall u \in H^{m+t}(\mathbb{R}^d), \quad |\mathrm{Op}(\sigma_R)u|_{H^{s+t-d/2}} \lesssim N^m_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{H^{m+t}}$$

and

$$\forall u \in C^{m+t}_*(\mathbb{R}^d), \quad |\mathrm{Op}(\sigma_R)u|_{H^{s+t}} \lesssim N^m_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{C^{m+t}_*}.$$

Remark 24. For the same reasons as in Remark 21, one can replace the quantity $N^m_{2[\frac{d}{2}]+2,s}(\sigma)$ by $N^m_{2[\frac{d}{2}]+2,s-k}(\nabla^k_x\sigma)$, $k \in \mathbb{N}$, in the estimates of the proposition, provided that one replaces σ_R by $\sigma_{R,1}$, where $\sigma_{R,1}$ is defined in (2.13).

Proof. We only prove the first of the two estimates given in the proposition. The second one is both easier and contained in Theorem B of [21]. The proof we present below is an adaptation of the corresponding result which gives control of the residual term in paraproduct theory (e.g. [5, Theorem 2.4.1]).

Using the expression of σ_R given in (2.12) and a Littlewood–Paley decomposition, one can write

$$\mathrm{Op}(\sigma_R)u = \sum_{r\geqslant -1} \varphi_r(D)\mathrm{Op}(\sigma_R)u = \sum_{r\geqslant -1} \varphi_r(D) \sum_{p\geqslant -1} R_p(\sigma)u,$$

where $R_p(\sigma)u := \sum_{|p-q|\leqslant N} \sigma_{(q)}(x, D)(1-\psi(D))\varphi_p(D)u$, *and with* $\sigma_{(q)}(\cdot,\xi) = \varphi_q(D)$ $\sigma(\cdot,\xi)$.

Since Sp $R_p(\sigma)u$ is included in $|\xi|\leqslant(1+2^N)2^{p+1}$, there exists $n_0\in\mathbb{N}$ such that $\varphi_r(D)R_p(\sigma)u = 0$ whenever $r > p+n_0$. Thus, one has in fact

$$\mathrm{Op}(\sigma_R)u = \sum_{r\geqslant -1} \varphi_r(D) \sum_{p\geqslant r-n_0} R_p(\sigma)u,$$

and the proposition follows from Lemma 5 and the estimate

$$\left(\sum_{r\geqslant -1} 2^{2r(s+t-d/2)}\left|\varphi_r(D) \sum_{p\geqslant r-n_0} R_p(\sigma)u\right|_2^2\right)^{1/2} \lesssim N^m_{2[\frac{d}{2}]+2,s}(\sigma)|u|_{m+t}. \tag{3.6}$$

The end of the proof is thus devoted to establishing (3.6).

Using Proposition 15—and with the same notations—one can see that it suffices to prove (3.6) with $R_p(\sigma)$ replaced by $R_p(p_k\langle\xi\rangle^m)$, provided that the estimate is uniform in $k\in\mathbb{Z}^d$. Without loss of generality, we can also assume that $m=0$.

Now, remark that

$$2^{r(s+t-d/2)}\left|\varphi_r(D) \sum_{p\geqslant r-n_0} R_p(p_k)u\right|_2 \leqslant \mathrm{Cst}\ 2^{r(s+t)}\left|\sum_{p\geqslant r-n_0} R_p(p_k)u\right|_{L^1} \tag{3.7}$$

and that

$$\left|R_p(p_k)u\right|_{L^1} = \left|\sum_{|p-q|\leqslant N} \varphi_q(D_x)p_k(x,D)(1-\psi(D))\varphi_p(D)u\right|_{L^1}. \tag{3.8}$$

Now, using the expression of $p_k(x,\xi)$ given in Proposition 15, one can write

$$\begin{aligned}&\varphi_q(D_x)p_k(x,D)(1-\psi(D))\varphi_p(D)u\\ &\quad = \sum_{l\geqslant -1} \varphi_q(D_x)c_{k,l}(x)\lambda_k(2^{-l}D)(1-\psi(D))\varphi_p(D)u,\end{aligned}$$

and since $\lambda_k(2^{-l}\xi)\varphi_p(\xi)=0$ if $|p-l|>n_1$, for some $n_1\in\mathbb{N}$, one deduces that the summation in the r.h.s. of the above inequality is over a finite number of integers l; therefore, by Cauchy–Schwartz's inequality and Proposition 15,

$$\left|\varphi_q(D_x)p_k(x,D)(1-\psi(D))\varphi_p(D)u\right|_{L^1}\lesssim N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(q)})\left|\varphi_p(D)u\right|_{L^2}. \tag{3.9}$$

From (3.8) and (3.9) one obtains

$$\left|R_p(p_k)u\right|_{L^1}\lesssim 2^{-p(s+t)}\sum_{|p-q|\leqslant N}2^{qs}N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(q)})2^{pt}\left|\varphi_p(D)u\right|_2,$$

and the l.h.s. of (3.7) is therefore bounded from above by

$$\sum_{p\geqslant r-n_0}2^{(r-p)(s+t)}\sum_{|p-q|\leqslant N}2^{qs}N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(q)})2^{pt}\left|\varphi_p(D)u\right|_2.$$

Since $s+t>0$, Hölder's inequality yields that the l.h.s. of (3.6) is bounded from above by

$$\left|\left(2^{ps}N^m_{2[\frac{d}{2}]+2,0}(\sigma_{(p)})2^{pt}\left|\varphi_p(D)u\right|_2\right)_{p\geqslant -1}\right|_{l^1}.$$

By Cauchy–Schwartz's inequality, Lemma 5 and an argument similar to the one used in (3.5), one obtains (3.6), which concludes the proof. □

A first important consequence of Propositions 20 and 23 is that one can control the action of the operator associated to the 'remainder' symbol $\sigma-\sigma_{lf}-\sigma_I$, which is more regular than the full operator if $\sigma(x,\xi)$ is smooth enough in the space variables.

Proposition 25. *Let $m\in\mathbb{R}$, $s_0>d/2$ and $d/2<t_0\leqslant s_0$. If for some $r\geqslant 0$, one has $\sigma\in\Gamma^m_{s_0+r}$ then,*

(i) *For all $-t_0<s\leqslant s_0$, the following estimate holds:*

$$\forall u\in H^{m+t_0-r}(\mathbb{R}^d),\quad |\mathrm{Op}(\sigma-\sigma_I-\sigma_{lf})u|_{H^s}\lesssim N^m_{2[\frac{d}{2}]+2,s+r}(\sigma)|u|_{H^{m+t_0-r}}.$$

(ii) *For all $r'\in\mathbb{R}$ (such that $t_0+r'\leqslant s_0+r$) and $-t_0<s\leqslant t_0+r'$, one has*

$$\forall u\in H^{s+m-r'}(\mathbb{R}^d),\quad |\mathrm{Op}(\sigma-\sigma_I-\sigma_{lf})u|_{H^s}\lesssim N^m_{2[\frac{d}{2}]+2,t_0+r'}(\sigma)|u|_{H^{s+m-r'}}.$$

(iii) *For symbols of nonnegative order, i.e. when* $m > 0$, *then for all* $s > 0$ *such that* $s + m \leqslant s_0$, *one also has*

$$\forall u \in C_*^{-r}(\mathbb{R}^d), \quad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, s+m+r}(\sigma)|u|_{C_*^{-r}};$$

this estimate still holds for slightly negative values of r, *namely, if* $-m < r$.

Proof. One has $\sigma - \sigma_I - \sigma_{lf} = \sigma_{II} + \sigma_R$, and we are therefore led to control $|\mathrm{Op}(\sigma_{II})u|_{H^s}$ and $|\mathrm{Op}(\sigma_R)u|_{H^s}$. We first prove point (i).

The estimate on $|\mathrm{Op}(\sigma_{II})u|_{H^s}$ is given by the first part of Proposition 20 when $r = 0$. When $r > 0$, taking $s = s + r$ and $t = r$ in the second part of this proposition gives the result. The estimate on $|\mathrm{Op}(\sigma_R)u|_{H^s}$ is given by taking $s = s + r$ and $t = t_0 - r$ in the first part of Proposition 23.

To establish (ii), take $s = t_0 + r'$ and $t = t_0 - s + r'$ in the second estimate of Proposition 20 to obtain that $|\mathrm{Op}(\sigma_{II})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, t_0+r'}(\sigma)|u|_{H^{s+m-r'}}$ for all $s < t_0 + r'$. Taking $t = s - r'$ and $s = t_0 + r'$ in the first estimate of Proposition 23 shows that $|\mathrm{Op}(\sigma_R)u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, t_0+r'}(\sigma)|u|_{H^{s+m-r'}}$ for all $s > -t_0$ and the proof of (ii) is complete (the endpoint $s = t_0 + r'$ being given by (i)).

To prove (iii), take $s = s+m+r$ and $t = m+r > 0$ in the second part of Proposition 20 and $s = s + m + r$ and $t = -m - r$ in the second estimate of Proposition 23. □

The first two points of the following proposition are a close variant of Proposition 25 which uses the decomposition (2.13) of the component σ_R, while the last point addresses the case when $\sigma(x, \xi) = \sigma(x)$ does not depend on ξ.

Proposition 26. *Let* $m \in \mathbb{R}$, $k \in \mathbb{N}$, $s_0 > d/2$ *and* $d/2 < t_0 \leqslant s_0$. *If for some* $r \geqslant 0$, *one has* $\sigma \in \Gamma^m_{s_0+r}$ *then,*

(i) *For all* $-t_0 < s \leqslant s_0$, *the following estimate holds*:

$$\forall u \in H^{m+t_0-r}, \quad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, s+r-k}(\nabla_x^k \sigma)|u|_{H^{m+t_0-r}}.$$

(ii) *For all* $r' \in \mathbb{R}$ (*such that* $t_0 + r' \leqslant s_0 + r$) *and* $-t_0 < s < t_0 + r'$, *one has*

$$\forall u \in H^{s+m-r'}, \quad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, t_0+r'-k}(\nabla_x^k \sigma)|u|_{H^{s+m-r'}}.$$

(iii) *For symbols of nonnegative order, i.e. when* $m > 0$, *then for all* $s > 0$ *such that* $s + m \leqslant s_0$, *one also has*

$$\forall u \in C_*^{-r}(\mathbb{R}^d), \qquad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})u|_{H^s} \lesssim N^m_{2[\frac{d}{2}]+2, s+m+r-k}(\nabla_x^k \sigma)|u|_{C_*^{-r}}.$$

(iv) *When σ is a function, $\sigma \in H^{s_0}(\mathbb{R}^d)$, one has*

$$\forall \quad 0 < s \leqslant s_0, \quad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})u|_{H^s} \lesssim |\nabla^k \sigma|_{H^{s-k}} |u|_\infty$$

and, if $\sigma \in H^{s_0+r}(\mathbb{R}^d)$, with $r > 0$,

$$\forall \quad 0 < s \leqslant s_0, \quad |\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})u|_{H^s} \lesssim |\nabla^k \sigma|_{H^{s+r-k}} |u|_{C_*^{-r}};$$

when $s = 0$, the above two estimates still hold if one adds $|\nabla^{n+1}\sigma|_{L^\infty} |u|_{H^{-n-1}}$ to the right-hand-side, for any $n \in \mathbb{N}$.

Remark 27. When $k = 0$, the estimates of (iv) still hold if one replaces $\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2})$ by $\mathrm{Op}(\sigma - \sigma_I - \sigma_{lf})$ (and $|\nabla^{n+1}\sigma|_\infty$ by $|\sigma|_{W^{n+1,\infty}}$ in the additional term when $s = 0$). This is a consequence of the definition of $\sigma_{R,2}$ and of Lemma 17.

Proof. One has $\sigma - \sigma_I - \sigma_{lf} - \sigma_{R,2} = \sigma_{II} + \sigma_{R,1}$, so that the first three points of the proposition are proved as in Proposition 25, using Remarks 21 and 24.

We now prove the fourth point of the proposition. Since σ is a function, we can write, as in Remark 13, $\mathrm{Op}(\sigma_{II} + \sigma_{R,1})u = T_{\widetilde{u}}\sigma + R(\widetilde{\sigma}, \widetilde{u})$, with $\widetilde{u} := (1 - \psi(D))u$ and $\widetilde{\sigma} := (1 - \psi(D))\sigma$. The estimate for $s > 0$ thus follows from the classical properties (e.g. [5, Theorem 2.4.1], and [17, Proposition 3.5.D] for the last one):

- for all $s \in \mathbb{R}$, $|T_f g|_{H^s} \lesssim |f|_\infty |g|_{H^s}$;
- for all $s \in \mathbb{R}$ and $r > 0$, $|T_f g|_{H^s} \lesssim |f|_{C_*^{-r}} |g|_{H^{s+r}}$;
- for all $s > 0$, $r \in \mathbb{R}$, $|R(f,g)|_{H^s} \lesssim |f|_{H^{s+r}} |g|_{C_*^{-r}}$;
- for all $n \in \mathbb{N}$, $|R(f,g)|_{L^2} \lesssim |f|_{W^{n+1,\infty}} |g|_{H^{-n-1}}$;

(we also use the fact that $|f|_{H^s} \leqslant \mathrm{Cst}\ |\nabla^k f|_{H^{s-k}}$ and $|f|_{W^{n,\infty}} \leqslant \mathrm{Cst}\ |\nabla^n f|_{L^\infty}$ for all f such that $\widehat{f}$ vanishes in a neighborhood of the origin). $\square$

Gathering the results of the previous propositions, one obtains the following theorem, which describes the action of the full operator $\mathrm{Op}(\sigma)$, which is of course of order m.

Theorem 1. *Let $m \in \mathbb{R}$, $d/2 < t_0 \leqslant s_0$ and $\sigma \in \Gamma^m_{s_0}$. Then for all $u \in \mathcal{S}(\mathbb{R}^d)$, the following estimates hold:*

$$\forall -t_0 < s < t_0, \quad |\mathrm{Op}(\sigma)u|_{H^s} \lesssim \Big(n_{0,t_0}(\sigma) + N^m_{2[\frac{d}{2}]+2,t_0}(\sigma)\Big)|u|_{H^{s+m}},$$

and

$$\forall t_0 \leqslant s \leqslant s_0, \quad |\mathrm{Op}(\sigma)u|_{H^s} \lesssim \big(n_{0,s}(\sigma) + N^m_{2[\frac{d}{2}]+2,s}(\sigma)\big)|u|_{H^{m+t_0}} + M^m_d(\sigma)|u|_{H^{s+m}}.$$

Proof. Recall that $\sigma = \sigma_{lf} + \sigma_I + (\sigma - \sigma_{lf} - \sigma_I)$; we use the first two estimates of Proposition 25 (with $r = 0$ and $r' = 0$) to control $\sigma - \sigma_{lf} - \sigma_I$ while $|\mathrm{Op}(\sigma_{lf})u|_{H^s}$ and

$|\mathrm{Op}(\sigma_I)u|_{H^s}$ are easily controlled using Propositions 16 and 18 and the observation that by a classical Sobolev embedding, $M_d^m(\sigma) \lesssim N^m_{2[\frac{d}{2}]+2,t_0}(\sigma)$. □

Remark 28. (i) If $\sigma(x,\xi) = \sigma(x) \in H^{s_0}(\mathbb{R}^d)$, then the results on the microlocal regularity of products (e.g. [8, p.240]) say that if $u \in H^s(\mathbb{R}^d)$, then $\sigma u \in H^s(\mathbb{R}^d)$ if $s + s_0 > 0$, $s \leqslant s_0$ and $s_0 > d/2$. This result can be deduced from Theorem 1 (note that the limiting case $s + s_0 = 0$ is also true, but the proof requires different tools [8, Theorem 8.3.1]).

(ii) We refer to Proposition 8.1 of [19] for another kind of estimate on the action of pseudo-differential operators; see also estimate (25) of [12].

One of the interests of Theorem 1 is that it gives control of $\mathrm{Op}(\sigma)u$ in Sobolev spaces of negative order. The price to pay is that for nonnegative values of the Sobolev index s, and when $\sigma(\cdot,\xi) = \sigma(\cdot) \in H^{s_0}(\mathbb{R}^d)$ does not depend on ξ, we do not recover the classical tame estimate $|\sigma u|_{H^s} \lesssim (|u|_{H^s}|\sigma|_\infty + |\sigma|_{H^s}|u|_\infty)$ but a weaker one, namely $|\sigma u|_{H^s} \lesssim |u|_{H^s}|\sigma|_\infty + |\sigma|_{H^s}|u|_{H^{\frac{d}{2}+\varepsilon}}$, for all $\varepsilon > 0$. The difference is slight because the embedding $H^{\frac{d}{2}+\varepsilon}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ is critical, but can be cumbersome. The next theorem can therefore be a useful alternative to Theorem 1.

Theorem 2. *Let $m \in \mathbb{R}$, $d/2 < t_0 \leqslant s_0$ and $\sigma \in \Gamma^m_{s_0+m}$. Assume that $m > 0$ and σ is $2[\frac{d}{2}]+2$-regular at the origin. Then for all $u \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s \leqslant s_0$, one has*

$$|\mathrm{Op}(\sigma)u|_{H^s} \lesssim \Big(n_{2[\frac{d}{2}]+2,s}(\sigma) + N^m_{2[\frac{d}{2}]+2,s+m}(\sigma)\Big)|u|_{C^0_*} + M^m_d(\sigma)|u|_{H^{s+m}}.$$

Proof. We just have to control the H^s-norm of the four components of $\mathrm{Op}(\sigma)u$ by the r.h.s. of the estimate given in the theorem.

For $\mathrm{Op}(\sigma_{lf})u$ and $\mathrm{Op}(\sigma_I)u$, this is a simple consequence of the second part of Propositions 16 and 18 respectively. The other components are controlled with the help of Proposition 25(iii). □

Remark 29. Using the fact that slightly negative values of r are allowed in Proposition 25(iii), one can check that the estimate given by Theorem 2 can be extended to $s = 0$ provided that the quantity $|u|_{C^0_*}$ which appears in the r.h.s. of the estimate is replaced by $|u|_{C^\varepsilon_*}$, for any $\varepsilon > 0$.

The following corollary deals with the case when the symbol σ is of the form $\sigma(x,\xi) = \Sigma(v(x),\xi)$.

Corollary 30. *Let $m \in \mathbb{R}$, $p \in \mathbb{N}$ and $s_0 \geqslant t_0 > d/2$. Consider $v \in H^{s_0}(\mathbb{R}^d)^p$ and assume that $\sigma(x,\xi) = \Sigma(v(x),\xi)$, with $\Sigma \in C^\infty(\mathbb{R}^p, \mathcal{M}^m)$. Then*:

(i) $\quad \forall -t_0 < s < t_0, \quad |\sigma(x,D)u|_{H^s} \lesssim C_\Sigma(|v|_\infty)|v|_{H^{t_0}}|u|_{H^{s+m}}$

and

$$\forall t_0 \leqslant s \leqslant s_0, \qquad |\sigma(x,D)u|_{H^s} \lesssim C_\Sigma(|v|_\infty)\left(|v|_{H^s}|u|_{H^{m+t_0}} + |u|_{H^{s+m}}\right).$$

(ii) *If moreover* $m > 0$, $v \in H^{s_0+m}(\mathbb{R}^d)$, *and* Σ *is* $2[\frac{d}{2}]+2$*-regular at the origin, then, for* $0 < s \leqslant s_0$,

$$|\sigma(x,D)u|_{H^s} \lesssim C_\Sigma(|v|_\infty)\big(|v|_{H^{s+m}}|u|_{C^0_*} + |u|_{H^{s+m}}\big).$$

In the above, $C_\Sigma(\cdot)$ *denotes a smooth nondecreasing function depending only on a finite number of derivatives of* Σ.

Proof. We write $\sigma(x,\xi) = [\sigma(x,\xi) - \Sigma(0,\xi)] + \Sigma(0,\xi)$. Owing to Lemma 8, the first component of this decomposition is in $\Gamma^m_{s_0}$ and we can use Theorem 1 to study the associated pseudo-differential operator. The estimates of the theorem transform into the estimates stated in the corollary thanks to Lemma 8.

Since the action of the Fourier multiplier $\Sigma(0,D)$ satisfies obviously these estimates, the first point of the corollary is proved. Using Theorem 2, one proves the second estimate in the same way. □

4. Composition and commutator estimates

The composition of two pseudo-differential operators is well-known for classical symbols, and one has $\mathrm{Op}(\sigma^1)\circ\mathrm{Op}(\sigma^2) \sim \mathrm{Op}(\sigma^1 \sharp \sigma^2)$, where the symbol $\sigma^1\sharp\sigma^2$ is given by an infinite expansion of σ^1 and σ^2. When dealing with symbols of limited regularity, one has to stop this expansion. Therefore, for all $n \in \mathbb{N}$, we define $\sigma^1\sharp_n\sigma^2$ as

$$\sigma^1\sharp_n\sigma^2(x,\xi) := \sum_{|\alpha|\leqslant n} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\sigma^1(x,\xi)\partial_x^\alpha\sigma^2(x,\xi). \tag{4.1}$$

Similarly, we introduce the Poisson brackets:

$$\{\sigma^1,\sigma^2\}_n(x,\xi) := \sigma^1\sharp_n\sigma^2(x,\xi) - \sigma^2\sharp_n\sigma^1(x,\xi). \tag{4.2}$$

In this section, we describe the composition or commutator of pseudo-differential operators of limited regularity with Fourier multipliers, or with another pseudo-differential operator. A key point in this analysis is the following proposition; the first two points are precise estimates for Meyer's classical result on the composition of paradifferential operators (e.g. [15, Theorem XVI.4]).

Proposition 31. *Let* $m_1, m_2 \in \mathbb{R}$, $s_0 > d/2$, $n \in \mathbb{N}$ *and* $\sigma^2 \in \Gamma^{m_2}_{s_0+n+1}$. *Then*

(i) *If* $\sigma^1(x,\xi) = \sigma^1(\xi) \in \mathcal{M}^{m_1}$, *there exists a symbol* $\rho_n(x,\xi)$ *such that*

$$\sigma^1(D)\circ\mathrm{Op}(\sigma^2_I) = \mathrm{Op}(\sigma^1\sharp_n\sigma^2_I) + \mathrm{Op}(\rho_n);$$

moreover $\rho_n(x,\xi)$ *vanishes for* $|\xi|\leqslant 1/2$ *and satisfies the spectral condition* (2.5) *and the estimate*

$$\sup_{|\xi|\geqslant 1/2}\ \sup_{|\beta|\leqslant d}\Big(\langle\xi\rangle^{|\beta|+n+1-m_1-m_2}|\partial_\xi^\beta\rho_n(\cdot,\xi)|_\infty\Big)\lesssim M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)M_d^{m_2}(\nabla_x^{n+1}\sigma^2).$$

(ii) *If* $\sigma^1\in\Gamma^{m_1}_{s_0}$ *then there exists a symbol* $\rho_n(x,\xi)$ *such that*

$$\mathrm{Op}(\sigma^1_I)\circ\mathrm{Op}(\sigma^2_I)=\mathrm{Op}(\sigma^1_I\sharp_n\sigma^2_I)+\mathrm{Op}(\rho_n),$$

and which satisfies the same properties as in case (i).

(iii) *If* σ^1 *is a function,* $\sigma^1\in C^r_*$ *for some* $r\geqslant 0$*, then the symbol* $\rho_n(x,\xi)$ *defined in* (ii) *is of order* $m_2-n-1-r$ *and*

$$M_d^{m_2-n-1-r}(\rho_n)\lesssim|\sigma^1|_{C^r_*}M_d^{m_2}(\nabla_x^{n+1}\sigma^2).$$

Remark 32. For the sake of simplicity, we stated the above proposition for paradifferential symbols σ^1_I (and σ^2_I in (ii) and (iii)) associated to symbols σ^1 and σ^2; the proof below shows that the only specific properties of σ^1_I and σ^2_I actually used are the spectral property (2.5) and the cancellation for frequencies $|\xi|\leqslant 1/2$. Thus, one can extend the result to all symbols satisfying these conditions.

Proof. We omit the proof of the first point of the proposition, which can be deduced from the proof below with only minor changes. The method we propose here is inspired by the proof of ([13, Theorem B.2.16]) rather than Meyer's classical one ([15, Theorem XVI.4]) which would lead to less precise estimates here.

First remark that since σ^1_I satisfies the spectral condition (2.5) and vanishes for frequencies $|\xi|\leqslant 1/2$, there exists an admissible cut-off function χ (in the sense of Definition 3) such that $\widehat{\sigma^2_I}(\eta,\xi)\chi(\eta,\xi)=\widehat{\sigma^2_I}(\eta,\xi)$; it is then both classical and easy to see that

$$\rho_n(x,\xi)=\sum_{|\gamma|=n+1}\int_{\mathbb{R}^d}G_\gamma(x,x-y,\xi)(\partial_x^\gamma\sigma^2_I)(y,\xi)\,dy,$$

with

$$G_\gamma(x,y,\xi):=(-i)^{|\gamma|}(2\pi)^{-d}\int_{\mathbb{R}^d}e^{iy\cdot\eta}\sigma_I^{1,\gamma}(x,\eta,\xi)\chi(\eta,\xi)\,d\eta$$

and $\sigma_I^{1,\gamma}(x,\eta,\xi):=\displaystyle\int_0^1\frac{(1-t)^n}{n!}\partial_\xi^\gamma\sigma^1_I(x,\xi+s\eta)\,ds$. Therefore, for all $0\leqslant|\beta|\leqslant d$,

$$\begin{aligned}|\partial_\xi^\beta\rho_n(x,\xi)|&\leqslant \mathrm{Cst}\sum_{\beta'+\beta''=\beta}\Big|\partial_\xi^{\beta'}G_\gamma(x,\cdot,\xi)\Big|_{L^1}\Big|\partial_x^\gamma\partial_\xi^{\beta''}\sigma^2_I(\cdot,\xi)\Big|_\infty\\&\leqslant \mathrm{Cst}\sum_{\beta'+\beta''=\beta}\Big|\partial_\xi^{\beta'}G_\gamma(x,\cdot,\xi)\Big|_{L^1}M_d^{m_2}(\nabla_x^{n+1}\sigma^2)\langle\xi\rangle^{m_2-|\beta''|},\end{aligned}\tag{4.3}$$

where we used Proposition 14 to obtain the last equality.

The proposition follows therefore from (4.3) and the estimate, for all $|\beta'| \leqslant d$,

$$\left|\partial_\xi^{\beta'} G_\gamma(x,\cdot,\xi)\right|_{L^1} \leqslant \mathrm{Cst}\ M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)\langle\xi\rangle^{m_1-|\beta'|-n-1}. \tag{4.4}$$

and, when σ^1 is a function (case (iii) of the lemma),

$$\left|\partial_\xi^{\beta'} G_\gamma(x,\cdot,\xi)\right|_{L^1} \leqslant \mathrm{Cst}\ |\sigma^1|_{C^r_*}\langle\xi\rangle^{-|\beta'|-n-1-r}. \tag{4.5}$$

Both (4.4) and (4.5) follow from the next two lemmas.

Lemma 33. *For all $\alpha,\beta\in\mathbb{N}^d$ such that $|\alpha|\leqslant[d/2]+1$ and $|\beta|\leqslant d$, one has*

$$|\partial_\eta^\alpha\partial_\xi^\beta(\sigma_I^{1,\gamma}(x,\cdot,\cdot)\chi)(\eta,\xi)|\leqslant \mathrm{Cst}\ M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)\langle\xi\rangle^{m_1-|\alpha|-|\beta|-n-1}.$$

If moreover the symbol is a function, $\sigma^1\in C^r_*$ for some $r\in\mathbb{R}$, then

$$|\partial_\eta^\alpha\partial_\xi^\beta(\sigma_I^{1,\gamma}(x,\cdot,\cdot)\chi)(\eta,\xi)|\leqslant \mathrm{Cst}\ |\sigma^1|_{C^r_*}\langle\xi\rangle^{-|\alpha|-|\beta|-n-1-r}.$$

Proof. It suffices to prove estimate of the lemma for $\partial_\eta^{\alpha'}\partial_\xi^{\beta'}\sigma_I^{1,\gamma}\partial_\eta^{\alpha''}\partial_\xi^{\beta''}\chi$ for all $\alpha'+\alpha''=\alpha$ and $\beta'+\beta''=\beta$. By definition of $\sigma_I^{1,\gamma}$, one has

$$\partial_\eta^{\alpha'}\partial_\xi^{\beta'}\sigma_I^{1,\gamma}\partial_\eta^{\alpha''}\partial_\xi^{\beta''}\chi(\eta,\xi)=\int_0^1\frac{(1-t)^n}{n!}\partial_\xi^{\alpha'+\beta'+\gamma}\sigma_I^1(x,\xi+s\eta)s^{|\alpha'|}ds\,\partial_\eta^{\alpha''}\partial_\xi^{\beta''}\chi(\eta,\xi).$$

Since on the support of $\partial_\eta^{\alpha''}\partial_\xi^{\beta''}\chi$ one has $\langle\xi+s\eta\rangle\sim\langle\xi\rangle$, the first estimate of the lemma follows from the definition of the seminorms $M_k(\cdot)$ and (1.9).

When σ^1 is a function, and since $|\alpha'|+|\beta'|+|\gamma|\geqslant 1$, we can use Proposition A.5 of [1] which asserts that $|\partial_\xi^{\alpha'+\beta'+\gamma}\sigma_I^1(x,\xi)|\leqslant \mathrm{Cst}\ |\sigma^1|_{C^r_*}\langle\xi\rangle^{-r-|\alpha'|-|\beta'|-|\gamma|}$, from which one easily obtains the second estimate of the lemma. □

Lemma 34. *Let $F(\cdot,\cdot)$ be a function defined on $\mathbb{R}^d_\eta\times\mathbb{R}^d_\xi$ and such that*

- *There exists $0<\delta<1$ such that $F(\eta,\xi)=0$ for all $|\eta|\geqslant\delta|\xi|$;*
- *For all $\alpha\in\mathbb{N}^d$, $|\alpha|\leqslant[d/2]+1$, there exists a constant C_α such that*

$$\forall\eta,\xi\in\mathbb{R}^d,\quad \left|\partial_\eta^\alpha F(\eta,\xi)\right|\leqslant C_\alpha\langle\xi\rangle^{\mu-|\alpha|}.$$

Then, one has

$$\forall\xi\in\mathbb{R}^d,\quad \left|\check{F}(\cdot,\xi)\right|_{L^1}\leqslant \mathrm{Cst}\left(\sup_{|\alpha|\leqslant[d/2]+1}C_\alpha\right)\langle\xi\rangle^\mu.$$

Proof. This result can be proved with the techniques used to prove estimate (2.21) of Appendix B in [13]. Briefly, and for the sake of completeness, we sketch the proof. Define $F^\flat(\eta, \xi) := F(\langle\xi\rangle\eta, \xi)$ and remark that $|\check{F}(\cdot, \xi)|_{L^1} = |\check{F^\flat}(\cdot, \xi)|_{L^1} \leqslant \text{Cst}\ |F^\flat(\cdot, \xi)|_{H^{[d/2]+1}}$. The first assumption made in the statement of the lemma shows that $F^\flat(\cdot, \xi)$ is supported in the ball $\{|\eta| \leqslant 1\}$ so that it is easy to conclude using the second assumption. □

4.1. Commutators with Fourier multipliers

We give in this section some commutator estimates between a Fourier multiplier and a pseudo-differential operator of limited regularity. We first set some notations:

Notations: For all $m \in \mathbb{R}$, $s_0 > d/2$, and all symbols $\sigma \in \Gamma^m_{s_0}$, we define

$$\forall s \leqslant s_0, \qquad \|\sigma\|_{H^s_{(m)}} := \frac{1}{2}\left(n_{0,s}(\sigma) + N^m_{2[\frac{d}{2}]+2,s}(\sigma)\right) \tag{4.6}$$

and, when σ is also $2[\frac{d}{2}]+2$-regular at the origin,

$$\forall s \leqslant s_0, \quad \|\sigma\|_{H^s_{\mathrm{reg},(m)}} := \frac{1}{2}\left(n_{2[\frac{d}{2}]+2,s}(\sigma) + N^m_{2[\frac{d}{2}]+2,s}(\sigma)\right). \tag{4.7}$$

Finally, if σ is d-regular at the origin, we set

$$\|\sigma\|_{\infty,(m)} := \frac{1}{2}\left(m_d(\sigma) + M^m_d(\sigma)\right). \tag{4.8}$$

When no confusion is possible, we omit the subscript m in the above definitions. Remark that when σ does not depend on ξ (i.e. when it is a function), then one has $\|\sigma\|_{H^s} = \|\sigma\|_{H^s_{\mathrm{reg}}} = |\sigma|_{H^s}$, and $\|\sigma\|_\infty = |\sigma|_\infty$.

The first commutator estimates we state are of Kato–Ponce type:

Theorem 3. *Let $m_1, m_2 \in \mathbb{R}$, $n \in \mathbb{N}$ and $d/2 < t_0 \leqslant s_0$. Let $\sigma^1(\xi) \in \mathcal{M}^{m_1}$ be n-regular at the origin and $\sigma^2(x, \xi) \in \Gamma^{m_2}_{s_0+m_1\wedge n+1}$. Then*:

(i) *For all $s \in \mathbb{R}$ such that* $\max\{-t_0, -t_0 - m_1\} < s \leqslant s_0 + 1$, *one has*

$$\begin{aligned} &\left|\left[\sigma^1(D), \mathrm{Op}(\sigma^2)\right]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\right|_{H^s} \\ &\quad\lesssim C(\sigma^1)\left(M^{m_2}_d(\nabla_x^{n+1}\sigma^2)|u|_{H^{s+m_1+m_2-n-1}} + \|\sigma^2\|_{H^{s+m_1\wedge n}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}\right), \end{aligned}$$

where $C(\sigma^1) := M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_n(\sigma^1)$.

(ii) *If moreover* σ^1 *is* $(n+2+[\frac{d}{2}]+d)$*-regular and* σ^2 *is d-regular at the origin, then the above estimate can be replaced by*

$$\big|\big[\sigma^1(D),\mathrm{Op}(\sigma^2)\big]u-\mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\big|_{H^s}$$
$$\lesssim C'(\sigma^1)\big(\|\nabla_x^{n+1}\sigma^2\|_\infty|u|_{H^{s+m_1+m_2-n-1}}+\|\nabla_x^{n+1}\sigma^2\|_{H^{s+m_1\wedge n-n-1}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}\big),$$

where $C'(\sigma^1):=M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)+m_{n+2+[\frac{d}{2}]+d}(\sigma^1)$.

Proof. (i) Remark that $\{\sigma^1,\sigma^2\}_n=\sigma^1\sharp_n\sigma^2-\sigma^1\sigma^2$ and that $\mathrm{Op}(\sigma^2)\circ\mathrm{Op}(\sigma^1)=\mathrm{Op}(\sigma^1\sigma^2)$ since $\mathrm{Op}(\sigma^1)$ is a Fourier multiplier. Therefore, one has

$$\Big[\mathrm{Op}(\sigma^1),\mathrm{Op}(\sigma^2)\Big]-\mathrm{Op}(\{\sigma^1,\sigma^2\}_n)=\mathrm{Op}(\sigma^1)\circ\mathrm{Op}(\sigma^2)-\mathrm{Op}(\sigma^1\sharp_n\sigma^2).$$

Write now $\sigma^1(D)\circ\mathrm{Op}(\sigma^2)-\mathrm{Op}(\sigma^1\sharp_n\sigma^2)=\sum_{j=1}^5\tau^j(x,D)$, with

$$\tau^1(x,D)=\sigma^1(D)\circ\mathrm{Op}(\sigma^2-\sigma^2_I-\sigma^2_{lf}),$$
$$\tau^2(x,D)=\sigma^1(D)\circ\mathrm{Op}(\sigma^2_{lf}),$$
$$\tau^3(x,D)=\sigma^1(D)\circ\mathrm{Op}(\sigma^2_I)-\mathrm{Op}(\sigma^1\sharp_n\sigma^2_I),$$
$$\tau^4(x,D)=\mathrm{Op}(\sigma^1\sharp_n\sigma^2_I-(1-\psi(\xi))\sigma^1\sharp_n\sigma^2),$$
$$\tau^5(x,D)=-\mathrm{Op}(\psi(\xi)\sigma^1\sharp_n\sigma^2).$$

We now turn to control the operator norms of $\tau^j(x,D)$, $j=1,\dots,5$.

• *Control of* $\tau^1(x,D)$: Since $\sigma^1(D)$ is a Fourier multiplier, one obtains easily that for all $s\in\mathbb{R}$ and $u\in\mathcal{S}(\mathbb{R}^d)$,

$$|\tau^1(x,D)u|_{H^s}\leqslant(m_0(\sigma^1)+M_0^{m_1}(\sigma^1))\Big|\mathrm{Op}(\sigma^2-\sigma^2_I-\sigma^2_{lf})u\Big|_{H^{s+m_1}}.$$

Using Proposition 25(i) (with $r=m_1\wedge n-m_1$) gives therefore, for all $-t_0-m_1<s\leqslant s_0+1$,

$$|\tau^1(x,D)u|_{H^s}\lesssim(m_0(\sigma^1)+M_0^{m_1}(\sigma^1))N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n}(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.9}$$

• *Control of* $\tau^2(x,D)$: One has

$$|\tau^2(x,D)u|_{H^s}\leqslant(m_0(\sigma^1)+M_0^{m_1}(\sigma^1))|\mathrm{Op}(\sigma^2_{lf})u|_{H^{s+m_1}},$$

so that it is a direct consequence of Proposition 16 that one has, for all $s \leqslant s_0 + 1$,

$$|\mathrm{Op}(\tau^2)u|_{H^s} \lesssim (m_0(\sigma^1) + M_0^{m_1}(\sigma^1))n_{0,s+m_1}(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.10}$$

• *Control of* $\tau^3(x, D)$: We have $\mathrm{Op}(\tau^3) = \mathrm{Op}(\rho_n)$ with ρ_n as given in the first part of Proposition 31. This lemma asserts that the symbol $\rho_n(x, \xi)$ satisfies the conditions of application of Lemma 19, which states that $|\mathrm{Op}(\rho_n)u|_{H^s} \lesssim M_d^{m_1+m_2-n-1}(\rho_n)$ $|u|_{H^{s+m_1+m_2-n-1}}$. Using the estimate of $M_d^{m_1+m_2-n-1}(\rho_n)$ given in Proposition 31 shows therefore that for all $s \in \mathbb{R}$,

$$|\mathrm{Op}(\tau^3)u|_{H^s} \lesssim M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)M_d^{m_2}(\nabla_x^{n+1}\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}. \tag{4.11}$$

• *Control of* $\tau^4(x, D)$: By definition of the product law $\sharp_n$, one has

$$\begin{aligned}\tau^4(x,\xi) &= -\sum_{|\alpha|\leqslant n} \frac{(-i)^\alpha}{\alpha!}\partial_\xi^\alpha\sigma^1(\xi)\partial_x^\alpha(\sigma^2 - \sigma^2_I - \sigma^2_{lf})(x,\xi)\\ &= -\sum_{|\alpha|\leqslant n} \frac{(-i)^\alpha}{\alpha!}\mathbf{1}_{[1/2,\infty)}(\xi)\partial_\xi^\alpha\sigma^1(\xi)\partial_x^\alpha(\sigma^2 - \sigma^2_I - \sigma^2_{lf})(x,\xi),\end{aligned}$$

where $\mathbf{1}_{[1/2,\infty)}(\cdot)$ denotes the characteristic function of the interval $[1/2, \infty)$. It follows that

$$\left|\tau^4(x, D)u\right|_{H^s} \lesssim \sum_{|\alpha|\leqslant n} \left|\mathrm{Op}(\partial_x^\alpha\sigma^2 - (\partial_x^\alpha\sigma^2)_I - (\partial_x^\alpha\sigma^2)_{lf})v\right|_{H^s}$$

with $v = \mathrm{Op}(\mathbf{1}_{[1/2,\infty)}\partial_\xi^\alpha\sigma^1)u$; we now use the first estimate of Proposition 25 (with $r = m_1 \wedge n - |\alpha|$) to obtain that the terms of the above sum are bounded from above by $N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n-|\alpha|}(\partial_x^\alpha\sigma^2)|v|_{H^{m_2+t_0-m_1\wedge n+|\alpha|}}$, for all $-t_0 < s \leqslant s_0 + 1$. It is now straightforward to conclude that for all $-t_0 < s \leqslant s_0 + 1$,

$$\left|\tau^4(x, D)u\right|_{H^s} \lesssim M_n^{m_1}(\sigma^1)N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n}(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.12}$$

• *Control of* $\tau^5(x, D)$: One has $\mathrm{Op}(\psi(\xi)\sigma^1\sharp_n\sigma^2)u = \mathrm{Op}((\sigma^1\sharp_n\sigma^2)_{lf})u$, so that Proposition 16 can be used to obtain for all $s \leqslant s_0 + 1$,

$$\left|\mathrm{Op}(\psi(\xi)\sigma^1\sharp_n\sigma^2)u\right|_{H^s} \lesssim m_n(\sigma^1)n_{0,s+n}(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.13}$$

Recalling that $\|\sigma^2\|_{H^s}$ is defined in (4.6), the estimate given in (i) of the theorem now follows directly from (4.9)–(4.13).

(ii) We use here another decomposition, namely $\sigma^1(D)\circ \mathrm{Op}(\sigma^2) - \mathrm{Op}(\sigma^1\sharp_n\sigma^2) = \sum_{j=1}^{6} \underline{\tau}^j(x,D)$, with

$$\underline{\tau}^1(x,D) = \sigma^1(D)\circ \mathrm{Op}(\sigma^2 - \sigma^2_I - \sigma^2_{lf} - \sigma^2_{R,2}),$$

$$\underline{\tau}^2(x,D) = \sigma^1(D)\circ \mathrm{Op}(\Psi(D_x)\sigma^2_{lf} + \sigma^2_{R,2}) - \mathrm{Op}(\sigma^1\sharp_n(\psi(D_x)\sigma^2_{lf} + \sigma^2_{R,2})),$$

$$\underline{\tau}^3(x,D) = \sigma^1(D)\circ \mathrm{Op}(\sigma^2_I) - \mathrm{Op}(\sigma^1\sharp_n\sigma^2_I),$$

$$\underline{\tau}^4(x,D) = -\mathrm{Op}(\sigma^1\sharp_n(\sigma^2 - \sigma^2_I - \sigma^2_{lf} - \sigma^2_{R,2})),$$

$$\underline{\tau}^5(x,D) = \sigma^1(D)\circ \mathrm{Op}((1-\psi(D_x)\sigma^2_{lf})),$$

$$\underline{\tau}^6(x,D) = -\mathrm{Op}(\sigma^1\sharp_n(1-\psi(D_x))\sigma^2_{lf}).$$

We now turn to control the operator norms of $\underline{\tau}^j(x,D)$, $j=1,\dots,6$.

• *Control of* $\underline{\tau}^1(x,D)$: Proceeding as for the control of $\tau^1(x,D)$ in (i) above, but using Proposition 26 instead of Proposition 25, one can replace $N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n}(\sigma^2)$ by $N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n-n-1}(\nabla_x^{n+1}\sigma^2)$ in (4.9).

• *Control of* $\underline{\tau}^2(x,D)$: We need here two lemmas:

Lemma 35. *Let $m_1\in\mathbb{R}$, $n\in\mathbb{N}$ and $\sigma^1(\xi)\in\mathcal{M}^{m_1}$ be $(n+2+[\frac{d}{2}]+d)$-regular at the origin. Let $\sigma^2(x,\xi)$ be a symbol d-regular at the origin and such that $\widehat{\sigma^2}(\eta,\xi)$ is supported in the ball $|\eta|+|\xi|\leqslant A$, for some $A>0$.*

Then, $\sigma^1(D)\circ\mathrm{Op}(\sigma^2) = \mathrm{Op}(\sigma^1\sharp_n\sigma^2) + \mathrm{Op}(\rho_n)$, where the symbol $\rho_n(x,\xi)$ is such that $\widehat{\rho_n}(\eta,\xi)$ vanishes outside the ball $|\eta|+|\xi|\leqslant A$ and satisfies the estimate

$$\sup_{|\xi|\leqslant A}\sup_{|\beta|\leqslant d}|\partial_\xi^\beta\rho_n(\cdot,\xi)|_\infty$$
$$\leqslant \mathrm{Cst}\sup_{|\xi|\leqslant 2A}\sup_{|\alpha|\leqslant n+2+[\frac{d}{2}]+d}|\partial_\xi^\alpha\sigma^1(\xi)|\sup_{|\xi|\leqslant A}\sup_{|\alpha|\leqslant d}|\nabla_x^{n+1}\partial_\xi^\alpha\sigma^2(\cdot,\xi)|_\infty.$$

Proof. The proof is a close adaptation of the proof of Proposition 31. First replace the admissible cut-off function $\chi(\eta,\xi)$ used there by a smooth function $\widetilde{\chi}(\eta,\xi)$ supported in the ball $|\eta|+|\xi|\leqslant A$. Inequality (4.3) must then be replaced by

$$|\partial_\xi^\beta\rho_n(x,\xi)|\leqslant \mathrm{Cst}\left(\sum_{|\beta'|\leqslant d}\left|\partial_\xi^{\beta'}G_\gamma(\cdot,\xi)\right|_{L^1}\right)\sup_{|\xi|\leqslant A}\sup_{|\alpha|\leqslant d}|\nabla_x^{n+1}\partial_\xi^\alpha\sigma^2(\cdot,\xi)|_\infty,$$

for all $|\beta|\leqslant d$ and $|\xi|\leqslant A$.

Finally, one concludes the proof as in Proposition 31 after remarking that (4.4) can be replaced here by

$$\left|\partial_\xi^{\beta'} G_\gamma(\cdot,\xi)\right|_{L^1} \leqslant \mathrm{Cst} \sup_{|\xi|\leqslant 2A} \sup_{|\alpha|\leqslant n+2+[\frac{d}{2}]+d} |\partial_\xi^\alpha \sigma^1(\xi)|. \qquad \square$$

The proof of the following lemma is a straightforward adaptation of the proof of Lemma 19.

Lemma 36. *Let $\rho(x,\xi)$ be a symbol such that $\widehat{\rho}(\eta,\xi)$ is supported in the ball $|\eta| + |\xi| \leqslant A$, for some $A > 0$. Then $\mathrm{Op}(\rho)$ extends as a continuous operator on every Sobolev space, with values in $H^\infty(\mathbb{R}^d)$. Moreover,*

$$\forall s,t\in\mathbb{R}, \forall u\in H^t(\mathbb{R}^d), \quad |\mathrm{Op}(\rho)u|_{H^s} \leqslant \mathrm{Cst} \sup_{|\xi|\leqslant A} \sup_{|\alpha|\leqslant d} |\partial_\xi^\alpha \rho(\cdot,\xi)|_\infty |u|_{H^t},$$

where the constant depends only on A, s and t.

From these two lemmas, one easily gets

$$|\underline{\tau}^2(x,D)u|_{H^s} \lesssim C'(\sigma^1)\|\nabla_x^{n+1}\sigma^2\|_\infty |u|_{H^{s+m_1+m_2-n-1}},$$

where $C'(\sigma^1)$ is as given in the statement of the theorem.

• *Control of* $\underline{\tau}^3(x,D)$: One has $\underline{\tau}^3 = \tau^3$, so that $\underline{\tau}^3(x,D)$ is controlled via (4.11).

• *Control of* $\underline{\tau}^4(x,D)$: To control this term, proceed exactly as for the control of $\tau^4(x,D)$ above, but use Proposition 26 instead of Proposition 25. This yields

$$\left|\underline{\tau}^4(x,D)u\right|_{H^s} \lesssim M_n^{m_1}(\sigma^1) N^{m_2}_{2[\frac{d}{2}]+2,s+m_1\wedge n-n-1}(\nabla_x^{n+1}\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}.$$

• *Control of* $\underline{\tau}^5(x,D)$ and $\underline{\tau}^6(x,D)$: The difference between $\underline{\tau}^5(x,D)$ and $\tau^2(x,D)$ is that the operator $(1-\psi(D_x))$ is applied to $\sigma^2_{lf}(\cdot,\xi)$ in the former. This allows us to replace $n_{0,s+m_1}(\sigma^2)$ by $n_{0,s+m_1-n-1}(\nabla_x^{n+1}\sigma^2)$ in (4.10).

A similar adaptation of (4.13) gives

$$\left|\mathrm{Op}(\underline{\tau}^6)u\right|_{H^s} \lesssim m_n(\sigma^1) n_{0,s-1}(\nabla_x^{n+1}\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}.$$

Point (ii) of the theorem thus follows from the estimates on $\underline{\tau}^j(x,D)$, $j = 1,\ldots,6$, proved above. $\square$

Remark 37. When $m_2 \geqslant 0$, an easy adaptation of the above proof shows that the quantity $\|\sigma^2\|_{H^{s+m_1\wedge n}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}$ which appears in the r.h.s. of the first estimate of the theorem can be replaced by $\|\sigma^2\|_{H^{s+m_1\wedge n+m_2}}|u|_{H^{t_0+m_1-m_1\wedge n}}$.

In the spirit of Theorem 2, the following two theorems can be a useful alternative to Theorem 3. Theorem 4 deals with the case of pseudo-differential operators $\sigma^2(x,D)$ of nonnegative order, while Theorem 5 addresses the case $\sigma^2(x,\xi)=\sigma^2(x)$.

Theorem 4. *Let $m_1\in\mathbb{R}$, $m_2>0$, $n\in\mathbb{N}$ and $s_0>d/2$. Let $\sigma^1(\xi)\in\mathcal{M}^{m_1}$ be n-regular at the origin and let $\sigma^2(x,\xi)\in\Gamma^{m_2}_{s_0+m_1\wedge n+1}$ be $2[\frac{d}{2}]+2$-regular at the origin. Then*:

(i) *For all s such that $0<s+m_1$, $0<s$ and $s+m_2\leqslant s_0+1$, one has*

$$\big|\big[\sigma^1(D),\mathrm{Op}(\sigma^2)\big]u-\mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\big|_{H^s}$$
$$\lesssim C(\sigma^1)\big(M_d^{m_2}(\nabla_x^{n+1}\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}+\|\sigma^2\|_{H_{\mathrm{reg}}^{s+m_1\wedge n+m_2}}|u|_\infty\big),$$

where $C(\sigma^1)=M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)+m_n(\sigma^1)$.

(ii) *If moreover σ^1 is $(n+2+[\frac{d}{2}]+d)$-regular at the origin, then the above estimate can be replaced by*

$$\big|\big[\sigma^1(D),\mathrm{Op}(\sigma^2)\big]u-\mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\big|_{H^s}$$
$$\lesssim C'(\sigma^1)\big(\|\nabla_x^{n+1}\sigma^2\|_\infty|u|_{H^{s+m_1+m_2-n-1}}+\|\nabla_x^{n+1}\sigma^2\|_{H_{\mathrm{reg}}^{s+m_1\wedge n+m_2-n-1}}|u|_\infty\big),$$

where $C'(\sigma^1):=M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1)+m_{n+2+[\frac{d}{2}]+d}(\sigma^1)$.

Proof. We only give a sketch of the proof of (i) which follows the same lines as the proof of Theorem 3(ii) is obtained similarly. The modifications to be made are:

- Inequality (4.9) must be replaced by

$$|\tau^1(x,D)u|_{H^s}\lesssim(m_0(\sigma^1)+M_0^{m_1}(\sigma^1))N^{m_2}_{2[\frac{d}{2}]+2,s+m_1+m_2}(\sigma^2)|u|_\infty,\tag{4.14}$$

  which holds for all $s\in\mathbb{R}$ such that $s+m_1>0$ and $s+m_2\leqslant s_0+1$. This is a consequence of Proposition 25(iii), which can be used since we assumed $m_2>0$.
- Similarly, the second estimate of Proposition 16 allows us to replace (4.10) by

$$|\mathrm{Op}(\tau^2)u|_{H^s}\lesssim(m_0(\sigma^1)+M_0^{m_1}(\sigma^1))n_{2[\frac{d}{2}]+2,s+m_1}(\sigma^2)|u|_\infty.$$

- Inequality (4.11) is left unchanged.
- Estimate (4.12) must be replaced by

$$\big|\tau^4(x,D)u\big|_{H^s}\lesssim M_n^{m_1}(\sigma^1)N^{m_2}_{2[\frac{d}{2}]+2,s+m_2+m_1\wedge n}(\sigma^2)|u|_\infty,\tag{4.15}$$

which holds for all $0 < s$ and $s + m_2 \leqslant s_0 + 1$. One proves (4.15) in the same way as (4.12), using the third point of Proposition 25 rather than the first one (this is possible because $m_2 > 0$ here).

- Finally, inequality (4.13) is replaced, using the second part of Proposition 16, by

$$\left|\mathrm{Op}(\psi(\xi)\sigma^1 \sharp_n \sigma^2)u\right|_{H^s} \lesssim m_n(\sigma^1) n_{2[\frac{d}{2}]+2,s+n}(\sigma^2)|u|_\infty. \qquad \square$$

An interesting particular case is obtained when the symbol $\sigma^2(x,\xi)$ does not depend on ξ (i.e., it is a function).

Theorem 5. *Let $m_1 \in \mathbb{R}$, $n \in \mathbb{N}$ and $s_0 > d/2$. Let $\sigma^1(\xi) \in \mathcal{M}^{m_1}$ be n-regular at the origin and let $\sigma^2 \in H^{s_0+m_1\wedge n+1}(\mathbb{R}^d)$.*

(i) *If $m_1 > n$ then for all s such that $0 \leqslant s \leqslant s_0 + 1$, one has*

$$\begin{aligned}&\left|\left[\sigma^1(D), \sigma^2\right]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\right|_{H^s}\\ &\quad \leqslant C(\sigma^1)\left(|\sigma^2|_{W^{n+1,\infty}}|u|_{H^{s+m_1-n-1}} + |\sigma^2|_{H^{s+m_1}}|u|_\infty\right),\end{aligned}$$

where $C(\sigma^1) = M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_n(\sigma^1)$.

(ii) *If moreover σ^1 is $(n+2+[\frac{d}{2}]+d)$-regular at the origin, then the above estimate can be replaced by*

$$\begin{aligned}&\left|\left[\sigma^1(D), \sigma^2\right]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\right|_{H^s}\\ &\quad \leqslant C'(\sigma^1)\left(|\nabla^{n+1}\sigma^2|_\infty |u|_{H^{s+m_1-n-1}} + |\nabla^{n+1}\sigma^2|_{H^{s+m_1-n-1}}|u|_\infty\right),\end{aligned}$$

where $C'(\sigma^1) := M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_{n+2+[\frac{d}{2}]+d}(\sigma^1)$.

Proof. Here again, we only prove the first point of the theorem since the proof of the second one can be deduced similarly from the proof of Theorem 3(ii).

Remark that for all $k \in \mathbb{N}$, $s \in \mathbb{R}$ and $v \in \mathcal{S}(\mathbb{R}^d)$, one has $N^0_{k,s}(v) = |v|_{H^s}$. Therefore, we just have to adapt the points of the proof of Theorem 4 which use the assumption $m_2 > 0$, namely the obtention of (4.14) and (4.15).

One can check that (4.14) remains true here. This is a consequence of Proposition 26(iv) and Remark 27.

We now prove that (4.15) can be replaced by

$$|\tau^4(x,D)u|_{H^s} \lesssim M^{m_1}_n(\sigma^1)|\sigma^2|_{H^{s+m_1}}|u|_\infty,$$

which holds for all $0 < s \leqslant s_0 + 1$ and provided that $m_1 > n$, and which remains true when $s = 0$ provided one adds $|\sigma^2|_{W^{n+1,\infty}}|u|_{H^{m_1-n-1}}$ to the right-hand side. To obtain this, we need to control in H^s-norm, and for all $0 \leqslant |\alpha| \leqslant n$, the terms

$\mathrm{Op}(\partial_x^\alpha\sigma^2 - (\partial_x^\alpha\sigma^2)_I - (\partial_x^\alpha\sigma^2)_{lf})(\partial_\xi^\alpha\sigma^1(D))u$, which is done using Proposition 26(iv) (with $r = m_1 - |\alpha| > 0$) and Remark 27. □

We finally give commutator estimates of Calderon–Coifman–Meyer type:

Theorem 6. *Let $m_1, m_2 \in \mathbb{R}$, $n \in \mathbb{N}$ and $d/2 < t_0 \leqslant s_0$. Let $\sigma^1(\xi) \in \mathcal{M}^{m_1}$ be n-regular at the origin and $\sigma^2(x,\xi) \in \Gamma^{m_2}_{s_0+m_1\wedge n+1}$. Then:*

(i) *For all $s \in \mathbb{R}$ such that $-t_0 < s \leqslant t_0 + 1$ and $-t_0 < s + m_1 \leqslant t_0 + n + 1$,*

$$\left|\left[\sigma^1(D), \mathrm{Op}(\sigma^2)\right]u - \mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\right|_{H^s} \lesssim C(\sigma^1)\|\sigma^2\|_{H^{t_0+n+1}}|u|_{H^{s+m_1+m_2-n-1}},$$

where $C(\sigma^1) := M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_n(\sigma^1)$.

(ii) *If moreover σ^1 is $(n+2+[\frac{d}{2}]+d)$-regular and σ^2 is d-regular at the origin, then the above estimate can be replaced by*

$$\left|\left[\sigma^1(D), \mathrm{Op}(\sigma^2)\right]u - \mathrm{Op}(\{\sigma^1,\sigma^2\}_n)u\right|_{H^s} \lesssim C'(\sigma^1)\|\nabla_x^{n+1}\sigma^2\|_{H^{t_0}}|u|_{H^{s+m_1+m_2-n-1}},$$

where $C'(\sigma^1) := M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_{n+2+[\frac{d}{2}]+d}(\sigma^1)$.

Proof. The proof follows closely the proof of Theorem 3, so that we just mention the adaptations that have to be made.

(i) Below is the list of changes one must perform in the control of the operators $\tau^j(x,D)$.
- *Control of* $\tau^1(x,D)$: For all $-t_0 < s+m_1 \leqslant t_0+n+1$ and using Proposition 25(ii) with $r' = n+1$ instead of Proposition 25(i), one obtains instead of (4.9) a control in terms of $N^{m_2}_{2[\frac{d}{2}]+2,t_0+n+1}(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}$.
- *Control of* $\tau^2(x,D)$: For $s+m_1 \leqslant t_0+n+1$, one just has to remark that Proposition 16 gives a control in terms of $n_{0,t_0+n+1}(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}$.
- *Control of* $\tau^4(x,D)$: If $-t_0 < s \leqslant t_0+1$, one can replace (4.12) by a control in terms of $N^{m_2}_{2[\frac{d}{2}]+2,t_0+n+1}(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}$ provided that one uses Proposition 25(ii) with $r' = n+1-|\alpha|$ instead of Proposition 25(i).
- *Control of* $\tau^5(x,D)$. When $s \leqslant t_0+1$ one gets easily a control in terms of $\|\sigma^2\|_{H^{t_0+1}}$ $|u|_{H^{s+m_1+m_2-n-1}}$.

(ii) One deduces the second point of the theorem from Theorem 3(ii) exactly as we adapted the proof of the first point from the proof of Theorem 3(i). □

Remark 38. (i) Extending results of Moser [16] and Kato–Ponce [10], Taylor proved in [17] the following generalized Kato–Ponce estimates (which also hold in L^p-based Sobolev spaces): for all Fourier multiplier $\sigma^1(D)$ of order $m_1 > 0$, and all

$\sigma^2 \in H^\infty(\mathbb{R}^d)$, one has for all $s \geqslant 0$,

$$\big|[\mathrm{Op}(\sigma^1), \sigma^2]u\big|_{H^s} \leqslant \underline{C}(\sigma^1)\big(|\nabla\sigma^2|_\infty |u|_{H^{s+m_1-1}} + |\sigma^2|_{H^{s+m_1}}|u|_\infty\big), \tag{4.16}$$

where $\underline{C}(\sigma^1)$ is some constant depending on σ^1 (in [17], Taylor also deals with classical pseudo-differential operators $\sigma^1(x,D)$—and not only Fourier multipliers—we address this problem in Section 4.2).

The estimate of Theorem 5 coincides with (4.16) when $n = 0$ (it is in fact more precise since it allows one to replace the term $|\sigma^2|_{H^{s+m_1}}$ by $|\nabla\sigma^2|_{H^{s+m_1-1}}$ [1]); the general case $n \in \mathbb{N}$ gives an extension of this result involving the Poisson bracket of σ^1 and σ^2. Theorem 4 extends (4.16) in another way, allowing $\sigma^2(x,D)$ to be a pseudo-differential operator of nonnegative order instead of a function. Finally, the most general extension of (4.16) is given by Theorem 3, since it contains the two improvements just mentioned, allows estimates in Sobolev spaces of negative order, and does not assume cumbersome restrictions on the order of $\sigma^1(\xi)$ and $\sigma^2(x,\xi)$. For instance, (4.16) does not hold when $s < 0$ or $m_1 \leqslant 0$ but can be replaced by: for all Fourier multiplier $\sigma^1(D)$ of order $m_1 \in \mathbb{R}$ regular at the origin, and all $\sigma^2 \in H^\infty(\mathbb{R}^d)$, one has for all $t_0 > d/2$ and $s > \max\{-t_0, -t_0 - m_1\}$,

$$\big|[\mathrm{Op}(\sigma^1), \sigma^2]u\big|_{H^s} \leqslant C(\sigma^1)\big(|\nabla\sigma^2|_\infty |u|_{H^{s+m_1-1}} + |\nabla\sigma^2|_{H^{s+(m_1)_+-1}}|u|_{H^{t_0+(m_1)_-}}\big),$$

with $(m_1)_+ = \max\{m_1, 0\}$ and $(m_1)_- = \min\{m_1, 0\}$.

(ii) The Calderon–Coifman–Meyer commutator estimate of Theorem 6 coincides with (1.3) when $n = 0$ and σ^1 is a Fourier multiplier (the general case is addressed in Section 4.2) but its range of validity is wider since it allows negative values of s and m_1. We also have the same kind of generalization as for the Kato–Ponce estimates.

In the particular case when the symbol σ^2 is of the form $\sigma^2(x,\xi) = \Sigma(v(x),\xi)$, one can now obtain easily, proceeding as in the proof of Corollary 30:

Corollary 39. *Let $m_1, m_2 \in \mathbb{R}$, $p \in \mathbb{N}$ and $d/2 < t_0 \leqslant s_0$. Let $\sigma^1(\xi) \in \mathcal{M}^{m_1}$ be n-regular at the origin and $\sigma^2(x,\xi) = \Sigma^2(v(x),\xi)$ with $\Sigma^2 \in C^\infty(\mathbb{R}^p, \mathcal{M}^{m_2})$ and $v \in H^{s_0+m_1\wedge n+1}(\mathbb{R}^d)^p$. Then,*

(i) *For all $s \in \mathbb{R}$ such that* $\max\{-t_0, -t_0 - m_1\} < s \leqslant s_0 + 1$

$$\begin{aligned}&\big|\big[\sigma^1(D), \sigma^2(x,D)\big]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\big|_{H^s}\\ &\quad\leqslant C(\sigma^1)C_{\Sigma^2}(|v|_{W^{n+1,\infty}})\big(|u|_{H^{s+m_1+m_2-n-1}} + |v|_{H^{(s+m_1\wedge n)_+}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}\big).\end{aligned}$$

[1] Such an improvement has been proved recently in [2] for $n = 0$ or $n = 1$, in the case where $\sigma^1(\xi) = \langle\xi\rangle^{m_1}$ and σ^2 does not depend on ξ.

(ii) *For all* $s \in \mathbb{R}$ *such that* $-t_0 < s < t_0 + 1$ *and* $-t_0 < s + m_1 \leqslant t_0 + n + 1$, *one also has*

$$\left|\left[\sigma^1(D), \sigma^2(x, D)\right]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\right|_{H^s} \\ \leqslant C(\sigma^1) C_{\Sigma^2}(|v|_\infty)|v|_{H^{t_0+n+1}}|u|_{H^{s+m_1+m_2-n-1}}.$$

(iii) *If moreover* $m_2 > 0$ *and* Σ^2 *is* $2[\frac{d}{2}] + 2$*-regular at the origin, one has, for all* $s \in \mathbb{R}$ *such that* $0 < s + m_1$, $0 < s$, *and* $s + m_2 \leqslant s_0 + 1$,

$$\left|\left[\sigma^1(D), \sigma^2(x, D)\right]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\right|_{H^s} \\ \leqslant C(\sigma^1) C_{\Sigma^2}(|v|_{W^{n+1,\infty}})\big(|u|_{H^{s+m_1+m_2-n-1}} + |v|_{H^{s+m_2+m_1\wedge n}}|u|_\infty\big).$$

In the above, $C_{\Sigma^2}(\cdot)$ *denotes a smooth nondecreasing function depending only on a finite number of derivatives of* Σ^2 *and* $C(\sigma^1) = M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) + m_n(\sigma^1)$.

Example 3. Let $\sigma(x, \xi)$ be the symbol given by (1.4), with $a \in H^\infty(\mathbb{R}^d)$, and let $m \geqslant 0$. Then for all $t_0 > d/2$ and $s > -t_0$, one has

$$\left|\left[\langle D\rangle^m, \sigma(x, D)\right] u\right|_{H^s} \leqslant C(|\nabla a|_{W^{1,\infty}}) \left(|u|_{H^{s+m}} + |\nabla a|_{H^{(s+m)_+}}|u|_{H^{1+t_0}}\right),$$

and, writing $\widetilde{m} := \max\{m, 1\}$

$$\left|\left[\langle D\rangle^m, \mathrm{Op}(\sigma)\right] u - \mathrm{Op}(\tau)u\right|_{H^s} \leqslant C(|\nabla a|_{W^{2,\infty}}) \left(|u|_{H^{s+m-1}} + |\nabla a|_{H^{(s+\widetilde{m})_+}}|u|_{H^{1+t_0}}\right),$$

where the symbol $\tau(x, \xi)$ is given by

$$\tau(x, \xi) := m\langle\xi\rangle^{m-2}\frac{|\xi|^2 d^2 a(\nabla a, \xi) - (\nabla a \cdot \xi) d^2 a(\xi, \xi)}{\sigma(x, \xi)}.$$

For $s > 0$, we also have

$$\left|\left[\langle D\rangle^m, \sigma(x, D)\right] u\right|_{H^s} \leqslant C(|\nabla a|_{W^{1,\infty}}) \left(|u|_{H^{s+m}} + |\nabla a|_{H^{1+s+m}}|u|_\infty\right).$$

Finally, if s and t_0 are such that $-t_0 < s$ and $s + m \leqslant t_0 + 1$, then

$$\left|\left[\langle D\rangle^m, \sigma(x, D)\right] u\right|_{H^s} \leqslant C(|\nabla a|_{W^{1,\infty}})|\nabla a|_{H^{t_0+1}}|u|_{H^{s+m}}.$$

4.2. Composition and commutators between two pseudo-differential operators of limited regularity

This section is devoted to the proof of composition and commutator estimates involving two pseudo-differential operators of limited regularity. We first introduce the following notations:

Notations: For all $s_0 > d/2$, $n \in \mathbb{N}$, and all symbols $\sigma \in \Gamma^m_{s_0}$ n-regular at the origin, we define

$$\forall s \leqslant s_0, \quad \|\sigma\|_{H^s_{n,(m)}} := \frac{1}{2}\left(n_{n,s}(\sigma) + N^m_{n+2+[\frac{d}{2}]+d,s}(\sigma)\right) \tag{4.17}$$

and

$$\forall k \in \mathbb{N}, \quad 0 \leqslant k \leqslant n, \quad \|\sigma\|_{W^{k,\infty}_{n,(m)}} := \frac{1}{2}\left(m_n(\sigma) + M^m_{n+2+[\frac{d}{2}]+d,k}(\sigma)\right). \tag{4.18}$$

When no confusion is possible, we omit the subscript m. Remark that when σ does not depend on ξ (i.e. when it is a function), then one has $\|\sigma\|_{H^s_n} = |\sigma|_{H^s}$ and $\|\sigma\|_{W^{k,\infty}_n} = |\sigma|_{W^{k,\infty}}$.

We first give commutator estimates of Kato–Ponce type:

Theorem 7. *Let $m_1, m_2 \in \mathbb{R}$, $n \in \mathbb{N}$ and $d/2 < t_0 \leqslant s_0$. Define $m := m_1 \wedge m_2$ and let $\sigma^j \in \Gamma^{m_j}_{s_0+m\wedge n+1}$ $(j = 1, 2)$ be two symbols n-regular at the origin.*

(i) *For all s such that* $\max\{-t_0, -t_0 - m_1\} < s \leqslant t_0 + 1$, *one has*

$$\begin{aligned}\left|\mathrm{Op}(\sigma^1)\circ\mathrm{Op}(\sigma^2)u - \mathrm{Op}(\sigma^1\sharp_n\sigma^2)u\right|_{H^s} \\ \lesssim \|\sigma^1\|_{W^{n+1,\infty}_n}\|\sigma^2\|_{W^{n+1,\infty}}|u|_{H^{s+m_1+m_2-n-1}} \\ +\left(\|\sigma^1\|_{H^{t_0+1}_n}\|\sigma^2\|_{H^{s_++m_1\wedge n}} + \|\sigma^1\|_{H^{s_++m_1\wedge n}_n}\|\sigma^2\|_{H^{t_0}}\right)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}},\end{aligned}$$

where $s_+ = \max\{s, 0\}$ *and with notations* (4.6)–(4.7) *and* (4.17)-(4.18).

(i.bis) *Under the same assumptions, one also has*

$$\begin{aligned}\left|\mathrm{Op}(\sigma^1)\circ\mathrm{Op}(\sigma^2)u - \mathrm{Op}(\sigma^1\sharp_n\sigma^2)u\right|_{H^s} \\ \lesssim \|\sigma^1\|_{H^{t_0+m_1\wedge n+1}_n}\left(\|\sigma^2\|_{H^{s+m_1\wedge n}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}} + \|\sigma^2\|_{W^{n+1,\infty}}|u|_{H^{s+m_1+m_2-n-1}}\right).\end{aligned}$$

(i.ter) *For* $s \in \mathbb{R}$ *such that* $\max\{t_0 + 1, -t_0 - m_1\} \leqslant s \leqslant s_0 + 1$, *the estimate of*

(i) *still holds if one adds* $\|\sigma^1\|_{H^s}\|\sigma^2\|_{H^{m_1\wedge n+t_0+1}}|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}$ *to the right-hand side.*

(ii) *For all* $s \in \mathbb{R}$ *such that* $\max\{-t_0, -t_0 - m_1, -t_0 - m_2\} < s \leqslant t_0 + 1$, *one has*

$$\big|\big[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)\big]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\big|_{H^s} \lesssim I(\sigma^1, \sigma^2, m_1, m_2) + I(\sigma^2, \sigma^1, m_2, m_1),$$

where $I(\sigma^1, \sigma^2, m_1, m_2)$ *denotes the r.h.s. or the estimate of* (i), (i.bis) *or* (i.ter) *In this latter case, the range of application of the estimate is bounded from above by* $s_0 + 1$ *instead of* $t_0 + 1$.

(iii) *If* $m_1 > 0$, σ^1 *is* $2[\frac{d}{2}] + 2$ *regular at the origin, and* $\sigma^2(x, \xi) = \sigma^2(x)$ *does not depend on* ξ *then, for all* $0 \leqslant s \leqslant t_0 + 1$,

$$\big|\big[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)\big]u\big|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}_{\mathrm{reg}}}\big(|\sigma^2|_{H^{s+m_1}}|u|_\infty + |\sigma^2|_{W^{1,\infty}}|u|_{H^{s+m_1-1}}\big).$$

Proof. One has

$$\begin{aligned}[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)] - \mathrm{Op}(\{\sigma^1, \sigma^2\}) &= \mathrm{Op}(\sigma^1) \circ \mathrm{Op}(\sigma^2) - \mathrm{Op}(\sigma^1 \sharp_n \sigma^2) \\ &\quad - \Big(\mathrm{Op}(\sigma^2) \circ \mathrm{Op}(\sigma^1) - \mathrm{Op}(\sigma^2 \sharp_n \sigma^1)\Big) \\ &:= \tau_1(x, D) - \tau_2(x, D).\end{aligned}$$

Controlling the operator norm of $\tau_1(x, D)$ gives the composition estimates of the theorem. To obtain the commutators estimates, we need also a control on $\tau_2(x, D)$; since this latter is obtained by a simple permutation of σ^1 and σ^2, we just have to treat $\tau_1(x, D)$. We decompose this operator into $\tau_1(x, D) = \sum_{j=1}^7 \tau_1^j(x, D)$ with

$$\begin{aligned}
\tau_1^1(x, D) &:= \mathrm{Op}(\sigma^1) \circ \mathrm{Op}(\sigma^2 - \sigma^2_{lf} - \sigma^2_I) \\
\tau_1^2(x, D) &:= \mathrm{Op}(\sigma^1) \circ \mathrm{Op}(\sigma^2_{lf}) \\
\tau_1^3(x, D) &:= \Big[\mathrm{Op}(\sigma^1_I) \circ \mathrm{Op}(\sigma^2_I) - \mathrm{Op}(\sigma^1_I \sharp_n \sigma^2_I)\Big] \\
\tau_1^4(x, D) &:= \Big[\mathrm{Op}(\sigma^1_I \sharp_n \sigma^2_I) - \mathrm{Op}((1 - \psi(\xi))\sigma^1 \sharp_n \sigma^2)\Big] \\
\tau_1^5(x, D) &:= \mathrm{Op}((\sigma^1 \sharp_n \sigma^2)_{lf}) \\
\tau_1^6(x, D) &:= \mathrm{Op}(\sigma^1_{lf}) \circ \mathrm{Op}(\sigma^2_I) \\
\tau_1^7(x, D) &:= \mathrm{Op}(\sigma^1 - \sigma^1_{lf} - \sigma^1_I) \circ \mathrm{Op}(\sigma^2_I).
\end{aligned}$$

The proof reduces therefore to the control of $|\tau_1^j(x, D)u|_{H^s}$ for all $j = 1, \ldots, 7$.

• *Control of* $\tau_1^1(x, D)$ *and* $\tau_1^2(x, D)$: Using the first estimate of Theorem 1, one gets that for all $-t_0 < s \leqslant t_0 + 1$,

$$|\tau_1^j(x, D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}}|v_j|_{H^{s+m_1}}, \quad (j = 1, 2) \tag{4.19}$$

with $v_1 := \mathrm{Op}(\sigma^2 - \sigma^2_{lf} - \sigma^2_I)u$ and $v_2 := \mathrm{Op}(\sigma^2_{lf})u$.

Proceeding as for the estimates of τ^1 and τ^2 in the proof of Theorem 3, one obtains, for all $\max\{-t_0, -t_0 - m_1\} < s \leqslant t_0 + 1$,

$$|\tau_1^j(x, D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}} \|\sigma^2\|_{H^{s+m_1\wedge n}} |u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.20}$$

• *Control of* $\tau_1^3(x, D)$: A direct use of Proposition 31 and Lemma 19 yields, for all $s \in \mathbb{R}$,

$$|\tau_1^3(x, D)u|_{H^s} \lesssim M^{m_1}_{n+2+[\frac{d}{2}]+d}(\sigma^1) M_d^{m_2}(\nabla_x^{n+1}\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}. \tag{4.21}$$

• *Control of* $\tau_1^4(x, D)$: Obviously, it suffices to control the operator norm of $A_\alpha(x, D) := \mathrm{Op}(\partial_\xi^\alpha \sigma_I^1 \partial_x^\alpha \sigma_I^2) - \mathrm{Op}((1-\psi(\xi))\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)$ for all $0 \leqslant |\alpha| \leqslant n$. Let us introduce here $\widetilde{N} := N+3$; we can assume that the paradifferential decomposition (2.8) used in this proof is done using the integer $\widetilde{N}$ instead of N. To enhance this fact, we write momentaneously $\sigma_{\widetilde{I}}$ the paradifferential symbol associated to any symbol σ using (2.9) with N replaced by $\widetilde{N}$. We denote σ_I when N is used. We can write $A_\alpha(x, D) = A_\alpha^1(x, D) + A_\alpha^2(x, D)$ with

$$A_\alpha^1(x, D) := \mathrm{Op}\left(\partial_\xi^\alpha \sigma^1_{\widetilde{I}} \partial_x^\alpha \sigma^2_{\widetilde{I}} - (\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)_I\right), \tag{4.22}$$

$$A_\alpha^2(x, D) := -\mathrm{Op}\left(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2 - (\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)_{lf} - (\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)_I\right). \tag{4.23}$$

The operator norm of $A_\alpha^1(x, D)$ is controlled using the next lemma, whose proof is postponed to Appendix B to ease the readability of the present proof. Note that this result is in the spirit of the main result of [22], but that the estimate given in this latter reference is not useful here.

Lemma 40. *Let $\sigma^1(x, \xi)$ and $\sigma^2(x, \xi)$ be as in the statement of the theorem, and let $\alpha \in \mathbb{N}^d$ be such that $0 \leqslant |\alpha| \leqslant n$. Then,*

$$\forall u \in \mathcal{S}(\mathbb{R}^d), \qquad \left|A_\alpha^1(x, D)u\right|_{H^s} \lesssim M^{m_1}_{n+d,n+1}(\sigma^1) M^{m_2}_{d,n+1}(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}},$$

where $A_\alpha^1(x, D)$ is defined in (4.22).

To control $A_\alpha^2(x, D)$, we use Proposition 25(i). with $r = n \wedge m_1 - |\alpha|$ to obtain, for all $-t_0 \leqslant s \leqslant s_0 + 1$,

$$\left|A_\alpha^2(x, D)u\right|_{H^s} \lesssim N^{m_1+m_2-|\alpha|}_{2[\frac{d}{2}]+2, s+m_1\wedge n-|\alpha|}(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}.$$

Using classical tame product estimates, one gets easily

$$\begin{aligned} &N^{m_1+m_2-|\alpha|}_{2[\frac{d}{2}]+2, s+m_1\wedge n-|\alpha|}(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2) \\ &\quad \lesssim \|\sigma^1\|_{W_n^{0,\infty}} \|\sigma^2\|_{H^{s++m_1\wedge n}} + \|\sigma^1\|_{H_n^{s++m_1\wedge n}} \|\sigma^2\|_{W_0^{0,\infty}}. \end{aligned} \tag{4.24}$$

We have thus proved the following estimate on $\mathrm{Op}(\tau_1^4)$, for all $-t_0 < s \leqslant s_0 + 1$:

$$|\mathrm{Op}(\tau_1^4)u|_{H^s} \lesssim M^{m_1}_{n+d,n+1}(\sigma^1) M^{m_2}_{d,n+1}(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}} \\ +\big(\|\sigma^1\|_{W^{0,\infty}_n}\|\sigma^2\|_{H^{s_+ + m_1\wedge n}} + \|\sigma^1\|_{H_n^{s_+ + m_1\wedge n}}\|\sigma^2\|_{W^{0,\infty}_0}\big)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.25}$$

• *Control of* $\tau_1^5(x, D)$: Using again classical tame product estimates, one gets

$$n_{0,s}((\sigma^1 \sharp_n \sigma^2)_{lf}) \lesssim m_n(\sigma^1) n_{0,s_+ + n}(\sigma^2) + n_{n,s_+ + n}(\sigma^1) m_{0,0}(\sigma^2)$$

so that by Proposition 16, one obtains, for all $s \leqslant s_0 + 1$,

$$|\mathrm{Op}(\tau_1^5)u|_{H^s} \lesssim \big(m_n(\sigma^1) n_{0,s_+ + n}(\sigma^2) + n_{n,s_+ + n}(\sigma^1) m_{0,0}(\sigma^2)\big)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.26}$$

• *Control of* $\tau_1^6(x, D)$: Using successively Propositions 16 and 18 one obtains, for all $s \leqslant s_0 + n + 1$,

$$|\mathrm{Op}(\tau_1^6)u|_{H^s} \lesssim n_{0,s}(\sigma^1) M^{m_2}_d(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.27}$$

• *Control of* $\tau_1^7(x, D)$: Using successively Propositions 25(i) (with $r = m_1 \wedge n$) and 18 one obtains, for all $-t_0 < s \leqslant s_0 + n + 1$,

$$|\mathrm{Op}(\tau_1^7)u|_{H^s} \lesssim N^{m_1}_{0,s+m_1\wedge n}(\sigma^1) M^{m_2}_d(\sigma^2)|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}. \tag{4.28}$$

• *Proof of (i)*. Gathering estimates (4.20)–(4.21) and (4.25)–(4.28), and using standard Sobolev embeddings yields the result.
• *Proof of* (*i.bis*): When $-t_0 < s \leqslant t_0 + 1$ one can replace the r.h.s. of (4.24) by $\|\sigma^1\|_{H_n^{t_0+m_1\wedge n+1}}\|\sigma^2\|_{H^{s+m_1\wedge n}}$ and modify subsequently (4.25). Similarly, one can modify (4.26) remarking that $n_{0,s}((\sigma^1 \sharp_n \sigma^2)_n) \lesssim \|\sigma^1\|_{H_n^{t_0+1}}\|\sigma^2\|_{H^{s+n}}$. Remarking also that in (4.27) on can replace $|u|_{H^{m_1+m_2+t_0-m_1\wedge n}}$ by $|u|_{H^{s+m_1+m_2-n-1}}$ and that (4.28) can be replaced by

$$|\mathrm{Op}(\tau_1^7)u|_{H^s} \lesssim N^{m_1}_{0,t_0+n+1}(\sigma^1) M^{m_2}_d(\sigma^2)|u|_{H^{s+m_1+m_2-n-1}}$$

if one uses Proposition 25(ii) (with $r' = n + 1$) rather than Proposition 25(i), one gets (i.bis).
• *Proof of* (*i.ter*): In the proof of (i), the only estimate which is not valid when $t_0 + 1 < s \leqslant s_0 + 1$ is (4.20). To give control of τ_1^1 and τ_1^2 one now has to use the second estimate of Theorem 1 instead of the first one, whence the additional term in the final estimate.

• *Proof of* (*ii*): As said above, the control of τ_2 is deduced from the control of τ_1 by a simple permutation of σ^1 and σ^2, whence the result.
• *Proof of* (*iii*): When $n = 0$, one has $\tau_1^4 = \tau_2^4$, so that the control of both term is not needed to estimate the commutator. We keep the same control of τ_1^3 as in the proof of (i) and explain the modifications that must be performed to control τ_1^j, $j = 1, 2, 5, 6, 7$ (of course, the components of τ_2 are treated the same way).
• *Control of* $\tau_1^1(x, D)$ and $\tau_1^2(x, D)$: Since σ^2 is a function, one can invoke Proposition 26(iv) (see also Remark 27) and Lemma 17 to deduce from (4.19) that for all $0 \leqslant s \leqslant t_0 + 1$,

$$|\tau_1^j(x, D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}} |\sigma^2|_{H^{s+m_1}} |u|_\infty \qquad (j = 1, 2). \tag{4.29}$$

• *Control of* $\tau_1^5(x, D)$: Since $\sigma_1 \sharp_0 \sigma^2 = \sigma^1\sigma^2$, and because σ^1 is $2[\frac{d}{2}] + 2$-regular at the origin, we can use Propositions 16 to get $|\tau_1^5(x, D)u|_{H^s} \lesssim N_{2[\frac{d}{2}]+2,s}(\sigma^1\sigma^2)|u|_\infty$. It follows easily that for all $0 \leqslant s \leqslant t_0 + 1$,

$$|\tau_1^5(x, D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}_{\mathrm{reg}}} |\sigma^2|_{H^s} |u|_\infty. \tag{4.30}$$

• *Control of* $\tau_1^6(x, D)$: Using successively Propositions 16 and 18 as in (i) one can also obtain, for all $s \leqslant t_0 + 1$,

$$|\mathrm{Op}(\tau_1^6)u|_{H^s} \lesssim n_{0,s}(\sigma^1)|\sigma^2|_\infty |u|_{H^{s+m_1-1}}. \tag{4.31}$$

• *Control of* $\tau_1^7(x, D)$: Using successively Propositions 25(ii) (with $r' = 1$) and 18 one obtains, for all $0 \leqslant s \leqslant t_0 + 1$,

$$|\mathrm{Op}(\tau_1^7)u|_{H^s} \lesssim N^{m_1}_{0,t_0+1}(\sigma^1)|\sigma^2|_\infty |u|_{H^{s+m_1-1}}. \tag{4.32}$$

Point (iii) of the theorem follows from (4.21) and (4.29)–(4.32). □

Remark 41. Taylor proved in [17] that for all classical pseudo-differential operator $\sigma^1(x, D)$ of order $m_1 > 0$ and $\sigma^2 \in H^\infty(\mathbb{R}^d)$, one has, for all $s \geqslant 0$,

$$\Big|[\mathrm{Op}(\sigma^1), \sigma^2]u\Big|_{H^s} \leqslant \underline{C}(\sigma^1)\left(|\sigma^2|_{W^{1,\infty}}|u|_{H^{s+m_1-1}} + |\sigma^2|_{H^{s+m_1}}|u|_\infty\right).$$

This is exactly the estimate of Theorem 7(iii), which also gives a description of the constant $\underline{C}(\sigma^1)$. The commutator estimate corresponding to (i.bis) generalizes this result taking into account the smoothing effect of the Poisson bracket. It turns out that this estimate is not tame with respect to m_1, while the commutator estimate corresponding to (i), which is not *stricto sensu* of Kato–Ponce type, is tame.

We also give commutator estimates of Calderon–Coifman–Meyer type:

Theorem 8. *Let $m_1, m_2 \in \mathbb{R}$, $n \in \mathbb{N}$ and $d/2 < t_0 \leqslant s_0$. Let $\sigma^j(x,\xi) \in \Gamma^{m_2}_{s_0+m_j\wedge n+1}$ $(j=1,2)$ be n-regular at the origin.*

For all $s \in \mathbb{R}$ such that $-t_0 < s + m_j \leqslant t_0 + n + 1$ $(j = 1,2)$ and $-t_0 < s \leqslant t_0 + 1$, one has

$$\big|\big[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)\big]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\big|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+n+1}_n} \|\sigma^2\|_{H^{t_0+n+1}} |u|_{H^{s+m_1+m_2-n-1}}.$$

Proof. The proof follows closely the proof of Theorem 7, so that we just mention the adaptations that have to be made.

- *Control of $\tau_1^j(x,D)$, j=1, 2, 4, 5*: One just has to do as in the proof of Theorem 6.
- *Control of $\tau_1^6(x,D)$*: When gets easily from Propositions 16 and 18 that for all $s \leqslant t_0 + 1$, one has $|\tau_1^6(x,D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+1}} \|\sigma^2\|_\infty |u|_{H^{s+m_1+m_2-n-1}}$.
- *Control of $\tau_1^7(x,D)$*: By Propositions 25(ii) (with $r' = n+1$) and Proposition 18, one obtains $|\tau_1^7(x,D)u|_{H^s} \lesssim \|\sigma^1\|_{H^{t_0+n+1}} \|\sigma^2\|_\infty |u|_{H^{s+m_1+m_2-n-1}}$, for all $-t_0 < s \leqslant t_0 + 1$. □

Remark 42. When $n = 0$ and σ^2 is a function, the estimate of Theorem 8 is exactly the Calderon–Coifman–Meyer estimate (1.3), with extended range of validity. Theorem 8 is also more general in the sense that it allows $n > 0$ and σ^2 to be a pseudo-differential operator.

When the symbols σ^1 and σ^2 are of form (2.1), one gets the following corollary:

Corollary 43. *Let $m_1, m_2 \in \mathbb{R}$, $m := m_1 \wedge m_2$, and $d/2 < t_0 \leqslant s_0$. Let also $\sigma^j(x,\xi) = \Sigma^j(v^j(x), \xi)$ with $p_j \in \mathbb{N}$, $\Sigma^j \in C^\infty(\mathbb{R}^{p_j}, \mathcal{M}^{m_j})$ and $v_j \in H^{s_0+m\wedge n+1}(\mathbb{R}^d)^{p_j}$ $(j = 1,2)$. Assume moreover that Σ^1 and Σ^2 are n-regular at the origin.*

(i) *For all $s \in \mathbb{R}$ such that $\min\{-t_0, -t_0 - m_1, -t_0 - m_2\} \leqslant s \leqslant s_0 + 1$ the following estimate holds (writing $v := (v^1, v^2)$):*

$$\begin{aligned}&\Big|[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)\Big|_{H^s} \\ &\quad\lesssim C(|v|_{W^{n+1,\infty}})|u|_{H^{s+m_1+m_2-n-1}} \\ &\qquad + C(|v|_{W^{n+1,\infty}})\big(|v^1|_{H^{t_0+1}}|v^2|_{H^{s_++m\wedge n}} + |v^2|_{H^{t_0+1}}|v^1|_{H^{s_++m\wedge n}}\big)|u|_{H^{m+t_0}};\end{aligned}$$

(ii) *For all $s \in \mathbb{R}$ such that $-t_0 < s + m_j \leqslant t_0 + n + 1$ $(j = 1,2)$ and $-t_0 < s \leqslant t_0 + 1$, one has*

$$\begin{aligned}&\big|\big[\mathrm{Op}(\sigma^1), \mathrm{Op}(\sigma^2)\big]u - \mathrm{Op}(\{\sigma^1, \sigma^2\}_n)u\big|_{H^s} \\ &\quad\lesssim C(|v|_\infty)|\sigma^1|_{H^{t_0+n+1}}|\sigma^2|_{H^{t_0+n+1}}|u|_{H^{s+m_1+m_2-n-1}}.\end{aligned}$$

Proof. Writing $\sigma^j(x,\xi) = \left[\sigma^j(x,\xi) - \Sigma^j(0,\xi)\right] + \Sigma^j(0,\xi)$, the result follows from Lemma 8, Theorems 3 and 7 (for (i)) and Theorems 6 and 8 (for (ii)). □

Appendix A. Proof of Proposition 15

Owing to (1.7)–(1.8), we can write $(1-\psi(\xi))\sigma(x,\xi) = \sum_{q\geqslant -1} \sigma_q(x,\xi)\langle\xi\rangle^m$, with $\sigma_q(x,\xi) := (1-\psi(\xi))\sigma(x,\xi)\varphi_q(\xi)\langle\xi\rangle^{-m}$. Obviously, for all $q \geqslant -1$, $\sigma_q(x,\xi) = 0$ if $|\xi| \geqslant 2^{q+1}$ or $|\xi| \leqslant 2^{q-1}$; it follows that the function $A_q(x,\xi) := \sum_{k\in\mathbb{Z}^d} \sigma_q(x, 2^{q+1}(\xi - 2k\pi))$ is 2π-periodic with respect to ξ and coincides with $\sigma_q(x, 2^{q+1}\xi)$ in the box $\mathcal{C} := \{\xi \in \mathbb{R}^d, -\pi \leqslant \xi_j \leqslant \pi, j = 1,\dots,d\}$. Therefore, we can write $\sigma_q(x, 2^{q+1}\xi) = A_q(x,\xi)\lambda(\xi)$, where $\lambda \in C_0^\infty(\mathbb{R}^d)$ is supported in $1/5 \leqslant |\xi| \leqslant 6/5$ and $\lambda(\xi) = 1$ in $1/4 \leqslant |\xi| \leqslant 1$.

Expending $A_q(x,\xi)$ into Fourier series, one obtains

$$A_q(x,\xi) = \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^{1+[d/2]}} c_{k,q}(x) e^{i\xi\cdot k},$$

with

$$c_{k,q}(x) = (1+|k|^2)^{1+[d/2]}(2\pi)^{-d} \int_{\mathcal{C}} e^{-i\xi\cdot k} \sigma_q(x, 2^{q+1}\xi)\, d\xi,$$

so that

$$\begin{aligned}\sigma_q(x,\xi) &= A_q(x, 2^{-q-1}\xi)\lambda(2^{-q-1}\xi) \\ &= \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^{1+[d/2]}} c_{k,q}(x)\lambda_k(2^{-q}\xi),\end{aligned}$$

where $\lambda_k(\xi) := e^{i\xi\cdot k/2}\lambda(\xi/2)$ and satisfies therefore the properties announced in the statement of the proposition.

The last step is therefore to obtain the desired estimates on the Fourier coefficients $c_{k,q}$. By integration by parts, one obtains first

$$c_{k,q}(x) = (2\pi)^{-d} \int_{\mathcal{C}} e^{-i\xi\cdot k} \left[(1 - 2^{2(q+1)}\Delta_\xi)^{1+[d/2]}\sigma_q\right](x, 2^{q+1}\xi)\, d\xi,$$

which we can rewrite as

$$c_{k,q}(x) = (2\pi)^{-d} \int_{\mathcal{C}} e^{-i\xi\cdot k} \sum_{|\alpha|\leqslant 2+2[d/2]} *_\alpha 2^{(q+1)|\alpha|} (\partial_\xi^\alpha \sigma_q)(x, 2^{q+1}\xi)\, d\xi, \tag{A.1}$$

where, here and in the following, $*_\alpha$ denotes some numerical coefficient depending on α and whose precise value is not important.

Recalling that for $q \geqslant 0$ (we omit the case $q = -1$ which does not raise any difficulty), $\sigma_q(x,\xi) = \widetilde{\sigma}(x,\xi)\varphi(2^{-q}\xi)$ with $\widetilde{\sigma}(x,\xi) := (1-\psi(\xi))\sigma(x,\xi)\langle\xi\rangle^{-m}$, one obtains, for all $\alpha \in \mathbb{N}^d$,

$$\partial_\xi^\alpha \sigma_q(x,\xi) = \sum_{\alpha'+\alpha''=\alpha} *_{\alpha',\alpha''}\partial_\xi^{\alpha'}\widetilde{\sigma}(x,\xi)2^{-q|\alpha''|}(\partial_\xi^{\alpha''}\varphi)(2^{-q}\xi);$$

it follows that

$$\begin{aligned}
&2^{(q+1)|\alpha|}\partial_\xi^\alpha \sigma_q(x, 2^{q+1}\xi)\\
&\quad = \sum_{\alpha'+\alpha''=\alpha} *_{\alpha',\alpha''}\frac{2^{(q+1)|\alpha'|}}{\langle 2^{q+1}\xi\rangle^{|\alpha'|}}\langle 2^{q+1}\xi\rangle^{|\alpha'|}\partial_\xi^{\alpha'}\widetilde{\sigma}(x,2^{q+1}\xi)(\partial_\xi^{\alpha''}\varphi)(2\xi).
\end{aligned} \tag{A.2}$$

Since $\dfrac{2^{(q+1)|\alpha'|}}{\langle 2^{q+1}\xi\rangle^{|\alpha'|}} \leqslant \mathrm{Cst}$ on the support of $\partial_\xi^{\alpha''}\varphi(2\cdot)$, it follows from (A.1) and (A.2) that

$$|c_{k,q}|_{H^s} \leqslant (2\pi)^{-d}\sum_{|\alpha|\leqslant 2+2[d/2]}\ \sum_{\alpha'+\alpha''=\alpha} *_{\alpha',\alpha''}\int_{\mathcal{C}}\langle 2^{q+1}\xi\rangle^{|\alpha'|}\left|\partial_\xi^{\alpha'}\widetilde{\sigma}(\cdot,2^{q+1}\xi)\right|_{H^s}d\xi,$$

from which the estimate on $c_{k,q}$ follows. The estimate on $\varphi_p(D)c_{k,q}$ is proved in a similar way.

Appendix B. Proof of Lemma 40

Throughout this proof, we write, $\psi_p(\cdot) := \psi(2^{-p}\cdot)$, for all $p \in \mathbb{Z}$. By definition of $\sigma^1_{\widetilde{I}}$ and $\sigma^2_{\widetilde{I}}$, one has

$$\begin{aligned}
&\partial_\xi^\alpha\sigma^1_{\widetilde{I}}(\cdot,\xi)\partial_x^\alpha\sigma^2_{\widetilde{I}}(\cdot,\xi)\\
&\quad=\sum_{q\geqslant -1}\partial_\xi^\alpha\Big(\psi_{q-\widetilde{N}}(D_x)\sigma^1(\cdot,\xi)\varphi_q(\xi)(1-\psi(\xi))\Big)\\
&\qquad\times\sum_{p\geqslant -1}\psi_{p-\widetilde{N}}(D_x)\partial_x^\alpha\sigma^2(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi))\\
&\quad=\sum_{p\geqslant -1}\sum_{h=0,\pm 1}\partial_\xi^\alpha\Big(\psi_{p+h-\widetilde{N}}(D_x)\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)(1-\psi(\xi))\Big)\\
&\qquad\times\psi_{p-\widetilde{N}}(D_x)\partial_x^\alpha\sigma^2(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi)),
\end{aligned}$$

the last equality being a consequence of the fact that for all $p \geqslant -1$, one has $\varphi_p \varphi_q = 0$ for all $q \neq p, p \pm 1$.

Remarking that for all $p \geqslant -1$, the pth term of the above summation has a spectrum included in the ball $\{|\eta| \leqslant 2^{p+2-\widetilde{N}}\}$, and recalling that $\widetilde{N} = N + 3$, one deduces that

$$\partial_\xi^\alpha \sigma_{\widetilde{I}}^1(\cdot,\xi)\partial_x^\alpha \sigma_{\widetilde{I}}^2(\cdot,\xi) = \sum_{p \geqslant -1} \psi_{p-N}(D_x)\theta_p(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi)) \tag{B.1}$$

with

$$\theta_p(\cdot,\xi) := \psi_{p-N-3}(D_x)\partial_x^\alpha \sigma^2(\cdot,\xi) \times \sum_{h=0,\pm 1} \partial_\xi^\alpha \Big(\psi_{p+h-N-3}(D_x)\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)(1-\psi(\xi))\Big).$$

We now turn to study the term $(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)_I$. By definition, one has

$$\begin{aligned}(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2)_I(\cdot,\xi) &= \sum_{p \geqslant -1} \psi_{p-N}(D_x)\Big(\partial_\xi^\alpha \sigma^1 \partial_x^\alpha \sigma^2\Big)(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi)) \\ &= \sum_{p \geqslant -1} \psi_{p-N}(D_x)\Theta_p(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi)),\end{aligned}$$

with

$$\Theta_p(\cdot,\xi) := \sum_{h=0,\pm 1} \partial_\xi^\alpha \Big(\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)\Big)\partial_x^\alpha \sigma^2(\cdot,\xi),$$

where we used (1.7) and the fact that $\varphi_p \varphi_q = 0$ if $|p-q| \geqslant 2$.

Decomposing σ^1 into $\sigma^1 = \psi_{p+h-N-3}(D_x)\sigma^1 + (1-\psi_{p+h-N-3}(D_x))\sigma^1$, one obtains

$$\begin{aligned}\Theta_p(\cdot,\xi) = &\sum_{h=0,\pm 1} \partial_\xi^\alpha \Big(\psi_{p+h-N-3}(D_x)\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)\Big)\partial_x^\alpha \sigma^2(\cdot,\xi) \\ &+ \sum_{h=0,\pm 1} \partial_\xi^\alpha \Big((1-\psi_{p+h-N-3}(D_x))\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)\Big)\partial_x^\alpha \sigma^2(\cdot,\xi).\end{aligned}$$

Remark that in the first term of the r.h.s. of the above identity, one can replace $\partial_x^\alpha \sigma^2$ by $\psi_{p-N+1}(D_x)\partial_x^\alpha \sigma^2$, so that one finally gets

$$\begin{aligned}\Theta_p(\cdot,\xi) = &\ \theta_p(\cdot,\xi) \\ &+ \sum_{h=0,\pm 1} \partial_\xi^\alpha \Big(\psi_{p+h-N-3}(D_x)\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi)\Big)\end{aligned}$$

$$\times(\psi_{p-N+1}(D_x)-\psi_{p-N-3}(D_x))\partial_x^\alpha\sigma^2(\cdot,\xi)$$
$$+\sum_{h=0,\pm1}\partial_\xi^\alpha((1-\psi_{p+h-N-3}(D_x))\sigma^1(\cdot,\xi)\varphi_{p+h}(\xi))\partial_x^\alpha\sigma^2(\cdot,\xi)$$
$$:=\theta_p(\cdot,\xi)+\Theta_p^1(\cdot,\xi)+\Theta_p^2(\cdot,\xi). \tag{B.2}$$

It follows therefore from (B.1) and (B.2) that

$$\mathrm{Op}\big((\partial_\xi^\alpha\sigma^1\partial_x^\alpha\sigma^2)_I-\partial_\xi^\alpha\sigma^1_{\tilde I}\partial_x^\alpha\sigma^2_{\tilde I}\big)=\mathrm{Op}\big(\Theta^1(x,\xi))+\mathrm{Op}(\Theta^2(x,\xi)\big),$$

where $\Theta^j(\cdot,\xi)=\sum_{p\geqslant -1}\psi_{p-N}(D_x)\Theta_p^j(\cdot,\xi)\varphi_p(\xi)(1-\psi(\xi)),\ j=1,2.$

Quite obviously, the symbols $\Theta^1(x,\xi)$ and $\Theta^2(x,\xi)$ satisfy the assumptions of Lemma 19. The result follows therefore from this lemma and the estimates

$$M_d^{m_1+m_2-n-1}(\Theta^1)\leqslant \mathrm{Cst}\ M_{n+d}^{m_1}(\sigma^1)M_{d,n+1}^{m_2}(\sigma^2) \tag{B.3}$$

and

$$M_d^{m_1+m_2-n-1}(\Theta^2)\leqslant \mathrm{Cst}\ M_{n+d,n+1}^{m_1}(\sigma^1)M_{d,n}^{m_2}(\sigma^2).$$

We only prove the first of these two estimates, the second one being obtained in a similar way. One easily obtains that

$$\begin{aligned}|\Theta^1(\cdot,\xi)|_\infty &\leqslant \mathrm{Cst}\sup_{p\geqslant -1}|\Theta_p^1(\cdot,\xi)\varphi_p(\xi)|_\infty\\ &\leqslant \mathrm{Cst}\sup_{p\geqslant -1}|\partial_\xi^\alpha(\sigma^1(\cdot,\xi)\varphi_p(\xi))|_\infty\\ &\quad\times\sup_{p\geqslant -1}|(\psi_{p-N-1}(D_x)-\psi_{p-N-3}(D_x))\partial_x^\alpha\sigma^2(\cdot,\xi)\varphi_p(\xi)|_\infty.\end{aligned}$$

Since for all $r\in\mathbb{N}$, $p\geqslant -1$, and $f\in\mathcal{S}(\mathbb{R}^d)$, one has $|(\psi_{p+2}(D)-\psi_p(D))f|_\infty\leqslant \mathrm{Cst}\ 2^{-pr}|f|_{W^{r,\infty}}$, it follows that (taking $r=n+1-|\alpha|$),

$$|\Theta^1(\cdot,\xi)|_\infty\leqslant \mathrm{Cst}\ \langle\xi\rangle^{m_1+m_2-n-1}M_n^{m_1}(\sigma^1)M_{0,n}^{m_2}(\nabla_x\sigma^2).$$

The derivatives of Θ^1 with respect to ξ can be handled in the same way, thus proving (B.3).

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