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Riccati Type Transformations for Second-Order Linear Difference Equations

JOHN W. HOOKER AND WILLIAM T. PATULA

Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, Illinois 62901

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Oscillation and comparison theorems for a linear homogeneous second-order difference equation are proved by employing various equivalent non-linear equations obtained by means of transformations analogous to the Riccati transformation for ordinary differential equations.

In several recent papers, [4, 6, 7, 9–11], oscillation and non-oscillation of solutions of linear difference equations have been investigated. In this paper, we study oscillation and non-oscillation of solutions of the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \ldots, \tag{1}$$

with $c_n > 0$, $n = 0, 1, 2, \ldots$. This is accomplished by means of transformations analogous to the well-known Riccati transformation, which is often useful in the study of ordinary differential equations. In a paper by Gautschi [3, p. 30], the substitution $r_n = x_{n+1}/x_n$ is presented, whereby (1) is transformed into a first-order non-linear difference equation. This is used by Gautschi in connection with the expression of solutions of (1) in terms of continued fractions. It is this transformation, along with some variants, which we employ. Before proceeding, we make some general observations. (We first remark that the letters m, n, M, N, i, and j below always denote non-negative integer variables.)

Equation (1) is equivalent to the self-adjoint equation

$$-\Delta(c_{n-1} \Delta x_{n-1}) + a_n x_n = 0, \qquad n = 1, 2, \ldots, \tag{2}$$

where $a_n = b_n - c_n - c_{n-1}$ and the forward difference operator Δ is defined by $\Delta x_n = x_{n+1} - x_n$.

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Equations (1) and (2) are also equivalent to the difference equation

$$\alpha_n x_{n+1} + \beta_n x_n + \gamma_n x_{n-1} = 0, \qquad \alpha_n > 0, \gamma_n > 0, \tag{2'}$$

since (2′) can be transformed to the form (1) by defining the coefficients c_n inductively as $c_0 = 1$, $c_n = c_{n-1}\alpha_n/\gamma_n$, $n \geqslant 1$, with $b_n = -c_n\beta_n/\alpha_n$, $n \geqslant 1$. A non-trivial solution of (1) is called oscillatory if for every N there exists $n \geqslant N$ such that $x_n x_{n+1} \leqslant 0$. If one non-trivial solution of (1) is oscillatory then all non-trivial solutions are oscillatory (see [2, p. 153]), so Eq. (1) may be classified as being oscillatory or non-oscillatory.

Furthermore, if $\{x_n\}$ is a solution of (1) then $\{-x_n\}$ is also a solution, so it is clear that non-oscillation of (1) is equivalent to the existence of a solution which is positive for all sufficiently large n. For these and additional properties of Eq. (1) and difference equations in general we refer the reader to the books of Atkinson [1] and Fort [2].

Elementary consideration of signs in (1) immediately gives the following result (see [9, Lemma 3] or [13, Thm. 1]):

THEOREM 1. *If* $b_{n_k} \leqslant 0$ *for a sequence* $n_k \to \infty$, *then* (1) *is oscillatory.*

Because of this fact, in addition to our assumption that $c_n > 0$ for all $n \geqslant 0$, we assume the following condition throughout the remainder of this paper.

ASSUMPTION. In Eq. (1), $b_n > 0$ for $n > 0$.

(It would suffice to assume $b_n > 0$ for all sufficiently large n, but we assume the condition for all n, for simplicity of proof in the theorems below.)

Suppose that $\{x_n\}$, $n \geqslant 0$, is a solution of (1) such that $x_n \neq 0$ for $n \geqslant N$ for some N. The substitution $r_n = x_{n+1}/x_n$, $n \geqslant N$, leads to the non-linear difference equation

$$c_n r_n + c_{n-1}/r_{n-1} = b_n, \qquad n > N. \tag{3}$$

Similarly, if we let $z_n = c_n x_{n+1}/x_n$, $n \geqslant N$, then z_n satisfies

$$z_n + c_{n-1}^2/z_{n-1} = b_n, \qquad n > N. \tag{4}$$

If we let $s_n = (b_{n+1}x_{n+1})/(c_n x_n)$, $n \geqslant N$, then s_n satisfies

$$q_n s_n + 1/s_{n-1} = 1, \qquad n > N, \tag{5}$$

where $q_n = c_n^2/b_n b_{n+1}$.

The transformation $z_n = c_n x_{n+1}/x_n$ which leads to (4) above is perhaps the nearest analogue for difference equations to the classical Riccati transformation $z(t) = c(t)y'(t)/y(t)$ which transforms the self-adjoint differential equation $(cy')' + py = 0$, $c(t) \neq 0$, into the Riccati equation $z' + (1/c)z^2 + p = 0$. It is well known that this self-adjoint differential equation has a non-vanishing solution on a given interval I if and only if the related Riccati equation has a solution on I (see, for example, [12, Theorem 2.1]). It is the discrete analogue of this result which we use repeatedly in this paper to obtain oscillation and non-oscillation criteria for (1). Equation (4) may be written in the alternate form

$$\Delta z_{n-1} + (1/b_n) z_n z_{n-1} - z_n + c_{n-1}^2/b_n = 0, \qquad n > N. \tag{4'}$$

Since this is a first-order equation with a $z_n z_{n-1}$ term, we will call this and any of the related equations (3), (4), (5) difference equations of Riccati type.

It follows readily from the above transformations that Eq. (1) is non-oscillatory if and only if Eqs. (3), (4), and (5) have positive solutions. (By a "positive solution" of (3) we mean a sequence $\{r_n\}$, $n \geqslant N$ for some $N > 0$, such that r_n satisfies (3) for all $n > N$ and $r_n > 0$ for all $n \geqslant N$.) Specifically, we have the following theorem as a discrete analogue of the ordinary differential equation result mentioned above:

THEOREM 2. *The following conditions are equivalent*:

(i) *Equation* (1) *is non-oscillatory.*

(ii) *Equation* (3) *has a positive solution* $\{r_n\}$, $n \geqslant N$, *for some* $N > 0$.

(iii) *Equation* (4) *has a positive solution* $\{z_n\}$, $n \geqslant N$, *for some* $N > 0$.

(iv) *Equation* (5) *has a positive solution* $\{s_n\}$, $n \geqslant N$, *for some* $N > 0$.

Proof. If Eq. (1) is non-oscillatory and $\{x_n\}$, $n \geqslant 0$, is any solution of (1), there exists $N \geqslant 0$ such that $x_n x_{n+1} > 0$ for all $n \geqslant N$. The necessity of conditions (ii), (iii), and (iv) then follows immediately from the transformations which lead to Eqs. (3), (4), and (5).

Conversely, if $\{r_n\}$, $n \geqslant N$, is a positive solution of (3), we may let $x_N = 1$, $x_{n+1} = r_n x_n$ for all $n \geqslant N$. This defines a positive solution of (1) for $n \geqslant N$. Given x_N and x_{N+1}, the terms $x_{N-1}, x_{N-2}, \ldots, x_0$ may then be constructed directly from (1) to give a non-oscillatory solution of (1) for $n \geqslant 0$. Similar arguments hold for Eqs. (4) and (5), which complete the proof.

We now use the Riccati difference equations of conditions (ii)–(iv) in Theorem 2 to develop various conditions for oscillation and non-oscillation in terms of the coefficients of (1).

THEOREM 3. *If* $b_n \leqslant c_{n-1}$ *for all sufficiently large* n, *and if* $\limsup c_n/c_{n-1} > 1/2$, *then* (1) *is oscillatory.*

Proof. Assume that (1) is non-oscillatory. Then (3) has a positive solution $\{r_n\}$, $n \geqslant N$, for some $N > 0$. Then from our hypotheses, for some $M \geqslant N$ we have

$$c_n r_n/c_{n-1} + 1/r_{n-1} = b_n/c_{n-1} \leqslant 1, \qquad n \geqslant M. \tag{6}$$

Since $\limsup c_n/c_{n-1} > 1/2$, for some $\alpha > 1/2$ there is a sequence $n_k \to \infty$ with $c_{n_k}/c_{n_k-1} > \alpha$ for all $k \geqslant 1$. Then (6) implies

$$\alpha r_{n_k} + 1/r_{n_k-1} < 1 \tag{7}$$

for all sufficiently large k. Since all terms in (6) are positive, we have $c_n r_n/c_{n-1} < 1$ and $1/r_{n-1} < 1$ for all sufficiently large n, hence $r_n > 1$ and $1/r_n > c_n/c_{n-1}$ for all sufficiently large n. In particular, $1/r_{n_k} > c_{n_k}/c_{n_k-1} > \alpha$, and $r_{n_k} > 1$ so $\alpha r_{n_k} > \alpha$ for all sufficiently large k. It follows that each term on the left in (7) is greater than α for all sufficiently large k, so $\alpha < 1/2$, a contradiction, from which the theorem follows.

It is to be noted that the condition $b_n \leqslant c_{n-1}$ of the above theorem is not in itself sufficient to imply that (1) is oscillatory, as the following example shows:

EXAMPLE 1. Let $c_n = 1/4^n$, $n \geqslant 0$, and $b_n = c_{n-1}$, $n \geqslant 1$, so Eq. (1) becomes $(1/4^n)\,x_{n+1} + (1/4^{n-1})\,x_{n-1} = (1/4^{n-1})\,x_n$. It is readily verified that $x_n = 2^n$, $n \geqslant 0$, defines a non-oscillatory solution of this equation.

As a generalization of Theorem 3, we have the following result; the proof is omitted, since it is similar to that of Theorem 3.

THEOREM 4. *If for some $K > 0$, $b_n \leqslant K c_{n-1}$ for all sufficiently large n, and if* $\limsup_{n\to\infty} c_n/c_{n-1} > (1/2)K^2$, *then* (1) *is oscillatory.*

The next two theorems provide a pair of related conditions, one for oscillation and one for non-oscillation. In preparation for these, we have the following comparison lemma:

LEMMA 1. *Let $q_n \geqslant p_n > 0$, $n > 0$, and let $\{u_n\}$, $n \geqslant 0$, be a solution of*

$$q_n u_n + 1/u_{n-1} = 1, \tag{8}$$

with $u_n > 0$ for all $n \geqslant 0$. Then the equation

$$p_n v_n + 1/v_{n-1} = 1 \tag{9}$$

has a solution $\{v_n\}$ satisfying $v_n \geqslant u_n > 1$, for all $n \geqslant 0$.

Proof. We note first that any positive solution $\{u_n\}$ of (8) is readily seen to satisfy $u_n > 1$ for all n. This follows because (8) implies $1/u_{n-1} < 1$ for all n, hence $u_{n-1} > 1$. Given such a solution of (8), define v_n, $n \geqslant 0$, inductively, by choosing $v_0 \geqslant u_0$ and letting v_n satisfy (9) for $n > 0$. In order to be assured that v_n, $n > 0$, is well defined by (9) we need to know that $v_n \neq 0$, $n > 0$. But if $u_{n-1} \leqslant v_{n-1}$ and (9) holds, then (8) and (9) together imply

$$p_n v_n = 1 - \frac{1}{v_{n-1}} = q_n u_n + \frac{1}{u_{n-1}} - \frac{1}{v_{n-1}} \geqslant q_n u_n,$$

so

$$v_n \geqslant \frac{q_n u_n}{p_n} \geqslant u_n > 1,$$

since $q_n \geqslant p_n > 0$ by hypothesis. Therefore the sequence $\{v_n\}$ is well defined and is a solution of (9) by definition. Thus, for all $n > 0$, v_n satisfies (9) and the inequality $v_n \geqslant u_n$, which completes the proof.

THEOREM 5. *If $b_n b_{n+1} \leqslant (4-\varepsilon)c_n^2$ for some $\varepsilon > 0$ for all sufficiently large n, then* (1) *is oscillatory.*

Proof. If the hypothesis holds for some $\varepsilon \geqslant 4$, this theorem follows trivially from Theorem 1. Thus we assume that $0 < \varepsilon < 4$. Suppose (1) is non-oscillatory. Then (5) has a positive solution $\{s_n\}$, $n \geqslant N$, for some $N > 0$, i.e., s_n satisfies $q_n s_n + 1/s_{n-1} = 1$ for $n \geqslant N$, where $q_n = c_n^2/b_n b_{n+1}$. Since $q_n \geqslant (4-\varepsilon)^{-1}$ by hypothesis, the preceding lemma implies that the equation

$$(4-\varepsilon)^{-1} v_n + 1/v_{n-1} = 1 \tag{10}$$

has a solution $\{v_n\}$, $n \geqslant N$, which satisfies $v_n \geqslant s_n > 1$ for all $n \geqslant N$. We now define a positive sequence $\{x_n\}$, $n \geqslant N$, inductively, by letting $x_N = 1$, $x_{n+1} = (4-\varepsilon)^{-1/2} v_n x_n$ for $n \geqslant N$. Then $v_n = (4-\varepsilon)^{1/2} x_{n+1}/x_n$, and substituting this into Eq. (10) we find that $\{x_n\}$ is a positive solution of the equation

$$x_{n+1} + x_{n-1} = (4-\varepsilon)^{1/2} x_n, \qquad n > N. \tag{11}$$

But this is impossible because Eq. (11) is oscillatory, since it has the solutions $\{\cos n\theta\}$ and $\{\sin n\theta\}$, $n \geqslant 1$, where $\theta = \tan^{-1}(\varepsilon/4-\varepsilon)^{1/2}$. (In fact, (11) is essentially the equation which determines the possible shapes of a weighted vibrating string with equally spaced weights and fixed endpoints [2, pp. 168–170].) Thus we have a contradiction, and the theorem follows.

The following example shows that the inequality condition in Theorem 5 cannot in general be replaced by the weaker condition $b_n b_{n+1} \leqslant (4-\varepsilon_n)c_n^2$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$.

EXAMPLE 2. *Consider the equation*

$$x_{n+1} + x_{n-1} = b_n x_n, \qquad n = 1, 2, \ldots,$$

where

$$b_n = \frac{(n+1)^{1/2} + (n-1)^{1/2}}{n^{1/2}}.$$

This equation is of the form (1) *and is non-oscillatory, since it obviously ha the solution* $x_n = n^{1/2}$, $n = 1, 2, \ldots$. *It is readily verified that* $b_n < 2$ *and* $b_n \to$: *as* $n \to \infty$, *hence* $b_n b_{n+1} < 4$ *and* $\varepsilon_n = 4 - b_n b_{n+1} \to 0$ *as* $n \to \infty$. *Thus w have* $b_n b_{n+1} = 4 - \varepsilon_n$, *but the difference equation is non-oscillatory. It i interesting to note that in self-adjoint form this equation become* $\Delta^2 x_{n-1} - a_n x_n = 0$, *where* $a_n = n^{-1/2}\Delta^2((n-1)^{1/2})$. *This may be thought o as a discrete analogue of the well-known example of a non-oscillator differential equation* $y'' + (1/4t^2)y = 0$; *this has* $t^{1/2}$ *and* $t^{1/2}\log t$ *a solutions, and the coefficient of* y *is* $1/4t^2 = -t^{-1/2}(t^{1/2})''$.

We now turn to a non-oscillation criterion which is a companion to th oscillation condition of Theorem 5. Theorem 5 and Theorem 6 together shov that the constant 4 in each of the theorems is the best possible.

THEOREM 6. *If* $b_n b_{n+1} \geqslant 4c_n^2$ *for all sufficiently large n, then* (1) *is non oscillatory.*

Proof. Assume $b_n b_{n+1} \geqslant 4c_n^2$ for all $n \geqslant N$. Then $q_n \leqslant 1/4$, where q_n i defined as in Eq. (5). Construct a solution $\{s_n\}$ of (5) inductively by definin $s_N = 2$ and

$$s_n = \frac{1}{q_n}\left(1 - \frac{1}{s_{n-1}}\right), \qquad n > N. \tag{12}$$

We note that if $s_{n-1} \geqslant 2$ for any $n \geqslant N$, then $q_n s_n = 1 - (1/s_{n-1}) \geqslant 1/2$, s $s_n \geqslant (1/2)q_n^{-1} \geqslant 4 \cdot 1/2 = 2$. Therefore the sequence $\{s_n\}$, $n \geqslant N$ is wel defined by (12), and it is readily verified that s_n satisfies (5). We thus have positive solution of (5), so (1) is non-oscillatory by Theorem 2.

COROLLARY 1. *If* $b_n \geqslant \max(c_{n-1}, 4c_n)$ *for all sufficiently large n, the* (1) *is non-oscillatory.*

Proof. If $b_n \geqslant \max(c_{n-1}, 4c_n)$ for all sufficiently large n, then $b_{n+1} \geqslant c$ and $b_n \geqslant 4c_n$, so $b_n b_{n+1} \geqslant 4c_n^2$ for all sufficiently large n, hence the corollar follows from Theorem 6.

We note that the inequality conditions of Theorem 5 and 6 may be restated to apply directly to Eq. (2′) as $\beta_n\beta_{n+1} \leqslant (4-\varepsilon)\,\alpha_n\gamma_{n+1}$, $\beta_n\beta_{n+1} \geqslant 4\alpha_n\gamma_{n+1}$, respectively. In this light, we see that these conditions are generalizations of discriminant conditions for the constant coefficient case. That is, $x_n = r^n$ is a solution of $\alpha x_{n+1} + \beta x_n + \gamma x_{n-1} = 0$ if and only if r is a root of the characteristic equation $\alpha r^2 + \beta r + \gamma = 0$, so this difference equation is non-oscillatory if and only if $\beta^2 \geqslant 4\alpha\gamma$, [2, pp. 125–126], an analogue of an elementary non-oscillation condition for constant coefficient linear second order ordinary differential equations.

We note also that in case $c_n \equiv 1$, the non-oscillation condition in Theorem 6 becomes $b_n b_{n+1} \geqslant 4$. In this case, the self-adjoint form (2) is

$$\Delta^2 x_{n-1} - (b_n - 2)x_n = 0,$$

and we see that in this sense Theorem 6 may be thought of as a generalization of the discrete analogue of the simple condition for ordinary differential equations that $x'' - p(t)x = 0$ is non-oscillatory if $p(t) \geqslant 0$.

THEOREM 7. *If* $c_{n_k}^2 \geqslant b_{n_k} b_{n_k+1}$ *for a sequence* $n_k \to \infty$, *then* (1) *is oscillatory.*

Proof. If (1) is non-oscillatory then Eq. (5) has a positive solution $\{s_n\}$, $n \geqslant N$ for some N. From (5), $q_n s_n < 1$ and $s_n > 1$ for all $n > N$, so $q_n < 1$ for $n > N$, i.e., $c_n^2 < b_n b_{n+1}$ for $n > N$. Thus if $c_n^2 \geqslant b_n b_{n+1}$ for arbitrarily large values of n, (1) must be oscillatory, which completes the proof.

From this theorem one readily obtains several sufficient conditions for (1) to be oscillatory. In Corollaries 2, 3, and 4 we define q_n as in Eq. (5), i.e.,

$$q_n = \frac{c_n^2}{b_n b_{n+1}}, \qquad n > 0.$$

COROLLARY 2. *If* $\limsup_{n\to\infty} q_n > 1$, *then* (1) *is oscillatory.*

Proof. Follows immediately from Theorem 7.

COROLLARY 3. *If* $\limsup_{n\to\infty} (1/n) \sum_{j=1}^{n} q_j > 1$, *then* (1) *is oscillatory.*

Proof. If (1) is non-oscillatory, then $c_n^2 < b_n b_{n+1}$ for all sufficiently large n, say $n \geqslant N$, by Theorem 7. For all $n \geqslant N$, we then have $q_n = c_n^2/(b_n b_{n+1}) < 1$, so $\sum_{j=N}^{n} q_j < n - N + 1$. It follows that $(1/n) \sum_{j=1}^{n} q_j < 1 + K/n$ for some constant K. This leads to a contradiction of our hypothesis, so the corollary follows.

COROLLARY 4. *If* $\sum_{j=1}^{\infty} q_j^{-k} < \infty$ *for some* $k > 0$, *then* (1) *is oscillatory.*

Proof. Let $\sum_{j=1}^{\infty} q_j^{-k} = B$, $0 < B < \infty$, for some $k > 0$. For any $p > 1$ choose p' such that $1/p + 1/p' = 1$. Using the Hölder inequality we obtain

$$n = \sum_{j=1}^{n} q_j^{1/p} q_j^{-1/p} \leqslant \left(\sum_{j=1}^{n} (q_j^{1/p})^p \right)^{1/p} \left(\sum_{j=1}^{n} (q_j^{-1/p})^{p'} \right)^{1/p'}.$$

Then

$$n^{1/p} n^{1/p'} = n \leqslant \left(\sum_{j=1}^{n} q_j \right)^{1/p} \left(\sum_{j=1}^{n} q_j^{1-p'} \right)^{1/p'},$$

from which it follows that

$$\frac{1}{n} \sum_{j=1}^{n} q_j \geqslant \left[\frac{n}{\sum_{j=1}^{n} q_j^{1-p'}} \right]^{p/p'}.$$

In particular, if we choose $p = (1+k)/k$ and $p' = 1 + k$, the preceding inequality implies that

$$\frac{1}{n} \sum_{j=1}^{n} q_j \geqslant \left(\frac{n}{B} \right)^{1/k},$$

so $\lim_{n\to\infty} (1/n) \sum_{j=1}^{n} q_j = \infty$. Therefore (1) is oscillatory, by Corollary 3.

COROLLARY 5. *If* $\sum_{n=1}^{\infty} (b_n/c_{n-1})^k < \infty$ *for some* $k > 0$, *and* $c_n/c_{n-1} \geqslant \varepsilon$ *for some* $\varepsilon > 0$, *for all sufficiently large* n, *then* (1) *is oscillatory.*

Proof. If $c_n/c_{n-1} \geqslant \varepsilon$ then $c_n \geqslant \varepsilon c_{n-1}$ for all $n \geqslant N$ for some N. We may also assume N is large enough so that $(b_{n+1}/c_n)^k \leqslant 1$ for $n \geqslant N$, since $\sum^{\infty} (b_n/c_{n-1})^k < \infty$. Thus

$$\sum_{n=N}^{\infty} \left(\frac{b_n b_{n+1}}{c_n^2} \right)^k \leqslant \sum_{n=N}^{\infty} \left(\frac{b_{n+1}}{c_n} \right)^k \cdot \left(\frac{b_n}{\varepsilon c_{n-1}} \right)^k < \infty,$$

hence $\sum_{j=1}^{\infty} q_j^{-k} < \infty$. Therefore (1) is oscillatory by Corollary 4.

COROLLARY 6. *If* $\sum_{n=1}^{\infty} (b_n/c_n)^k < \infty$ *for some* $k > 0$, *and* $c_{n-1}/c_n \geqslant \varepsilon$ *for some* $\varepsilon > 0$, *for all sufficiently large* n, *then* (1) *is oscillatory.*

Proof. Similar to the proof of Corollary 5.

COROLLARY 7. *If* $\sum_{n=1}^{\infty} (b_n/c_{n-1})^k < \infty$ *for some* $k > 0$, *and* $b_n \leqslant c_n$ *for all sufficiently large* n, *then* (1) *is oscillatory.*

Proof. If $b_n \leqslant c_n$, then $b_n b_{n+1}/c_n^2 \leqslant b_{n+1}/c_n$, and the corollary then follows immediately from Corollary 4. Similarly we have:

COROLLARY 8. *If* $\sum_{n=1}^{\infty}(b_n/c_n)^k < \infty$ *for some* $k > 0$, *and* $b_{n+1} \leqslant c_n$ *for all sufficiently large* n, *then* (1) *is oscillatory.*

Our next theorem is related to an earlier result in [9]. We will need the following lemma:

LEMMA 2. *If* (1) *is non-oscillatory and* $\prod_{i=1}^{n} b_i/\prod_{i=1}^{n} c_i$ *is bounded as* $n \to \infty$, *then all solutions of* (1) *are bounded.*

Proof. Let (1) be non-oscillatory and let $\{x_n\}$, $n \geqslant 1$, be a solution of (1) with $x_n > 0$ for all $n \geqslant N$. Then Eq. (4) has a solution $\{z_n\}$ with $z_n = c_n x_{n+1}/x_n > 0$ for all $n \geqslant N$. From (4) we have

$$z_n = b_n - c_{n-1}^2/z_{n-1} < b_n$$

for all $n > N$, so $\prod_{i=N+1}^{n} z_i < \prod_{i=N+1}^{n} b_i$, $n > N$, that is,

$$\prod_{i=N+1}^{n} c_i x_{i+1}/x_i < \prod_{i=N+1}^{n} b_i, \qquad n > N.$$

It follows that

$$x_{n+1} < x_{N+1} \prod_{i=N+1}^{n} b_i/ \prod_{i=N+1}^{n} c_i, \qquad n > N,$$

and therefore $\{x_n\}$ is bounded. If $\{x_n\}$ is a solution of (1) which is eventually negative, then $\{-x_n\}$ is a solution which is eventually positive, hence all solutions of (1) are bounded, which completes the proof.

THEOREM 8. *If* $\sum_{n=1}^{\infty} c_n^{-1} = \infty$ *and* $\prod_{i=1}^{n} b_i/\prod_{i=1}^{n} c_i$ *is bounded as* $n \to \infty$, *then* (1) *is oscillatory.*

Proof. Suppose the above conditions hold and (1) is non-oscillatory. Then all solutions of (1) are bounded, by Lemma 2. But by Corollary 2 of [9], if $\sum_{n=1}^{\infty} c_n^{-1} = \infty$ and if all solutions of (1) are bounded, then (1) must be oscillatory. This contradicts our supposition, and the theorem follows.

(Example 1 above and a similar example with $b_n = c_n = 4^n$ and $x_n = 2^{-n}$ show that neither of these conditions alone implies oscillation.)

Finally, we prove a comparison result similar to [8, Thm. 1] and [10, Thm. 2]. For ordinary differential equations, some comparison results of this type can be found in a paper by Read [11]. Here we compare solutions of Eq. (1) with those of the equation

$$r_n y_{n+1} + r_{n-1} y_{n-1} = d_n y_n, \qquad n \geqslant 1, \tag{1'}$$

with $r_n > 0$, $n \geqslant 0$, and $c_n \geqslant r_n$ and $b_n \leqslant d_n$ for all sufficiently large n. In light of Theorem 1, we continue to assume $b_n > 0$ and hence $d_n > 0$ for large n.

although these conditions are not explicitly used. We assume below that (1) is non-oscillatory, while in [10, Thm. 2] it is assumed that $b_n \geqslant c_n + c_{n-1}$ and $d_n \geqslant r_n + r_{n-1}$. As noted in [9], these inequalities imply non-oscillation of (1) and (1′), respectively; thus the hypotheses below are weaker than those of [10, Thm. 2].

THEOREM 9. *Assume $c_n \geqslant r_n$ and $b_n \leqslant d_n$ for all sufficiently large n. Then if* (1) *is non-oscillatory,* (1′) *is also non-oscillatory. Furthermore, if $\{x_n\}$ is a solution of* (1) *with $x_n > 0$ for all $n \geqslant N$ and if $\{y_n\}$ is a solution of* (1′) *satisfying $r_N y_{N+1}/y_N \geqslant c_N x_{N+1}/x_N$ with $y_N > 0$, then $y_{n+1}/y_n \geqslant x_{n+1}/x_n$ for all $n \geqslant N$. If, in addition, $y_N \geqslant x_N$, then $y_n \geqslant x_n$ for all $n \geqslant N$.*

Proof. Assume $c_n \geqslant r_n$ and $b_n \leqslant d_n$ for all $n \geqslant N$. Given $\{x_n\}$ and $\{y_n\}$ as above, let $z_n = c_n x_{n+1}/x_n$ for all $n \geqslant N$. Then (1) implies

$$z_{n+1} = b_{n+1} - c_n^2/z_n, \qquad n \geqslant N. \tag{13}$$

Let $w_n = r_n y_{n+1}/y_n$ for all $n \geqslant N$ such that $y_n \neq 0$. Then for $n \geqslant N$ such that y_n and y_{n+1} are non-zero, w_n and w_{n+1} are defined, $w_n \neq 0$, and (1′) implies

$$w_{n+1} = d_{n+1} - r_n^2/w_n. \tag{14}$$

For such values of n, we subtract (13) from (14), adding and subtracting c_n^2/w_n, to obtain

$$w_{n+1} - z_{n+1} = \left(d_{n+1} - b_{n+1} + \frac{c_n^2 - r_n^2}{w_n}\right) + \frac{c_n^2}{z_n w_n}(w_n - z_n). \tag{15}$$

From the hypotheses, $z_n > 0$ for all $n \geqslant N$. If $w_n \geqslant z_n$, the right-hand side of (15) is then non-negative, hence $w_{n+1} \geqslant z_{n+1} > 0$. In particular, from the hypotheses, $w_N \geqslant z_N$ and y_N and y_{N+1} are positive. Thus w_{N+1} is defined, and (15) implies $w_{N+1} \geqslant z_{N+1} > 0$. Furthermore, $y_{N+2} > 0$ since $y_{N+2} = w_{N+1} y_{N+1}/r_{N+1}$, and hence w_{N+2} is defined. Proceeding inductively, we conclude that $w_n \geqslant z_n$ and $y_n > 0$ for all $n \geqslant N$. It follows that (1′) is non-oscillatory and that

$$y_{n+1}/y_n \geqslant (c_n/r_n)\, x_{n+1}/x_n \geqslant x_{n+1}/x_n \tag{16}$$

for all $n \geqslant N$. Finally, if $y_N \geqslant x_N$, (16) implies that $y_{N+1} \geqslant (y_N/x_N) x_{N+1} \geqslant x_{N+1}$. Proceeding inductively, we obtain $y_n \geqslant x_n$ for all $n \geqslant N$, which completes the proof.

Theorem 9 affords an immediate proof of the following result involving difference inequality conditions (cf. [4, Thm. 9.1]).

COROLLARY 9. *Let* (1) *be non-oscillatory. Suppose there exist positive sequences* $\{u_n\}$ *and* $\{v_n\}$ *satisfying*

$$c_n u_{n+1} + c_{n-1} u_{n-1} \leqslant b_n u_n, \qquad n \geqslant N, \tag{17}$$

and

$$c_n v_{n+1} + c_{n-1} v_{n-1} \geqslant b_n v_n, \qquad n \geqslant N. \tag{18}$$

If $v_{N+1}/v_N \geqslant u_{N+1}/u_N$, *then* (1) *has a solution* $\{x_n\}$ *satisfying*

$$v_{n+1}/v_n \geqslant x_{n+1}/x_n \geqslant u_{n+1}/u_n, \qquad n \geqslant N.$$

If, in addition, $v_N \geqslant x_N \geqslant u_N$, *then* $v_n \geqslant x_n \geqslant u_n$, $x \geqslant N$.

Proof. Given $\{u_n\}$ and $\{v_n\}$ as stated, we define sequences $\{B_n\}$ and $\{D_n\}$ by

$$c_n u_{n+1} + c_{n-1} u_{n-1} = B_n u_n, \qquad n \geqslant N,$$

and

$$c_n v_{n+1} + c_{n-1} v_{n-1} = D_n v_n, \qquad n \geqslant N.$$

Then $B_n \leqslant b_n \leqslant D_n$, $n \geqslant N$. Let $\{x_n\}$ be the solution of (1) satisfying $x_N = u_N$, and $x_{N+1} = u_{N+1}$. The conclusion then follows immediately from Theorem 9.

Similarly, one may use Theorem 9 to obtain the following version of [2, Thm. IV, p. 223] (where, as noted by Hartman in [4], the sense of the inequality should be reversed):

COROLLARY 10. *Equation* (1) *is non-oscillatory if and only if there exists a sequence* $\{u_n\}$ *satisfying* $u_n > 0$ *and* $c_n u_{n+1} + c_{n-1} u_{n-1} \leqslant b_n u_n$ *for all sufficiently large n.*

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REFERENCES

1. F. V. ATKINSON, "Discrete and Continuous Boundary Problems," Academic Press, New York, 1964.
2. T. Fort, "Finite Differences and Difference Equations in the Real Domain," Oxford University Press, London, 1948.

3. W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Rev.* **9** (1967), 24–82.
4. P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, *Trans. Amer. Math. Soc.* **246** (1978), 1–30.
5. P. Hartman and A. Wintner, On linear difference equations of the second order, *Amer. J. Math.* **72** (1950), 124–128.
6. D. Hinton and R. Lewis, Spectral analysis of second order difference equations, *J. Math. Anal. Appl.* **63** (1978), 421–438.
7. P. J. McCarthy, Note on oscillation of solutions of second-order linear difference equations, *Port. Math.* **18** (1959), 203–205.
8. F. W. J. Olver, Bounds for the solutions of second order linear difference equations, *J. Res. Nat. Bur. Standards Sect. B* **71** (1967), 161–166.
9. W. T. Patula, Growth and oscillation properties of second order linear difference equations, *SIAM J. Math. Anal.* **10** (1979), 55–61.
10. W. T. Patula, Growth, oscillation and comparison theorems for second order linear difference equations, *SIAM J. Math. Anal.* **10** (1979), 1272–1279.
11. T. T. Read, Bounds and qualitative comparison theorems for non-oscillatory second order differential equations, *Pacific J. Math.* **63** (1) (1976), 231–245.
12. W. T. Reid, "Riccati Differential Equations," Academic Press, New York, 1972.
13. A. Wouk, Difference equations and *J*-matrices, *Duke Math. J.* **20** (1953), 141–159.