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Regular 2-Graphs and Extensions of Partial Geometries

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We study so-called 2-graph geometries. These are geometries that carry a regular 2-graph, but also constitute 1-point extensions of partial geometries. First we develop some theory; then we go through lists of known regular 2-graphs and partial geometries in order to find examples. Some are found, including one that extends the partial geometry with parameters $s = 4$, $t = 17$ and $\alpha = 2$.

1. INTRODUCTION

Extensions of t-designs, especially 1-point extensions, have been studied a lot in the past. More recently, people have become interested in extensions of finite geometries, such as generalised quadrangles (there exist various papers on this subject; Cameron [7] gives a survey) or, more generally, partial geometries (see [18]). The points of a partial geometry carry a strongly regular graph, while regular 2-graphs are, in a certain sense, extensions of strongly regular graphs by one point. For this reason it seemed worthwhile to investigate a combination of these objects, being 1-point extensions of partial geometries with the structure of a regular 2-graph on the points. We call such structures 2-graph geometries. The present paper is a first attempt to study these geometries.

The reader is assumed to be familiar with the theory of designs and strongly regular graphs (see, for instance, [8]). We shall briefly survey the relevant results on 2-graphs and partial geometries. A *2-graph* (Ω, Δ) consists of a finite set Ω, together with a set Δ of unordered triples (called *coherent* triples) from Ω, such that every 4-subset of Ω contains an even number of coherent triples. Let ∇ denote the set of non-coherent triples. Then (Ω, ∇) is also a 2-graph, called the *complement* of (Ω, Δ). The 2-graph (Ω, Δ) is *empty* if Δ is empty and *complete* if ∇ is empty; (Ω, Δ) is *regular* if every pair of points from Ω is contained in a constant number a of coherent triples. For any point $\omega \in \Omega$ of a regular 2-graph (Ω, Δ), the matrix A_ω, defined by

$$(A_\omega)_{\beta\gamma} = \begin{cases} -1 & \text{if } \{\beta, \gamma, \omega\} \in \Delta, \text{ or } \omega \in \{\beta, \gamma\}, \text{ and } \beta \neq \gamma \\ 0 & \text{if } \beta = \gamma, \\ 1 & \text{if } \{\beta, \gamma, \omega\} \in \nabla, \end{cases}$$

has just 2 eigenvalues, ρ_1 and ρ_2 ($\rho_1 > \rho_2$). These eigenvalues have opposite sign and are odd integers if $\rho_1 \neq -\rho_2$; furthermore,

$$|\Omega| = 1 - \rho_1\rho_2, \qquad a = -(\rho_1 + 1)(\rho_2 + 1)/2.$$

The *derived graph* Γ_ω of (Ω, Δ) with respect to ω has vertex set $\Omega\backslash\{\omega\}$, 2 vertices β and γ being adjacent if $\{\beta, \gamma, \omega\} \in \Delta$. So, by deleting row and column ω from A_ω, we obtain the $(-1, 1, 0)$ adjacency matrix of Γ_ω. For any ω, the derived graph of a regular 2-graph (not complete or empty) is strongly regular with parameters (V, K, Λ, M), where

$$\begin{aligned} V &= |\Omega| - 1 = -\rho_1\rho_2, \qquad K = a = -(\rho_1 + 1)(\rho_2 + 1)/2, \\ \Lambda &= 1 - (\rho_1 + 3)(\rho_2 + 3)/4, \qquad M = K/2 = -(\rho_1 + 1)(\rho_2 + 1)/4. \end{aligned} \tag{1}$$

Conversely, a strongly regular graph with parameters $(V, K, \Lambda, K/2)$, extended by an

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isolated vertex, gives a regular 2-graph when triples are defined to be coherent if they have 1 or 3 edges. A *clique* (or coherent set) of (Ω, Δ) is a subset c of Ω, such that every triple from c is coherent. If $\omega \in c$, $c\backslash\{\omega\}$ is a clique (in the normal sense) of Γ_ω. If $\omega \notin c$, then c is a clique of Γ_ω, or consists of 2 disconnected cliques in Γ_ω. A clique c of (Ω, Δ) satisfies:

$$|c| \leq 1 - \rho_2. \tag{2}$$

2-graphs have been introduced by G. Higman, and were studied mainly by Seidel and Taylor [25, 26, 29].

A design is denoted by the pair (Φ, B), where Φ is the set of points and B is the set of blocks. An *anti-flag* of a design (Φ, B) is a pair (φ, b) with $\varphi \in \Phi$, $b \in B$ and $\varphi \notin b$.

A *partial geometry* $pg(s, t, \alpha)$ is a $1-(V, s+1, t+1)$ design (Φ, B), where any 2 distinct lines (= blocks) meet in at most 1 point, such that for every anti-flag (φ, b) there are precisely α points on b collinear with φ. It follows that $V = |\Phi| = (s+1)(st+\alpha)/\alpha$, $|B| = (t+1)(st+\alpha)/\alpha$. If we interchange the roles of points and lines we obtain the *dual* partial geometry $pg(t, s, \alpha)$. The *point graph* of a $pg(s, t, \alpha)$ has vertex set Φ; 2 vertices are adjacent if they are collinear. The point graph of a $pg(s, t, \alpha)$ is strongly regular with parameters $(V, K, \Lambda, M) = (V, s(t+1), s-1+t(\alpha-1), \alpha(t+1))$. A *1-point extension* of $pg(s, t, \alpha)$ is a design for which the derived design with respect to any point is a $pg(s, t, \alpha)$.

Partial geometries were introduced by Bose [2] and have been studied considerably. Some general references are [3] and [11].

2. 2-Graph Geometries

Definition 1. A *2-graph geometry* is a 2-(v, k, λ) design (Ω, C) satisfying the following properties:

(i) two distinct blocks of C have at most 2 points in common (therefore blocks are called circles);

(ii) any set of 4 points contains an even number of cocircular triples,

(iii) $v = 1 + (k-1)(2\lambda - 1)$.

Proposition 2. *Let Δ be the set of cocircular triples of a 2-graph geometry (Ω, C). Then (Ω, Δ) is a regular 2-graph with eigenvalues*

$$\rho_1 = 2\lambda - 1, \qquad \rho_2 = 1 - k.$$

Proof. By (i) and (ii) (Ω, Δ) is a 2-graph. Since (Ω, C) is a 2-(v, k, λ) design, (Ω, Δ) is regular with $\alpha = -(\rho_1+1)(\rho_2+1)/2 = \lambda(k-2)$. Using $v = |\Omega| = 1 + (k-1)(2\lambda-1) = 1 - \rho_1\rho_2$ and $\rho_1 > 0 > \rho_2$ the values of ρ_1 and ρ_2 follow. □

Note that we did not use property (iii) to prove that (Ω, Δ) is a regular 2-graph; it is only used to compute ρ_1 and ρ_2. In fact, once (Ω, Δ) is defined, property (iii) can be replaced by:

(iii′) the circles of C are maximal cliques of (Ω, Δ).

Herein maximal means that the bound $-\rho_2 + 1$ given in (2) is met. As usual, the number of circles is denoted by b and the number of circles through a fixed point by r. Then $\lambda(v-1) = r(k-1)$ and $bk = vr$ yield

$$r = \lambda(2\lambda - 1) = \rho_1(\rho_1 + 1)/2,$$
$$b = \lambda(2\lambda-1)^2 - 2\lambda(2\lambda-1)(\lambda-1)/k = \rho_1^2(\rho_1+1)/2 + \rho_1(\rho_1^2-1)/2(\rho_2-1).$$

We call a regular 2-graph *geometric* if it corresponds to a 2-graph geometry. Clearly, for a 2-graph to be geometric the following divisibility condition must be satisfied:

$$2(-\rho_2+1) \mid \rho_1(\rho_1^2-1). \tag{3}$$

The following result is straightforward:

PROPOSITION 3. *A regular 2-graph with eigenvalues ρ_1 and ρ_2 is geometric iff there exists a set C of cliques of size $(1-\rho_2)$, such that every coherent triple is covered by a unique clique of C.*

A regular 2-graph with $\rho_2=-1$ is empty. Also for the next case, $\rho_2=-3$, 2-graph geometries are nothing special, because of the following result:

PROPOSITION 4. *Let C be the set of all 4-cliques of a regular 2-graph (Ω, Δ) with $\rho_2=-3$. Then (Ω, C) is the unique 2-graph geometry corresponding to (Ω, Δ).*

PROOF. In a regular 2-graph with eigenvalues ρ_1 and ρ_2 each coherent triple is contained in exactly $\Lambda=1-(\rho_1+3)(\rho_2+3)/4$ cliques of size 4, by use of (1). This number Λ equals 1 if $\rho_2=-3$; hence Proposition 3 gives the result. □

Seidel's [23] determination of all regular 2-graphs with $\rho_2=-3$ leads to:

COROLLARY 5. *2-graph geometries with $\rho_2=-3$ (i.e. $k=4$) exist iff $\rho_1=1$, 3, 5 or 9 (i.e. $\lambda=1$, 2, 3 or 5) and are unique.*

Note that 2-graph geometries with $\rho_1=1$ are degenerate: there is just one circle of size $k=v$, and the 2-graphs are complete.

Let (ω, c) be an anti-flag of a design (Ω, C). The *anti-flag graph* $\Gamma_{\omega,c}$ has vertex set c: 2 vertices β and γ $(\beta\neq\gamma)$ are adjacent whenever ω, β and γ are covered by a block of C.

PROPOSITION 6. *A 2-(v, k, λ) design (Ω, C) with block intersection sizes at most 2 is a 2-graph geometry iff each anti-flag graph is the disjoint union of 2 complete graphs of size $k/2$.*

PROOF. Suppose that (Ω, C) is a 2-graph geometry. Let (ω, c) be an anti-flag of (Ω, C) and let β, γ and δ be 3 distinct points of c. Since $\{\beta, \gamma, \delta, \omega\}$ contains an even number of cocircular triples, the subgraph of $\Gamma_{\omega,c}$ induced by β, γ and δ is either a triangle or has just one edge. Thus $\Gamma_{\omega,c}$ is the complete graph or the disjoint union of 2 complete graphs. Conversely, it is easily seen that any 4-set contains 0, 2 or 4 cocircular triples if each anti-flag graph is the disjoint union of two complete graphs.

Next fix $c\in C$. For $\omega\in\Omega\backslash c$, let m_ω denote the size of a component of $\Gamma_{\omega,c}$. Counting in two ways the total number of triples (ω, β, γ) with $\omega\in\Omega\backslash c$, and β, γ adjacent vertices in $\Gamma_{\omega,c}$ gives

$$\sum_{\omega\notin c}(m_\omega(m_\omega-1)+(k-m_\omega)(k-m_\omega-1))=k(k-1)(\lambda-1)(k-2).$$

The left-hand side is at least $\hat{l}=(v-k)k(\frac{1}{2}k-1)$ with equality iff $m_\omega=k/2$ for all $\omega\notin c$. This proves the result, because $\hat{l}$ equals the right-hand side, precisely when $v=1+(k-1)(2\lambda-1)$. □

By definition, a 2-design is a 1-point extension of a partial geometry $pg(s, t, \alpha)$ iff any 2 distinct blocks meet in at most 2 points and each anti-flag graph is regular of degree α. Therefore we have:

THEOREM 7. *A 2-graph geometry (Ω, C) with eigenvalues ρ_1 and ρ_2 is a 1-point extension of a partial geometry with parameters*

$$s = -\rho_2 - 1, \qquad t = (\rho_1 - 1)/2, \qquad \alpha = (-\rho_2 - 1)/2.$$

So, only partial geometries with $s = 2\alpha$ occur. Clearly, the point graph of the partial geometry with respect to $\omega \in \Omega$ (say) is the derived graph Γ_ω of (Ω, Δ). Such strongly regular graphs satisfy $K = 2\Lambda$ ((V, K, Λ, M) is the set of parameters), which is equivalent to $s = 2\alpha$.

Suppose that a $pg(2\alpha, t, \alpha)$ (P, L) can be extended to a 2-graph geometry $(P \cup \{\omega\}, C)$. Then, of course, each line of L extended by ω is a circle of C. Each other circle c of C has to be an arc (no three points are collinear) of (P, L) of size $2\alpha + 2$. Property 6 implies that c admits a partitioning into 2 classes of size $\alpha + 1$, such that 2 points are collinear iff they are in the same class.

It is clear that the anti-flag graph of a 1-point extension of a $pg(s, t, 1)$ (i.e. a generalised quadrangle) consists of disjoint edges. So by Proposition 6 we have:

PROPOSITION 8. *A 1-point extension of a $pg(2, t, 1)$ is a 2-graph geometry.*

This result need not be true for $\alpha > 1$. For instance, there exist 1-point extensions of $pg(4, 1, 2)$ for which some anti-flag graphs are hexagons: so they are not 2-graph geometries. But no other exceptions are known (to us). The 2-graph geometries corresponding to the above proposition have $\rho_2 = -3$: so, by Corollary 5, we have the following result due to Buekenhout [4] (see also [30]).

COROLLARY 9. *1-point extensions of $pg(2, t, 1)$ exist and are unique.*

3. REGULAR 2-GRAPHS AND PARTIAL GEOMETRIES

Next we investigate known or feasible regular 2-graphs and partial geometries with $s = 2\alpha$. We more or less follow the surveys of [25] and [11]. Since the point graph of a partial geometry with $s = 2\alpha$ is a strongly regular graph with $K = 2M$, the regular 2-graph exists if the partial geometry exists. However, the converse is not true.

Case 1: *$\rho_1 = -\rho_2 - 2$ or, equivalently, $t = \alpha - 1$.* The regular 2-graph corresponds to a regular symmetric Hadamard matrix with constant diagonal. The corresponding partial geometries are duals of block designs with $\lambda = 1$. In such a partial geometry any 2 lines meet. This implies that 2 circles of the 2-graph geometry can only have no or 2 points in common. The 2-graph geometry is therefore a quasi-symmetric block design. The divisibility condition (3) leads to $\rho_1 = 1$, 3 or 9. The case $\rho_1 = 1$, $\rho_2 = -3$ is treated in Corollary 5. For the other two cases the parameters $(\rho_1, \rho_2, v, b, k, r, \lambda, s, t, \alpha)$ are $(3, -5, 16, 16, 6, 6, 2, 4, 1, 2)$ and $(9, -11, 100, 375, 12, 45, 5, 10, 4, 5)$. The first one is a 2-$(16, 6, 2)$ design. There exist precisely 3 such designs, but only one satisfies condition 2.1(ii), viz. the unique 2-$(16, 6, 2)$ design with characteristic 3 (see [6]). Nothing is known about the second case. Mavron and Shrikhande [22] also found the above-mentioned possibilities in their classification of quasi-symmetric block designs with block intersection sizes 0 and 2 and an additional requirement, a little weaker than condition 2.1(ii).

Case 2: $\rho = -\rho_2$ *or, equivalently,* $t = \alpha$. The regular 2-graphs are the ones associated to conference matrices with integral eigenvalues. The partial geometries $pg(s, s/2, s/2)$ are dual nets; they correspond to $(s-2)/2$ mutually orthogonal latin squares of order $s+1$. The parameters are:

$$v = \rho_1^2 + 1, \qquad b = \rho_1(\rho_1^2+1)/2, \qquad k = \rho_1 + 1, \qquad r = \rho_1(\rho_1+1)/2,$$
$$\lambda = (\rho_1+1)/2, \qquad s = \rho_1 - 1, \qquad t = \alpha = (\rho_1 - 1)/2.$$

It was observed by Fisher [15] that such 2-graph geometries can be constructed from the inversive plane over the field with ρ_1 elements: so they exist whenever ρ_1 is an odd prime power. The required set of circles is just one orbit of the group generated by the inversions acting on the blocks (circles) of the inversive plane. Wilbrink [33] proved that the corresponding 2-graphs are the Paley 2-graphs. The derived partial geometry is the corresponding half of the affine plane derived from the original inversive plane. The lines of this partial geometry are the only maximal cliques in the Paley graph (the Paley graph is the derived graph of the Paley 2-graph with respect to any point), see Blokhuis [1]. This implies that the Paley 2-graph is geometric in a unique way; the circles are just all maximal cliques.

The above cases together with Corollary 5 cover all 2-graph designs the corresponding partial geometry of which is improper (a $pg(s, t, \alpha)$ is improper if $\alpha = 1$, s, $s+1$, t or $t+1$; in our case $s = 2\alpha$, so $\alpha = s$ or $s+1$ is impossible). For the remaining cases $\alpha < t$ holds, which implies $\rho_1 > -\rho_2$.

complements of the ones considered in Case 1. *Partial geometries with* $t = \alpha + 1$ are classified by De Clerck [10]. The parameters for this case are:

$$v = (\rho_1 - 1)^2, \qquad b = \rho_1(\rho_1^2 - 1)/2, \qquad k = \rho_1 - 1, \qquad r = \rho_1(\rho_1+1)/2,$$
$$\lambda = (\rho_1+1)/2, \qquad s = \rho_1 - 3, \qquad t = (\rho_1 - 1)/2, \qquad \alpha = (\rho_1 - 3)/2.$$

A $pg(2\alpha, \alpha+1, \alpha)$ can be constructed from a projective plane of order $2\alpha+2$ possessing a hyperoval. Such planes do exist if the order is a power of 2. If $\alpha = 1$ the 2-graph geometry exists (Corollary 5). If $\alpha = 2$ or 4 the partial geometry does not exist (by [10] and [20] respectively). The regular 2-graph is known for many more values of $\rho_1 = 2\alpha + 3$ than the corresponding partial geometry, including for $\alpha = 2$ and $\alpha = 4$. The symplectic 2-graphs, for instance, belong to this case. By [14] they are not geometric if $\rho_1 = 9$ or 17. The smallest candidate has parameters $\rho_1 = 9$, $\rho_2 = -7$, $v = 64$, $b = 360$, $k = 8$, $r = 45$, $\lambda = 5$, $s = 6$, $t = 4$ and $\alpha = 3$. By [21], there are just two such partial geometries. Storme [28] showed by computer that both $pg(6, 4, 3)$ do not have arcs of size 8 with the structure required for circles in the extension (one $pg(6, 4, 3)$ was also done by Tonchev [32]). Therefore this 2-graph geometry does not exist. Storme [28] also showed that the $pg(14, 8, 7)$ corresponding to the hyperoval in the Desarguesian plane of order 16 cannot be extended to a 2-graph geometry.

Case 4: $\rho_1 = 2^m - 1$, $\rho_2 = -2^{m-1} - 1$ *or, equivalently,* $\alpha = 2^{m-2}$, $t = 2^{m-1} - 1$. By [13], partial geometries with these parameters are known if m is even. The 2-graphs exist for all $m > 1$, they are the complements of the orthogonal 2-graphs $\Omega^+(2m, 2)$ (see [24]). By condition (3) only $\rho_1 = 3$, 7 and 15 are possible. The first possibility exists (Corollary 5), the second one does not (see Case 3), and the remaining one has parameters $\rho_2 = -9$, $v = 136$, $k = 10$, $b = 1632$, $r = 120$, $\lambda = 8$, $s = 8$, $t = 7$ and $\alpha = 4$. Tonchev [32] showed by computer that the known $pg(8, 7, 4)$ (see [9] and [17] for other ways to construct this partial geometry) does not extend to a 2-graph geometry. It is conjectured that $pg(8, 7, 4)$ is unique; therefore the 2-graph geometry probably does not exist.

Case 5: $\rho_1 = 2^m + 1$, $\rho_2 = -2^{m-1} + 1$ *or, equivalently,* $t = 2^{m-1}$, $\alpha = 2^{m-2} - 1$. For the other parameters we find

$$v = 2^{2m-1} - 2^{m-1}, \qquad b = (2^{2m} - 1)(2^m + 1), \qquad k = 2^{m-1},$$
$$r = (2^m + 1)(2^{m-1} + 1), \qquad \lambda = 2^{m-1} + 1, \qquad s = 2^{m-1} - 2.$$

The corresponding regular 2-graphs are the orthogonal 2-graphs $\Omega^-(2m, 2)$ (see [24]). Corollary 5 takes care of $m = 3$. For $m > 3$ existence of the partial geometry is still open. De Clerck and Tonchev [12] showed that for $m = 4$ the corresponding $pg(6, 8, 3)$ can only have automorphisms of order 2 and 3, not leaving much hope of finding the 2-graph geometry.

Case 6: $\rho_1 = \rho_2^2$ *or, equivalently,* $t = 2\alpha(\alpha + 1)$. Regular 2-graphs with these eigenvalues were constructed by Taylor [29] whenever $\sqrt{\rho_1}$ is an odd prime power. The remaining parameters are:

$$v = \rho_1\sqrt{\rho_1} + 1, \qquad b = \rho_1(\rho_1 + 1)(\rho_1 - \sqrt{\rho_1} + 1)/2, \qquad k = \sqrt{\rho_1} + 1,$$
$$r = \rho_1(\rho_1 + 1)/2, \qquad \lambda = (\rho_1 + 1)/2, \qquad s = 2\alpha = \sqrt{\rho_1} - 1, \qquad t = (\rho_1 - 1)/2.$$

Again, the first one exists ($\rho_1 = 9$, $\rho_2 = -3$) by Corollary 5. For $\rho_2 = -5$ and $\rho_2 = -7$ Spence [27] proved that the derived strongly regular graph is not geometric. For $-\rho_2 > 7$ nothing is known. Also, the Ree groups provide regular 2-graphs with these eigenvalues whenever $-\rho_2$ is an odd power of 3: we have no idea whether these 2-graphs can be geometric.

Case 7: $\rho_2 = -5$ *or, equivalently,* $\alpha = 2$. Then $k = 6$, $s = 4$ and $\alpha = 2$. For ρ_1, 15, 19, 35 and 55 are the only possible values that have not been considered before. If $\rho_1 = 15$ or 19 neither the regular 2-graph nor the partial geometry is known to exist. For $\rho_1 = 35$ a 2-graph geometry is realised in the next section. For $\rho_1 = 55$ there is a unique regular 2-graph and a unique derived strongly regular graph. Nevertheless, the existence of the partial geometry and the 2-graph geometry is as yet unsolved.

4. A Sporadic 2-Graph Geometry

In this section we construct a 2-graph geometry (Ω, C) with parameters $\rho_1 = 35$, $\rho_2 = -5$, $v = 176$, $b = 18\,480$, $k = 6$, $r = 360$, $\lambda = 18$, $s = 4$, $t = 17$ and $\alpha = 2$. The regular 2-graph (Ω, Δ) is the one having the Higman–Sims group HS acting on Ω as a 2-transitive automorphism group (see [29]). The partial geometry (with respect to any point) is the one constructed by the author [16]. The group of the 2-graph geometry will be the Mathieu group M_{22}, which is a subgroup of HS, and the corresponding action on Ω is rank 3. For the construction we need some properties of this action.

LEMMA 10. *The action of* M_{22} *on* (Ω, Δ) *satisfies the following*:
(*i*) *There exists an orbit* C *of size* 18 480 *on the* 6-*cliques of* (Ω, Δ).
(*ii*) *Every triple from* Δ *is contained in a* 6-*clique of* C.

PROOF. Fix a point $\omega \in \Omega$. The subgroup of M_{22} stabilising ω is A_7. It is an automorphism group of Γ_ω (the full automorphism group of Γ_ω is $P\Sigma U(3, 5^2)$). We can define Γ_ω on the edges of the Hoffman–Singleton graph ('HoSi' for short), where 2 edges are adjacent whenever they are disjoint and possess an interconnecting edge (see [19]). The group of automorphisms of HoSi that fixes (setwise) a distinguished 15-coclique is A_7. Its action on the edges is the action on Γ_ω, just mentioned. This description is worked out in some detail in [16], in order to construct $pg(4, 17, 2)$. Using this description, the following facts are straightforward:

(a) The 5-cliques of Γ_ω are 1-factors in Petersen subgraphs of HoSi.

(b) The group A_7 has 2 orbits on the Petersen subgraphs of HoSi. The sizes are 105 (the 'special Petersen graphs' in [16]) and 420.

(c) The subgroup of A_7 that stabilises any Petersen subgraph P of HoSi acts transitively on the 1-factors of P.

So A_7 has 2 orbits on the 5-cliques of Γ_ω; one of size 630 (the lines of $pg(4, 17, 2)$), and one of size 2520. Any 5-clique of Γ_ω extended by ω is a 6-clique of (Ω, Δ). Thus (remembering that M_{22} acts transitively on Ω) there are $176 \times 3150/6 = 92\,400$ 6-cliques in (Ω, Δ), and M_{22} is either transitive on these 6-cliques or has two orbits, one of size 18 480 and one of size 73 920 respectively. However, the first option cannot occur, since 92 400 does not divide the order of M_{22} $(=443\,520)$. This proves (i). Next let $\{\omega, \beta, \gamma\} \in \Delta$. Then $\{\beta, \gamma\}$ is an edge of Γ_ω, which is contained in a (unique) 5-clique of the smaller orbit (i.e. a line of $pg(4, 17, 2)$). Hence $\{\omega, \beta, \gamma\}$ is contained in a 6-clique of C, proving (ii). □

THEOREM 11. *With C as in Lemma* 10, (Ω, C) *is a 2-graph geometry.*

PROOF. By (i) of the lemma, the cliques of C cover at most $18\,480 \times 20 = 369\,600$ coherent triples. However, this is precisely the total number of coherent triples. Therefore, by (ii), every triple is covered exactly once by a clique of C, so (Ω, C) is a 2-graph geometry by Proposition 3. □

It is clear that the parameters of this 2-graph geometry are the ones mentioned above. There must be several ways to prove Lemma 10. For instance, an alternative proof could follow the lines of the construction by Calderbank and Wales [5] of $pg(4, 17, 2)$; unlike HoSi, they start from the Steiner system $S(5, 8, 24)$. An approach that does not use either of these two construction methods of $pg(4, 17, 2)$, would give a new way of describing this partial geometry.

It has been checked (using a computer) that there is a unique way to extend the $pg(4, 17, 2)$ to a 2-graph geometry such that all automorphisms (A_7) of the partial geometry are preserved.

5. CONCLUDING REMARKS

Apart from the existence questions mentioned in Section 3, there are several other problems that look interesting. We mention a few here.

We do not know examples of non-isomorphic 2-graph geometries with the same parameters. For $\rho_2 = -3$ they are unique (Corollary 5). It is premature to conjecture that 2-graph geometries are necessarily unique, but it seems safe to do so for the sporadic one of the previous section, because of the remark at the end (it is even conceivable that the regular 2-graph and the partial geometry are unique).

The relation between 2-graphs and Seidel switching leads to a class of $(-1, 1, 0)$ incidence matrices of a 2-graph geometry in the following way. Consider the incidence matrix N or (Ω, C). Let Γ be a (strong) graph in the switching class of (Ω, Δ). Each circle of C corresponds to a disjoint union of 2 complete graphs in Γ. Sign the non-zero entries of each column of N with + and − according to the partition of the corresponding circle, just described. The matrix obtained in this manner has some interesting properties; for instance, its rank equals the multiplicity of ρ_2. It is not clear how this can be explored.

If, for a 2-graph geometry, $2(-\rho_2+1)$ divides ρ_1^2-1 (compare with (3)), then it is feasible that a subset of the circles forms a partial geometry $pg(-\rho_2, (\rho_1-1)/2, (-\rho_2+1)/2)$ $(=pg(s+1, t, \alpha+1))$. For Fisher's 2-graph geometries (Section 3, Case 2) this would mean that a subset of the circles is the dual of a 2-$((\rho_1^2+1)/2, (\rho_1+1)/2, 1)$ design. If $\rho_1=3$, this is possible: however, Thas [31] proved that it is impossible for $\rho_1>3$. An affirmative answer for the sporadic example of Section 4 would give a new partial geometry $pg(5, 17, 3)$.

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