The fixed point property for weakly admissible compact convex sets: searching for a solution to Schauder’s conjecture

Nguyen To Nhu

Institute of Mathematics at Hanoi, P.O. Box 631, Bo Ho, Hanoi, Viet Nam

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Abstract

A compact convex set $X$ in a linear metric space is weakly admissible if for every $\varepsilon > 0$ there exist compact convex subsets $X_1, \ldots, X_n$ of $X$ with $X = \text{conv}(X_1 \cup \cdots \cup X_n)$ and continuous maps $f_i$ from $X_i$ into finite dimensional subsets $E_i$, $i = 1, \ldots, n$, of $X$ such that $\sum_{i=1}^n \|f_i(x_i) - x_i\| < \varepsilon$ for every $x_i \in X_i$, and $i = 1, \ldots, n$.

Theorem: Any weakly admissible compact convex set has the fixed point property.

Question: Is every weakly admissible compact convex set an AR?

Keywords: Compact convex sets; The fixed point property; Admissible convex sets; Weakly admissible convex sets; AR-property

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1. Introduction

We are concerned with the following problem known as Schauder’s conjecture:

Problem 1 (Schauder). Does every compact convex set in a linear metric space have the fixed point property?

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1 Current address: Department of Mathematics, Rawles Hall, Indiana University, Bloomington, IN 47405, USA. E-mail: nhnguyen@indiana.edu.

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Schauder's conjecture is one of the most resistant open problems in the theory of nonlocally convex linear metric spaces: In fact Schauder posed Problem 1 in The Scottish Book in 1935 (see [8, Problem 54]) but it remains open, despite great efforts by analysts and topologists for many decades.

It is of interest to know that Schauder's conjecture is still unproved even in some very special cases: For instance, it is not known whether compact convex sets in the spaces $L_p, 0 < p < 1$, have the fixed point property. We recall that $L_p, 0 < p < 1$, are defined by:

$$L_p = \left\{ f : [0, 1] \to R, \|f\| = \int_0^1 |f(t)|^p dt < \infty \right\}, \text{ if } 0 < p < 1; \text{ and}$$

$$L_0 = \left\{ f : [0, 1] \to R, \|f\| = \int_0^1 \frac{|f(t)|}{1 + |f(t)|} dt < \infty \right\}, \text{ if } p = 0.$$

In previous paper [13] we have shown that all Roberts spaces have the fixed point property. (By a Roberts space we mean any compact convex set with no extreme points constructed by Roberts' method of needle point spaces [14,15,3,5]. See [12,9,11] for some results about the AR-property for Roberts spaces.) Let us observe that after Roberts constructed an example of a compact convex set with no extreme points [14] in 1975, his example became the main target for attacking Schauder's conjecture.

In this paper we continue our search for a solution to Problem 1. We introduce the notion of weak admissibility and prove that weak admissibility implies the fixed point property for compact convex sets in linear metric spaces. Our result reveals a new class of compact convex sets with the fixed point property. We hope that the result of this paper will pave the way for a solution to Schauder's conjecture, one of the most important problems in fixed point theory.

**Notation and conventions.** By a linear metric space we mean a topological vector space $X$ which is metrizable. The zero element of $X$ is denoted by $\theta$. We equip $X$ with an $F$-norm $\| \cdot \|$ such that, see [16]

$$\|\lambda x\| \leq \|x\| \text{ for every } x \in X \text{ and } \lambda \in R \text{ with } \|\lambda\| \leq 1.$$

Let $A$ be a subset of a linear metric space $X$. By $\text{conv } A$ we denote the convex hull of $A$ in $X$. We also use the following notation

$$\|x - A\| = \inf\{\|x - y\| : y \in A\} \text{ for } x \in X.$$

Let $X$ be a linear metric space. By a convex polyhedron we mean the convex hull of a finite set in $X$. By a polyhedron we mean a finite union of convex polyhedra. Therefore any polyhedron used in this paper is finite dimensional.
2. Weak admissibility and the fixed point property for weakly admissible compact convex sets

In this section we introduce the notion of weak admissibility. Our definition, of course, comes from the notion of admissibility of Klee [6,7].

**Definition.** We say that a compact convex set $X$ in a linear metric space is *weakly admissible* if for every $\varepsilon > 0$ there exist compact convex subsets $X_1, \ldots, X_n$ of $X$ with $X = \text{conv}(X_1 \cup \cdots \cup X_n)$ and continuous maps $f_i$ from $X_i$ into finite dimensional subsets $E_i$, $i = 1, \ldots, n$, of $X$ such that

$$\sum_{i=1}^{n} \|f_i(x_i) - x_i\| < \varepsilon \quad \text{for every} \quad x_i \in X_i, \quad \text{and} \quad i = 1, \ldots, n.$$  

Observe that $n = n(\varepsilon)$ depends on $\varepsilon$. We say that $X$ is *admissible*, see [6,7], if one can take $n = n(\varepsilon) = 1$ for any $\varepsilon > 0$.

Our result in this paper is the following:

**Theorem.** Any weakly admissible compact convex set has the fixed point property.

Our Theorem reduces Problem 1 to:

**Question 1.** Is every compact convex set weakly admissible?

**Remark.** Klee [6,7] proved that any admissible compact convex set is an AR. We do not know whether this stronger property holds true for weak admissibility.

**Question 2.** Is every weakly admissible compact convex set: (i) an AR? (ii) admissible?

Let us observe that the AR-problem is still unsolved even in the class of compact convex sets with the fixed point property, see [13]. Namely, we ask the following question which is somewhat more general than Question 2(i).

**Question 3** [13]. Assume that $X$ is a compact convex set with the fixed point property. Is $X$ an AR?

Following Kalton and Peck [3] let us say that a compact convex set $X$ in a linear metric space is *small generated* if there exists a point $x_0 \in X$ such that $X = \text{conv} S(x_0, \varepsilon)$ for every $\varepsilon > 0$, where

$$S(x_0, \varepsilon) = \{x \in X: \|x - x_0\| < \varepsilon\}.$$  

Observe that all Roberts spaces are small generated and weakly admissible. It is of interest to know whether the result of [13] can be extended for small generated compact convex sets. Our Theorem suggests the following question.

**Question 4.** Is every small generated compact convex set weakly admissible?
From the Theorem and from [10] we get

**Corollary.** A compact convex set $X$ has the fixed point property if the following condition holds: For every $\varepsilon > 0$ there exist compact convex sets $X_1, \ldots, X_n$ and finite dimensional convex subsets $E_1, \ldots, E_n$ of $X$ such that $X = \text{conv}(X_1 \cup \cdots \cup X_n)$ and

$$\sum_{i=1}^{n} (1 + \dim E_i) \|x_i - E_i\| < \varepsilon$$

for every $x_i \in X_i, i = 1, \ldots, n.$

**Proof.** Since $E_i, i = 1, \ldots, n,$ are finite dimensional convex sets, we have $\dim E_i = \dim \overline{E_i},$ where $\overline{E_i}$ denotes the closure of $E_i$ in $X.$ The following result, established in [10], shows that $X$ is weakly admissible.

**Proposition 1.** Let $X$ be an $n$-dimensional closed convex subset in a linear metric space $E.$ Then there is a retraction $r : E \to X$ such that

$$\|r(x) - x\| \leq 2(n + 1)\|x - X\| \text{ for every } x \in X.$$

3. **Proof of the main result**

In this section we prove the Theorem. Our approach is very elementary and self-contained: In fact we do not use any special results other than some basic definitions in the theory of linear metric spaces. Therefore our proof can be easily understood by nonexperts.

We start with the following fact.

**Lemma.** Let $X$ be an infinite dimensional convex set in a linear metric space and let $A_i, i = 1, \ldots, n,$ be finite subsets of $X$ and $\varepsilon > 0.$ Then there exist disjoint finite sets $B_i \subset X, i = 1, \ldots, n,$ such that

(i) $B = \bigcup_{i=1}^{n} B_i$ is a linearly independent subset of $X.$

(ii) $\|x - \text{conv } A_i\| < \varepsilon$ for every $x \in \text{conv } B_i, i = 1, \ldots, n.$

(iii) There exists an affine map $h : \text{conv } A \to \text{conv } B,$ where $A = \bigcup_{i=1}^{n} A_i,$ such that

$$\|x - h(x)\| < \varepsilon \text{ for every } x \in \text{conv } A.$$

**Proof.** For every $i = 1, \ldots, n,$ let $K_i$ be a triangulation of $\text{conv } A_i,$ and let $K$ be a triangulation of $\text{conv } A$ such that $A_i \subset K_i^0,$ $A \subset K^0,$ where $K^0$ denotes the set of all vertices of $K,$ and

$$\bigcup_{i=1}^{n} K_i^0 = K^0. \quad (1)$$

Observe that $\text{card } \sigma^0 \leq \text{card } K^0$ for every $\sigma \in K,$ where $\sigma^0$ denotes the set of all vertices of a simplex $\sigma \in K.$

Since $\dim X = \infty,$ there exists a linearly independent subset $B_1 = \{h(v) : v \in K_1^0\} \subset X.$
such that
$$\|h(v) - v\| < \left(\text{card } \mathcal{K}^0\right)^{-1} \varepsilon \quad \text{for every } v \in \mathcal{K}^0.$$

By induction we choose linearly independent subsets $B_k \subseteq X$ such that
$$B_k = \left\{ h(v): \quad v \in \mathcal{K}_k^0 \setminus \bigcup_{i=1}^{k-1} \mathcal{K}_i^0 \right\} \subseteq X \setminus \text{span } \bigcup_{i=1}^{k-1} B_i, \quad \text{for } k = 2, \ldots, n; \quad (2)$$
$$\|h(v) - v\| < \left(\text{card } \mathcal{K}^0\right)^{-1} \varepsilon \quad \text{for every } v \in \mathcal{K}_k^0. \quad (3)$$

Let $B = \bigcup_{i=1}^n B_i$. Since $B_i, i = 1, \ldots, n,$ are linearly independent, from (2) it follows that $B$ is a linearly independent finite subset of $X$. Let us check (ii): For every $x \in \text{conv } B_i$ we have
$$x = \sum_{j=1}^k \alpha_j h(v_j), \quad \text{where } v_j \in \text{conv } A_i, \alpha_j \in [0,1], j = 1, \ldots, k, \text{ and } \sum_{j=1}^k \alpha_j = 1.$$

Let $y = \sum_{j=1}^k \alpha_j v_j \in \text{conv } A_i$. Observe that $k \leq \text{card } \mathcal{K}^0$. Then from (3) we get
$$\|x - y\| = \left\| \sum_{j=1}^k \alpha_j (h(v_j) - v_j) \right\| \leq \sum_{j=1}^k \|h(v_j) - v_j\| < k(\text{card } \mathcal{K}^0)^{-1} \varepsilon \leq \varepsilon.$$

Therefore
$$\|x - \text{conv } A_i\| < \varepsilon \quad \text{for every } x \in \text{conv } B_i, \quad i = 1, \ldots, n.$$

Consequently condition (ii) is established.

We now extend $h$ linearly on each simplex $\sigma \in \mathcal{K}$ to an affine map
$$h: \text{conv } A \rightarrow \text{conv } B.$$

We need to verify condition (iii).

For every $x \in \text{conv } A$, where $A = \bigcup_{i=1}^n A_i \subseteq \mathcal{K}^0$, we have
$$x = \sum_{i=1}^k \alpha_i v_i, \quad \text{where } v_i \in A, \alpha_i \in [0,1], \quad i = 1, \ldots, k, \text{ and } \sum_{i=1}^k \alpha_i = 1.$$

Since $k \leq \text{card } \mathcal{K}^0 = \text{card } \bigcup_{i=1}^n \mathcal{K}_i^0$, from (3) we get
$$\|x - h(x)\| = \left\| \sum_{i=1}^k \alpha_i (v_i - h(v_i)) \right\| < k(\text{card } \mathcal{K}^0)^{-1} \varepsilon \leq \varepsilon,$$
and condition (iii) holds. Consequently the lemma is proved.

**Proof of Theorem.** Let $X$ be a weakly admissible compact convex set. Assume to the contrary that there exists a continuous map $f: X \rightarrow X$ such that $f(x) \neq x$ for every $x \in X$. By the compactness of $X$ there exists an $\varepsilon > 0$ such that
$$\|f(x) - x\| \geq \varepsilon \quad \text{for every } x \in X. \quad (4)$$
Our strategy is to construct, under condition (4), a finite dimensional compact convex set \( F \subset X \) and a continuous map \( \psi: F \to F \) without any fixed points. This contradicts Brower's fixed point theorem for finite dimensional compact convex sets.

Since \( X \) is weakly admissible there exist compact convex subsets \( X_1, \ldots, X_n \) of \( X \) with \( X = \text{conv}(X_1 \cup \cdots \cup X_n) \) and continuous maps \( f_i \) from \( X_i \) into finite dimensional convex subsets \( E_i \) of \( X \), \( i = 1, \ldots, n \), such that

\[
\sum_{i=1}^{n} \|f_i(x_i) - x_i\| < 2^{-4} \varepsilon \quad \text{for every } x_i \in X_i, \text{ and } i = 1, \ldots, n. \tag{5}
\]

Claim 1. For every \( i = 1, \ldots, n \), there exists a continuous map \( g_i \) from \( f_i(X_i) \) into a convex polyhedron \( F_i \subset X \) such that

\[
\sum_{i=1}^{n} \|g_i(y) - y\| < 2^{-4} \varepsilon \quad \text{for every } y \in f_i(X_i), \text{ and } i = 1, \ldots, n. \tag{6}
\]

Proof. Since \( f_i(X_i) \) is a finite dimensional compact set there exists a finite open cover \( \mathcal{U}_i \) of \( f_i(X_i) \) such that

\[
\text{diam } U < (1 + D_i)^{-1} n^{-1} 2^{-4} \varepsilon \quad \text{for every } U \in \mathcal{U}_i, \tag{7}
\]

where \( D_i = \dim f_i(X_i) \), \( i = 1, \ldots, n \), and

\[
\text{ord } \mathcal{U}_i \leq 1 + D_i. \tag{8}
\]

Let \( \{ \lambda_U: U \in \mathcal{U}_i \} \) be a partition of unity inscribed into \( \mathcal{U}_i \), see, for instance [1]. For every \( U \in \mathcal{U}_i \), we select \( x_U \in U \) and denote

\[
F_i = \text{conv}\{ x_U: U \in \mathcal{U}_i \}.
\]

We now define a map \( g_i: f_i(X_i) \to F_i \) by the formula:

\[
g_i(y) = \sum_{U \in \mathcal{U}_i} \lambda_U(y) x_U \quad \text{for every } y \in f_i(X_i).
\]

From (7), (8) we obtain

\[
\sum_{i=1}^{n} \|g_i(y) - y\| = \sum_{i=1}^{n} \left\| \sum_{U \in \mathcal{U}_i} \lambda_U(y) (x_U - y) \right\|
\leq \sum_{i=1}^{n} \sum_{U \in \mathcal{U}_i(y)} \|x_U - y\| \quad \text{(where } \mathcal{U}_i(y) = \{ U \in \mathcal{U}_i: y \in U \})
\leq \sum_{i=1}^{n} \text{ord } \mathcal{U}_i \|x_U - y\|
< n \text{ ord } \mathcal{U}_i (1 + D_i)^{-1} n^{-1} 2^{-4} \varepsilon
\leq 2^{-4} \varepsilon
\]

for every \( y \in f_i(X_i) \), and \( i = 1, \ldots, n \). The claim is proved.
From (5), (6) we get
\[ \sum_{i=1}^{n} \left\| g_i f_i(x_i) - x_i \right\| < 2^{-3} \varepsilon \quad \text{for every } x_i \in X_i, \text{ and } i = 1, \ldots, n. \]  

(9)

Denote \( F' = \text{conv} \bigcup_{i=1}^{n} F_i \). Then \( F \) is a finite dimensional compact convex polyhedron in \( X \). Let \( f|F: F \to X \). Our next step is to approximate \( f \) by an affine map \( g \) from \( F \) into a polyhedron in \( X \).

**Claim 2.** There exists an affine map \( g \) from \( F \) into a convex polyhedron \( G \subset X \) such that
\[ \left\| f(x) - g(x) \right\| < 2^{-3} \varepsilon \quad \text{for every } x \in F. \]  

(10)

**Proof.** Let \( \mathcal{K} \) be a triangulation of \( F \) such that
\[ \text{diam } f(\sigma) < 2^{-3}(1 + \text{dim } F)^{-1} \varepsilon \quad \text{for every } \sigma \in \mathcal{K}. \]  

(11)

Let \( g(v) = f(v) \) for every \( v \in \mathcal{K}^0 \) and let \( G = \text{conv} \{ g(v): v \in \mathcal{K}^0 \} \). Then we extend \( g \) linearly on each simplex \( \sigma \in \mathcal{K} \) to an affine map \( g: F \to G \). We claim that \( g \) satisfies condition (10).

Observe that for every \( x \in F \) there exists a simplex \( \sigma \in \mathcal{K} \) such that
\[ x = \sum_{i=1}^{n} \alpha_i v_i, \quad \text{where } v_i \in \sigma^0, \alpha_i \in [0, 1], \text{ and } \sum_{i=1}^{n} \alpha_i = 1. \]

Since \( n \leq 1 + \text{dim } F \), from (11) we get
\[ \left\| f(x) - g(x) \right\| = \left\| f(x) - \sum_{i=1}^{n} \alpha_i g(v_i) \right\| \leq \sum_{i=1}^{n} \left\| \alpha_i (f(x) - f(v_i)) \right\| \leq \sum_{i=1}^{n} \left\| f(x) - f(v_i) \right\| \leq n \text{diam } f(\sigma) < 2^{-3} \varepsilon \]

for every \( x \in F \). The claim is proved.

Let \( G^0 = \{ g(v): v \in \mathcal{K}^0 \} \). Then \( G = \text{conv} G^0 \). Observe that for every \( x \in G^0 \) we have \( x = g(v) \) for some \( v \in \mathcal{K}^0 \). Since
\[ G^0 \subset X = \text{conv}(X_1 \cup \cdots \cup X_n), \]

we get
\[ x = \sum_{i=1}^{k} \alpha_i(v) x_i(v), \]

where \( x_i(v) \in X_i, \alpha_i(v) \in [0, 1], i = 1, \ldots, k, \text{ and } \sum_{i=1}^{k} \alpha_i(v) = 1. \]
Let $A_i = \{x_i(v) : v \in K^0\} \subset X_i$, $i = 1, \ldots, n$. Then $G^0 \subset \text{conv} \bigcup_{i=1}^n A_i$. Therefore

$$G \subset \text{conv} A, \quad \text{where } A = \bigcup_{i=1}^n A_i. \quad (12)$$

For every $i = 1, \ldots, n$, let $r_i : X \rightarrow \text{conv} A_i$ be a retraction. Take $\delta, 0 < \delta < 2^{-3}\varepsilon$, such that

$$\|r_i(x) - x\| < 2^{-3}n^{-1}\varepsilon \quad \text{whenever } \|x - \text{conv} A_i\| < \delta, i = 1, \ldots, n. \quad (13)$$

From (4) it follows that $\dim X = \infty$. Therefore by the Lemma there exist disjoint finite sets $B_i = \{b_{ij}, j = 1, \ldots, n(i)\}, i = 1, \ldots, n$, of $X$ with the following properties:

$$B = \bigcup_{i=1}^n B_i = \{b_{ij}, j = 1, \ldots, n(i), i = 1, \ldots, n\} \quad (14)$$

is a linearly independent subset of $X$;

$$\|x - \text{conv} A_i\| < \delta \quad \text{for every } x \in \text{conv} B_i, i = 1, \ldots, n. \quad (15)$$

There exists an affine map $h : \text{conv} A \rightarrow \text{conv} B$, where $A = \bigcup_{i=1}^n A_i$, such that

$$\|x - h(x)\| < 2^{-3}\varepsilon \quad \text{for every } x \in \text{conv} A. \quad (16)$$

From (13), (15) we get

$$\|x - r_i(x)\| < 2^{-3}n^{-1}\varepsilon \quad \text{for every } x \in \text{conv} B_i, i = 1, \ldots, n. \quad (17)$$

**Proposition 2.** There exists a continuous map $\varphi : \text{conv} B \rightarrow F$ such that

$$\|x - \varphi(x)\| < 2^{-2}\varepsilon \quad \text{for every } x \in \text{conv} B.$$

**Proof.** Since $B$ is linearly independent, for every $x \in \text{conv} B$ there exist unique $\lambda_{ij} \in [0, 1]$ and $x_{ij} \in B_i, j = 1, \ldots, n(i), i = 1, \ldots, n$, with

$$\sum_{i=1}^n \sum_{j=1}^{n(i)} \lambda_{ij} = 1 \quad \text{such that } x = \sum_{i=1}^n \sum_{j=1}^{n(i)} \lambda_{ij} x_{ij}. \quad (18)$$

Also, from the linear independence of

$$B = \bigcup_{i=1}^n B_i = \{b_{ij}, j = 1, \ldots, n(i), i = 1, \ldots, n\},$$

see (14), we have

$$\text{conv} B_i \cap \text{conv} B_j = \emptyset \quad \text{for every } i, j \in \{1, \ldots, n\} \text{ with } i \neq j. \quad (19)$$

For every $i = 1, \ldots, n$, denote $\lambda_i = \sum_{j=1}^{n(i)} \lambda_{ij}$, and

$$x_i = \begin{cases} (\lambda_i)^{-1} \sum_{j=1}^{n(i)} \lambda_{ij} x_{ij} \in \text{conv} B_i & \text{if } \lambda_i > 0, \\ \theta & \text{if } \lambda_i = 0. \end{cases} \quad (20)$$
Observe that $\sum_{i=1}^{n} \lambda_i = 1$ and

$$x = \sum_{i=1}^{n} \lambda_i x_i,$$

where $x_i \in \text{conv} B_i$ if $\lambda_i > 0$.

Therefore we can define $\varphi : \text{conv} B \to F$ by the formula:

$$\varphi(x) = \sum_{i=1}^{n} \lambda_i \varphi_i(x_i)$$  \hspace{1cm} (21)

where

$$\varphi_i(x_i) = \begin{cases} g_i f_i r_i(x_i) & \text{if } x_i \neq \theta, \\ \theta & \text{if } x_i = \theta. \end{cases}$$  \hspace{1cm} (22)

We claim that

**Claim 3.** $\varphi$ is continuous.

**Proof.** Assume that $\{x_k\} \subset \text{conv} B$ and $x_k \to x \in \text{conv} B$ as $k \to \infty$. Let, see (18),

$$x_k = \sum_{i=1}^{n} \sum_{j=1}^{n(i)} \lambda_{ij}(k)x_{ij}, \quad \text{and} \quad x = \sum_{i=1}^{n} \sum_{j=1}^{n(i)} \lambda_{ij} x_{ij},$$  \hspace{1cm} (23)

where $\lambda_{ij}(k), \lambda_{ij} \in [0, 1]$ and $x_{ij}(k), x_{ij} \in B, \ j = 1, \ldots, n(i), \ i = 1, \ldots, n$, with

$$\sum_{i=1}^{n} \sum_{j=1}^{n(i)} \lambda_{ij}(k) = 1 \quad \text{and} \quad \sum_{i=1}^{n} \sum_{j=1}^{n(i)} \lambda_{ij} = 1.$$

We claim that $\lambda_{ij}(k) \to \lambda_{ij}$ as $k \to \infty$. In fact if it is not the case, then there exists a subsequence $\{k(\ell)\} \subset \mathbb{N}$ such that $\lambda_{ij}(k(\ell)) \to \lambda_{ij}$ as $\ell \to \infty$ and $\lambda_{ij} \neq \lambda_{ij}$ for some $j \in \{1, \ldots, n(i)\}, \ i \in \{1, \ldots, n\}$. Since $x_k \to x$, we get

$$x = \sum_{i=1}^{n} \sum_{j=1}^{n(i)} \lambda_{ij} x_{ij}.$$  \hspace{1cm} (24)

Therefore from (23) we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n(i)} (\lambda_{ij} - \lambda_{ij}) x_{ij} = \theta.$$  \hspace{1cm} (25)

By the linear independence of $B = \{x_{ij}, \ j = 1, \ldots, n(i), \ i = 1, \ldots, n\}$ we get $\lambda_{ij} = \lambda_{ij}$ for every $j = 1, \ldots, n(i), \ i = 1, \ldots, n$, a contradiction.

For every $i = 1, \ldots, n$, denote

$$\lambda_i(k) = \sum_{j=1}^{n(i)} \lambda_{ij}(k), \quad \lambda_i = \sum_{j=1}^{n(i)} \lambda_{ij}.$$
Then we get
\[ \sum_{i=1}^{n} \lambda_i(k) = 1, \quad \sum_{i=1}^{n} \lambda_i = 1. \]

Let
\[ x_i(k) = \begin{cases} \frac{(\lambda_i(k))^{-1} \sum_{j=1}^{n} x_{ij}(k)}{\theta} & \text{if } \lambda_i(k) > 0, \\ \theta & \text{if } \lambda_i(k) = 0, \end{cases} \]
\[ x_i = \begin{cases} \frac{(\lambda_i)^{-1} \sum_{j=1}^{n} x_{ij}}{\theta} & \text{if } \lambda_i > 0, \\ \theta & \text{if } \lambda_i = 0. \end{cases} \]

Then we have
\[ x_k = \sum_{i=1}^{n} \lambda_i(k) x_i(k), \quad \text{where } x_i(k) \in \text{conv } B_i \text{ if } \lambda_i(k) > 0, \tag{24} \]
\[ x = \sum_{i=1}^{n} \lambda_i x_i, \quad \text{where } x_i \in \text{conv } B_i \text{ if } \lambda_i > 0. \tag{25} \]

Since \( \lambda_{ij}(k) \to \lambda_{ij} \) as \( k \to \infty \), it follows that \( \lambda_i(k) \to \lambda_i \) and \( x_i(k) \to x_i \) as \( k \to \infty \) for every \( i = 1, \ldots, n \). Therefore by the continuity of \( f_i, g_i \) and \( r_i \) we get, for \( x_i \neq \theta \)
\[ \varphi_i(x_i(k)) - g_i f_i r_i(x_i) \to g_i f_i r_i(x_i) - \varphi_i(x_i) \quad \text{for every } i = 1, \ldots, n \]
as \( k \to \infty \).

Also, if \( x_i = \theta \), then \( \lambda_i = 0 \) and \( \lambda_i(k) \to 0 \) as \( k \to \infty \). Therefore
\[ \lambda_i(k) \varphi_i(x_i(k)) \to \theta = \lambda_i \varphi_i(x_i) \]
as \( k \to \infty \). Consequently, from (21), (22) and (24) we get \( \varphi(x_k) \to \varphi(x) \) as \( k \to \infty \).

The claim is proved.

From (9) and (17) we get
\[
\| \varphi(x) - x \| = \left\| \sum_{i=1}^{n} \lambda_i \left( g_i f_i r_i(x_i) - x_i \right) \right\|
\leq \sum_{i=1}^{n} \| g_i f_i r_i(x_i) - x_i \|
\leq \sum_{i=1}^{n} \| g_i f_i r_i(x_i) - r_i(x_i) \| + \sum_{i=1}^{n} \| r_i(x_i) - x_i \|
< 2^{-3} \varepsilon + n \left( n^{-1} 2^{-3} \varepsilon \right) = 2^{-2} \varepsilon.
\]

The proposition is proved.

Now we are able to complete the proof of our Theorem. Define \( \psi = \varphi h g : F \to F \).

Then from (10) and (16) and from Proposition 2 we get
\[
\|f(x) - x\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\| \\
+ \|h(x) - \varphi h(x)\| + \|\varphi h(x) - x\| \\
< 2^{-3} \varepsilon + 2^{-3} \varepsilon + 2^{-2} \varepsilon + \|\psi(x) - x\| \\
= 2^{-1} \varepsilon + \|\psi(x) - x\|
\]
for every \(x \in F\).

Therefore from (4) we get
\[
\|\psi(x) - x\| \geq \|f(x) - x\| - 2^{-1} \varepsilon \geq 2^{-1} \varepsilon
\]
for every \(x \in F\).

Consequently \(\psi\) does not have any fixed points. This contradicts Brower’s fixed point theorem stating that any finite dimensional compact convex set has the fixed point property, see, for instance [2]. The theorem is proved.

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References
