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Free vibrations for some Koiter shells of revolution

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Abstract

The asymptotic behaviour of the smallest eigenvalue in linear Koiter shell problems is studied, as the thickness parameter tends to zero. In particular, three types of shells of revolution are considered. A result concerning the ratio between the bending and the total elastic energy is also provided, by using the general theory detailed in [L. Beirão da Veiga, C. Lovadina, An interpolation theory approach to Shell eigenvalue problems (submitted for publication); L. Beirão da Veiga, C. Lovadina, Asymptotics of shell eigenvalue problems, C.R. Acad. Sci. Paris 9 (2006) 707–710].

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1. Introduction and problem description

In considering the free vibrations of shells using the Koiter model (see [8,6,5], for instance), one is led to study the following *eigenvalue* problem in variational form:

$$\begin{cases} \text{Find } (\mathbf{u}_t, \lambda_t) \in V \times \mathbf{R} \text{ such that} \\ t a^m(\mathbf{u}_t, \mathbf{v}) + t^3 a^b(\mathbf{u}_t, \mathbf{v}) = \lambda_t m_t(\mathbf{u}_t, \mathbf{v}) \quad \forall \mathbf{v} \in V \\ \|\mathbf{u}_t\|_0 = 1. \end{cases} \tag{1}$$

Above, t is the shell thickness parameter and V is the space of admissible displacements, incorporating also the kinematic boundary conditions. The bilinear forms $a^m(\cdot,\cdot)$ and $a^b(\cdot,\cdot)$ are independent of t and are associated with the membrane and bending energy, respectively. Finally, $m_t(\cdot,\cdot)$ is the mass bilinear form. We notice that for an eigenvalue λ_t, the corresponding shell vibration frequency is given by $\omega_t = \sqrt{\lambda_t}$.

In this work we are interested in the *smallest* eigenvalue of problem (1), still denoted by λ_t, and in particular we focus on the asymptotic behaviour of the function $t \to \lambda_t$, as $t \to 0^+$. We will also consider the percentage of the

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elastic energy stored in the bending part. Accordingly, for $(\mathbf{u}_t, \lambda_t) \in V \times \mathbf{R}$ solution of (1), we define the function $R(t, \mathbf{u}_t)$ as

$$R(t, \mathbf{u}_t) := \frac{t^3 a^b(\mathbf{u}_t, \mathbf{u}_t)}{\lambda_t}. \tag{2}$$

We examine a set of shells of revolution, whose mid-surfaces are all defined as follows. Let $I \subset \mathbf{R}$ be a bounded closed interval, and let $f : I \to \mathbf{R}^+$ be a regular function. The shell mid-surface is parametrised by means of the mapping

$$\phi : \Omega = I \times [0, 2\pi] \longrightarrow \mathbf{R}^3; \qquad \phi(\xi^1, \xi^2) = (\xi^1, f(\xi^1) \sin \xi^2,\ f(\xi^1) \cos \xi^2). \tag{3}$$

In particular, we study the following shells, which cover the three fundamental types of mid-surface geometry.

$$\text{Parabolic cylinder:} \qquad f''(\xi^1) = 0 \qquad \forall \xi^1 \in I \tag{4}$$

$$\text{Elliptic cylinder:} \qquad f''(\xi^1) < 0 \qquad \forall \xi^1 \in I \tag{5}$$

$$\text{Hyperbolic cylinder:} \qquad f''(\xi^1) > 0 \qquad \forall \xi^1 \in I. \tag{6}$$

For all the shells, we impose clamped boundary conditions at both ends $(\xi^1, \xi^2) \in \partial I \times [0, 2\pi]$. Accordingly, the space of admissible displacements is

$$V = [H_0^1(\Omega)]^2 \times H_0^2(\Omega). \tag{7}$$

We do not need now to explicitly describe the bilinear forms: it is sufficient to recall that:

1. The bilinear forms $a^m(\cdot, \cdot)$ and $a^b(\cdot, \cdot)$ are symmetric and continuous on V.
2. The sum $a^m(\cdot, \cdot) + a^b(\cdot, \cdot)$ is coercive on V.
3. The symmetric and positive-definite mass bilinear form $m_t(\cdot, \cdot)$ satisfies

$$m_t(\mathbf{v}, \mathbf{v}) \sim t \|\mathbf{v}\|_0^2 . \tag{8}$$

We now introduce the following definition (cf. [2]).

Definition 1.1. We say that the eigenvalue problem (1) is of *order* α if

$$\alpha = \inf \left\{ \beta : t^\beta \lambda_t^{-1} \in L^\infty(0, 1) \right\} . \tag{9}$$

Remark 1.1. Definition 1.1 means that if the eigenvalue problem is of order α, then α is the "best" exponent in order to have $\lambda_t \sim t^\alpha$. Furthermore, it is easily seen that if the eigenvalue problem (1) is of order α, then $0 \le \alpha \le 2$.

Remark 1.2. In [2,3] a different scaling has been employed for the right-hand side of problem (1). More precisely, the term $\lambda_t m_t(\mathbf{u}_t, \mathbf{v})$ is there replaced by a term of the type $\lambda_t^*(\mathbf{u}_t, \mathbf{v})_0$, where λ_t^* denotes the corresponding eigenvalue. As a consequence of (8), we have $\lambda_t \sim t^{-1}\lambda_t^*$. Accordingly, the problem order α^* is given by $\alpha^* = \alpha + 1$. This shift should be taken into account when comparing the results of the present note with those given in [2,3].

2. Asymptotic behaviour of λ_t and of $R(t, \mathbf{u}_t)$

We first notice that for all the shells under consideration $a^m(\cdot, \cdot)$ defines a norm on V. Indeed, using the clamped boundary conditions, it is easy to see that $a^m(\mathbf{v}, \mathbf{v}) = 0$ if and only if $\mathbf{v} = \mathbf{0}$. We set $H := [L^2(\Omega)]^3$ and W as the completion of V with the norm $a^m(\mathbf{v}, \mathbf{v})^{1/2} := \|\mathbf{v}\|_W$. Therefore, we have the dense inclusion $V \subseteq W$, which implies $W' \subseteq V'$ densely. We have the following result, whose proof involves the interpolation theory (see [4,9], for instance) and can be found in [2].

Theorem 2.1. *Suppose that* $a^m(\mathbf{v}, \mathbf{v}) = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$. *The order* α *of the eigenvalue problem* (1) *is given by*

$$\alpha = \inf \left\{ 2\theta : H \subseteq (W', V')_{\theta,1} \right\} = \inf \left\{ 2\theta : (V, W)_{1-\theta,2} \subseteq H \right\} . \tag{10}$$

Concerning the ratio $R(t, \mathbf{u}_t)$ defined by (2), in [2] the following result is proved.

Proposition 2.1. *Let the eigenvalue problem (1) be of order α. Suppose also that there exist*

$$\lim_{t\to 0^+} \left(t^{-\alpha}\lambda_t\right) = l_0 > 0 \quad \text{and} \quad \lim_{t\to 0^+} R(t, \mathbf{u}_t) \geq 0, \tag{11}$$

where $\mathbf{u}_t$ is any eigenfunction associated with the smallest eigenvalue λ_t. Then it holds

$$\lim_{t\to 0^+} R(t, \mathbf{u}_t) = \frac{\alpha}{2}. \quad \square \tag{12}$$

Proposition 2.2. *For the parabolic, elliptic and hyperbolic cylinders (cf. (4)–(6)), the problem orders and the corresponding asymptotic behaviour of the bending energy percentage (see Proposition 2.1) are the following:*

$$\textit{Parabolic cylinder:} \qquad \alpha = 1 \qquad \lim_{t\to 0^+} R(t, \mathbf{u}_t) = 1/2\,; \tag{13}$$

$$\textit{Elliptic cylinder:} \qquad \alpha = 0 \qquad \lim_{t\to 0^+} R(t, \mathbf{u}_t) = 0\,; \tag{14}$$

$$\textit{Hyperbolic cylinder:} \qquad \alpha = 2/3 \qquad \lim_{t\to 0^+} R(t, \mathbf{u}_t) = 1/3. \tag{15}$$

Proof. Since for the *parabolic cylinder* the result has already been proved in [2], we focus on the other two cases. *Elliptic cylinder*. Using the results of [7], we get

$$W \subseteq H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega). \tag{16}$$

As a consequence, recalling that $V \subseteq W$, we immediately get

$$(V, W)_{1-\theta,2} \subseteq W \subseteq H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \subseteq H \tag{17}$$

for all $\theta \in (0, 1)$. Therefore, from Theorem 2.1, it immediately follows that $\alpha = 0$.
Hyperbolic cylinder. It has been proven in [10] that

$$W \subseteq L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega). \tag{18}$$

As a consequence, recalling (7) and using classical results from interpolation theory, it follows that

$$(V, W)_{1-\theta,2} \subseteq (H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega), L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega))_{1-\theta,2}. \tag{19}$$

It holds that

$$(H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega), L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega))_{1-\theta,2} = H^\theta(\Omega) \times H^\theta(\Omega) \times H^{3\theta-1}(\Omega) \subseteq H \tag{20}$$

for every $\theta \geq \frac{1}{3}$. Therefore, from (19), (20) and Theorem 2.1 we get $\alpha \leq 2/3$.

In order to prove that the problem order is exactly $\alpha = 2/3$, we thus have to show that

$$(W, V)_{\theta,2} = (V, W)_{1-\theta,2} \not\subseteq H \quad \text{if} \quad \theta < 1/3\,. \tag{21}$$

Therefore, it is sufficient to find a sequence $\{\mathbf{v}^{(n)}\}_{n\in\mathbb{N}}$ in V such that

$$\|\mathbf{v}^{(n)}\|_0 \geq C > 0; \qquad \|\mathbf{v}^{(n)}\|_{(W,V)_{\theta,2}} \longrightarrow 0 \quad \text{as } n \to \infty \tag{22}$$

for $0 < \theta < 1/3$. In order to achieve this, we assume that we can write the shell map with respect to its asymptotic coordinates (η^1, η^2); this is always possible for sufficiently smooth hyperbolic shells. In these coordinates, which are different from the ones introduced in (3), the membrane strains are written as

$$\gamma_{11} = u_{1,1} - \Gamma^1_{11}u_1 - \Gamma^2_{11}u_2\,, \qquad \gamma_{22} = u_{2,2} - \Gamma^1_{22}u_1 - \Gamma^2_{22}u_2 \tag{23}$$

$$\gamma_{12} = \tfrac{1}{2}[u_{1,2} + u_{2,1} - \Gamma^1_{12}u_1 - \Gamma^2_{12}u_2] - b_{12}u_3 \tag{24}$$

where $0 < c_1 \leq b_{12} \in L^\infty(\Omega)$ is the off-diagonal term in the (symmetric) curvature tensor for the surface, and $\Gamma^\delta_{\alpha\beta} \in L^\infty(\Omega)$ are the Christoffel symbols for the shell mid-surface (see [5], for instance).

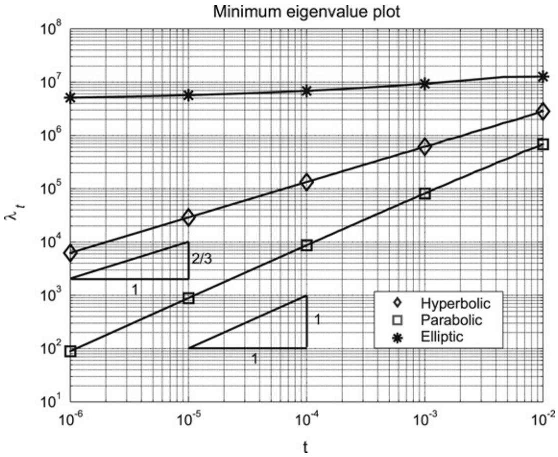

Fig. 1. Minimum eigenvalue as a function of t for three shells of revolution.

Let $g(\eta^1, \eta^2)$ be any fixed function in C^∞ with compact support. We then define $\mathbf{v}^{(n)} = (v_1^{(n)}, v_2^{(n)}, v_3^{(n)}) \in V$ as:

$$v_1^{(n)} = 0; \quad v_2^{(n)} = 2n^{-1} g(\eta^1, \eta^2) \sin(n\eta^1); \quad v_3^{(n)} = b_{12}^{-1} g(\eta^1, \eta^2) \cos(n\eta^1). \tag{25}$$

It is easy to check that, as $n \to \infty$,

$$\begin{aligned} &\|\mathbf{v}^{(n)}\|_0 \geq \|v_3^{(n)}\|_{L^2(\Omega)} \geq C > 0 \\ &\|\mathbf{v}^{(n)}\|_W \simeq \|\gamma_{11}^{(n)}\|_{L^2(\Omega)} + \|\gamma_{22}^{(n)}\|_{L^2(\Omega)} + \|\gamma_{12}^{(n)}\|_{L^2(\Omega)} \lesssim n^{-1} \\ &\|\mathbf{v}^{(n)}\|_V \simeq \|v_{3,11}^{(n)}\|_{L^2(\Omega)} \lesssim n^2 . \end{aligned} \tag{26}$$

Since it holds (see for example [4,9]) that

$$\|\mathbf{v}^{(n)}\|_{(W,V)_{\theta,2}} \leq c \|\mathbf{v}^{(n)}\|_W^{1-\theta} \|\mathbf{v}^{(n)}\|_V^{\theta}, \tag{27}$$

from (26) we easily infer (22), for all $0 < \theta < 1/3$. As a consequence, it holds that $\alpha = 2/3$. □

Finally, in Fig. 1 we report a few numerical results obtained with a collocation method, for the three choices (see [1] for further details):

Parabolic cylinder:	$f(\xi^1) = 1,$	$I = [-1, 1];$
Elliptic cylinder:	$f(\xi^1) = 1 - \left(\xi^1\right)^2/2,$	$I = [-0.892668, 0.892668];$
Hyperbolic cylinder:	$f(\xi^1) = 1 + \left(\xi^1\right)^2/2,$	$I = [-0.892668, 0.892668].$

We plot the value of the minimum eigenvalue λ_t as a function of the thickness t on a log–log scale. As can be appreciated, the rates of the three graphs are in accordance with Proposition 2.2. More exhaustive numerical results, also regarding other shell models, can be found in [1].

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