Triangulated Graphs and the Elimination Process

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A triangulated graph is a graph in which for every cycle of length \( \ell > 3 \), there is an edge joining two nonconsecutive vertices. In this paper we study triangulated graphs and show that they play an important role in the elimination process. The results have application in the study of the numerical solution of sparse positive definite systems of linear equations by Gaussian elimination.

1. INTRODUCTION

In this paper we discuss the combinatorial aspects of the elimination process which is regarded as vertex elimination on a graph. We show that triangulated graphs (introduced by Berge [1]) play an essential role, and we analyze in detail their structure with respect to elimination.

The graph theoretic results in this paper are related to the study of the following question. Given an \( n \times n \) positive definite matrix \( M \) that is sparse (many zero entries), which of the matrices \( PMPT \) (\( P \) an \( n \times n \) permutation matrix) should we use to solve a system equivalent to \( Mx = b \) by Gaussian elimination? Parter [2] discussed this question when \( M \) was represented by a tree, and he showed that an ordering, \( P \), could be found which resulted in a “perfect” elimination scheme. Our results show this is true more generally when \( M \) can be represented by a triangulated graph. We present here only the theoretical aspects of elimination and will present elsewhere the applications of this analysis to the study of efficient numerical solution of sparse positive definite systems of equations.

2. PRELIMINARIES

For our purposes a graph will be a pair, \( G = (X, E) \) where \( X \) is a finite set of \(|X|\) elements called vertices and

\[
E \subseteq \{\{x, y\} \mid x, y \in X \text{ and } x \neq y\}
\]

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is a set of $|E|$ vertex pairs called edges. Given $x \in X$, the set,

$$\text{Adj}(x) = \{y \in X \mid \{x, y\} \in E\},$$

is the set of vertices adjacent to $X$. For distinct vertices $x, y \in X$ a chain from $x$ to $y$ (of length $\ell = n$) is an ordered set of distinct vertices

$$\mu = [p_1, p_2, \ldots, p_{n+1}], \quad p_1 = x, \quad p_{n+1} = y$$

such that $p_{i+1} \in \text{Adj}(p_i), i = 1, \ldots, n$. Similarly a cycle (of length $\ell = n$) is an ordered set of $n$ distinct vertices

$$\mu = [p_1, p_2, \ldots, p_n, p_1]$$

such that $p_{i+1} \in \text{Adj}(p_i), i = 1, \ldots, n - 1$ and $p_1 \in \text{Adj}(p_n)$. In this paper we always assume that the graph $G$ is connected; i.e., for each pair of distinct vertices $x, y \in X$ there is a chain from $x$ to $y$.

For a graph $G = (X, E)$ and subset $A \subseteq X$, the section graph $G(A)$ is the subgraph

$$G(A) = (A, E(A)); \quad E(A) = \{(x, y) \in E \mid x, y \in A\}.$$  

A separator of a graph $G = (X, E)$ is a subset $S \subseteq X$ such that the section graph $G(X - S)$ consists of two or more connected components, say $C_i = (V_i, E_i)$. The section graphs $G(S \cup V_i)$ are then the leaves of $G$ with respect to $S$. A minimal separator is a separator no subset of which is also a separator. Similarly, given $a, b \in X$ with $a \not\in \text{Adj}(b)$ an $a, b$ separator is a separator such that $a$ and $b$ are in distinct components, say $C_a$ and $C_b$, respectively. Note that a minimal separator is a minimal $a, b$ separator for some $a, b \in X$, but a minimal $a, b$ separator is not, in general, a minimal separator. A clique, $C$, of a graph is a subset of vertices which are pairwise adjacent, and a separation clique is a separator which is also a clique.

### 3. Monotone Transitivity and Elimination

Let $G = (X, E)$ be a graph with $|X| = n$. An ordering of $X$ is a bijection

$$\alpha : \{1, 2, \ldots, n\} \leftrightarrow X.$$ 

We sometimes indicate an ordering by the shorthand $X = \{x_i\}_{i=1}^n$. If $X$ is ordered by $\alpha$, then $G_\alpha = (X, E, \alpha)$ is an ordered graph associated with $G$.

For any $x \in X$ of $G_\alpha = (X, E, \alpha)$ the set of vertices monotonely adjacent to $x$ is

$$\text{MAdj}(x) = \text{Adj}(x) \cap \{z \mid \alpha^{-1}(z) > \alpha^{-1}(x)\}.$$
The deficiency, $D(x)$, is the set of all pairs of $\text{Adj}(x)$ which are not themselves adjacent; i.e.,

$$D(x) = \{ \{y, z\} \mid y, z \in \text{Adj}(x) \text{ and } y \notin \text{Adj}(z) \}.$$ 

Similarly, the monotone deficiency, $\text{MD}(x)$, is the set

$$\text{MD}(x) = \{ \{y, z\} \mid y, z \in \text{MAdj}(x) \text{ and } y \notin \text{Adj}(z) \}.$$ 

Given a vertex $y$ of a graph $G$, the graph $G_y$ obtained from $G$ by

(a) deleting $y$ and its incident edges

(b) adding edges such that all vertices in the set $\text{Adj}(y)$ are adjacent

is the $y$-elimination graph of $G$ (compare Parter ([2], p. 120)). Thus

$$G_y = (X - \{y\}, E(X - \{y\}) \cup D(y)).$$

For an ordered graph $G = (X, E, \alpha)$ the order sequence of elimination graphs $G_1, \ldots, G_{n-1}$ is defined recursively by $G_1 = G_{x_1}$ and

$$G_i = (G_{i-1})_{x_i}, \quad i = 2, \ldots, n - 1.$$ 

Since the graphs $G_i$ determine the evolution of the process of vertex elimination we formally define the elimination process on a graph $G = (X, E)$ with ordering $\alpha$ as the ordered set

$$P(G; \alpha) = [G = G_0, G_1, \ldots, G_{n-1}].$$

An elimination process, $P(G; \alpha)$, is perfect if

$$G_i = G \left( X - \bigcup_{j=1}^{i} \{x_j\} \right).$$

Whether or not an elimination process is perfect depends only on the ordered graph $G_0$.

DEFINITION. The ordered graph $G = (X, E, \alpha)$ is monotone transitive when for all $x \in X$ we have

$$y \in \text{MAdj}(x) \quad \text{and} \quad z \in \text{MAdj}(x) \Rightarrow y \in \text{Adj}(z).$$

The significance of monotone transitivity is given in the following lemma which merely summarizes our definitions.
Lemma 1. Let $G = (X, E, \alpha)$ be an ordered graph. Then the following are equivalent:

1. $G$ is monotone transitive;
2. $\text{MD}(x) = \emptyset$ for all $x \in X$;
3. $P(G; \alpha)$ is a perfect elimination process.

Thus in a monotone transitive graph vertex elimination adds no edges. Even if the graph $G = (X, E, \alpha)$ is not monotone transitive the study of monotone transitive graphs is interesting because the elimination process may be regarded as transforming a graph $G = (X, E, \alpha)$ into its monotone transitive extension $MTE(G; \alpha)$ where

$$MTE(G; \alpha) = \left( X, E \bigcup_{i=1}^{n-1} \text{MD}(x_i) \right).$$

We show in the next section that monotone transitive graphs can be characterized by their cycle structure and their minimal $a, b$ separators.

4. Triangulated Graphs

The following definition is due to Berge ([1] p. 158).

Definition. A graph $G$ is triangulated if for every cycle $\mu = [p_1, ..., p_n, p_1]$ of length $\ell > 3$ there is an edge of $G$ joining two nonconsecutive vertices of $\mu$; such edges are called chords of the cycle.

We caution that the notion of triangulated graphs here is not the same as the notion of a triangular planar graph ([3] p. 89); for example, Fig. 1 shows a triangular but not triangulated graph. In this figure the cycle $[2, 4, 6, 3, 2]$ has no chord.

![Figure 1](image)

The main results of this section will now be stated.
Theorem 1. For a graph $G = (X, E)$ the following statements are equivalent:

1. There exists an ordering $\alpha$ of $X$ such that $G_\alpha = (X, E, \alpha)$ is monotone transitive;
2. The graph $G = (X, E)$ is triangulated;
3. Every minimal $a, b$ separator of $G$ is a clique.

Theorem 2. Let $G = (X, F)$ be triangulated with subgraph $\hat{G} = (X, E)$, $E \subseteq F$. Then $\hat{G}$ is triangulated if and only if for each $e = \{x, y\} \in F - E$ there exists an $x, y$ separation clique, $S_{x, y}$, of $\hat{G}$.

Proofs and Other Results

We begin the proof of Theorem 1 by generalizing slightly Theorem 3 of Berge ([1] p. 160) in the following:

Lemma 2. In a triangulated graph $G = (X, E)$ every minimal $a, b$ separator is a clique.

Proof. Let $S$ be a minimal $a, b$ separator and $C_a$ and $C_b$ be the components of $G(X - S)$ containing $a$ and $b$, respectively. Since $S$ is minimal each $s \in S$ is adjacent to some vertex in $C_a$ and some vertex in $C_b$. Let $x, y \in S$ and let $\mu_i$ be the shortest chains of the type $[x, c_{i,1}, c_{i,2}, \ldots, c_{i,\ell}, y]$ $i = 1, 2$ and $c_{i,j} \in C_a$ and $c_{i,j} \in C_b$. The cycle containing $x$ and $y$ formed by $\mu_1$ and $\mu_2$ has length $\ell \geq 4$ and the only possible chord is $\{x, y\}$.

The proof of the following lemma is immediate from the definition of section graphs.

Lemma 3 (Berge ([1] p. 161)). If $G = (X, E)$ is triangulated and $A \subseteq X$, then $G(A)$ is triangulated.

Lemma 4. A monotone transitive graph is a triangulated graph.

Proof. Let $\alpha$ be the ordering and $\mu$ be any cycle with $\ell > 3$. Let $p^* \in \mu$ be the vertex such that

$$\alpha^{-1}(p^*) = \min_{p \in \mu} \alpha^{-1}(p).$$

Since $p^*$ is adjacent to two nonconsecutive vertices by monotone transitivity $\mu$ has a chord.

Lemma 5. Let $G = (X, E)$ be a graph with separation clique $S$ and leaves
L_i, i = 1, ..., n. If S_0 is a separator of some L_i, then S_0 is a separator of G. Furthermore, if S_0 is a minimal a, b separator of L_i, then S_0 is a minimal a, b separator of G.

Proof. Let D_j, j = 1, ..., m be the components of L_i with respect to S_0. Since S is a clique vertices in S can be in only one component, say D_k. Thus S_0 is a separator of G because any chain from a vertex x ∈ (L_j − S_0), j ≠ i to a vertex y ∈ D_k, j ≠ k must contain a vertex of S_0. This proves the first statement.

For the second statement note that S_0 is a separator of G as we have just shown; it must be an a, b separator (for the same a, b) because D_i ∩ S ≠ ∅ for at most one i. Finally S_0 must be minimal in G since any a, b separator S_0 ⊂ S_0 in G must be an a, b separator in L_i.

LEMMA 6. Let G = (X, E) satisfy property (3) of Theorem 1. Then either X is a clique or given any clique C ⊂ X, there exists a vertex x ∉ C such that D(x) = ∅.

Proof. The proof is by induction on |X| and the case |X| = 1 is clear. Assuming any case with |X| ≤ k, let G = (X, E) be such a graph with |X| = k + 1 and C be any clique. Either X is a clique or there exists by Lemma 2 some a, b separation clique of G, say C_1. Let D_a, D_b and L_a, L_b be the corresponding components and leaves of G containing a and b, respectively. Clearly, the vertices in C − C_1 can be in at most one component; suppose such vertices are in D_a. Consider the leaf L_a; by Lemma 5 it inherits property (3), Theorem 1. Writing L_a = (W, F) we have |W| ≤ k and hence, by induction, either W is a clique or there exists a vertex x ∉ C_1 such that D(x) = ∅ in L_a. In either case, then, since W must contain at least one vertex not in C_1, there exists an x ∉ C_1 with D(x) = ∅ in L_a. Finally D(x) = ∅ in G because x is not adjacent to a vertex in any component other than D_b. Clearly x ∉ C, the original clique.

Lemmas 5 and 6 yield the following two corollaries concerning the existence of "empty deficiency" vertices.

COROLLARY 1. Let G be as in Lemma 6 and S be any separation clique of G with components C_i and leaves L_i. Then for each component, C_i, there exists a vertex c_i ∈ C_i with D(c_i) = ∅ in G.

Proof. By Lemma 5 each L_i has property (3) of Theorem 1. Thus by Lemma 6 for each L_i there exists a vertex of L_i with c_i ∉ S and D(c_i) = ∅.

COROLLARY 2. Let G be as in Lemma 6. Then for any x ∈ X one and only one of the following statements is true:

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(ii) $x \in S$, $S$ a minimal $a$, $b$ separation clique.

Proof. If (ii) is true clearly (i) must be false. We show by induction on $|X|$ that (i) or (ii) must be true. The case $|X| = 1$ is clear and we suppose the case $|X| \leq k$. Note that if $X$ is a clique the result is immediate; assuming otherwise let $S$ be a minimal $a$, $b$ separation clique of $G$. Let $x \in X$; if $x \in S$ we are finished so let $x \in (L_a - S)$. By the induction hypothesis and Lemma 5 either $D(x) = \phi$ in $L_a$ (and hence in $G$) or $x \in S$, $S$ a minimal $c$, $d$ separation clique of $L_a$ (and hence of $G$).

LEMMA 7. Let $G = (X, E)$ be as in Lemma 6. Then there exists an ordering $\alpha$ of $X$ such that for all $x \in X$, $MD(x) = \phi$.

Proof. Induction on $|X|$; the case $|X| = 1$ is clear and we suppose the case $|X| = k$. If $G$ is such a graph with $k + 1$ vertices there exists a vertex $x_1$ (say) such that $D(x_1) = \phi$. Let $G_1$ be the $x_1$-elimination graph; by Lemma 5 (since Adj($x_1$) is a separation clique) $G_1$ satisfies the hypothesis of the lemma and has $|X| = k$. By induction there exists an ordering $\alpha_1$ of the vertices of $G_1$ such that

$$\alpha_1(i) = x_{i+1}, \quad i = 1, \ldots, k$$

with $MD(x_i) = \phi$. Finally in $G$ choose the ordering $\alpha(i) = x_i$, $i = 1, \ldots, k + 1$. Then $MD(x_i) = \phi$ with this ordering in $G$.

Note that the ordering $\alpha$ assured by Lemma 7 is not unique in view of Corollary 1. Also note that another way of stating Lemma 7 is that there exists an ordering $\alpha$ such that the order sequence of elimination graphs of $G$; i.e., $G = G_0$, $G_1, \ldots, G_{n-1}$ have $D(x_i) = \phi$ in $G_{i-1}$. Finally we shall call any ordering guaranteed by Lemma 7 a monotone transitive ordering.

Proof of Theorem 1. Property (1) $\Rightarrow$ Property (2) by Lemma 4; Property (2) $\Rightarrow$ Property (3) by Lemma 2; and Property (3) $\Rightarrow$ Property (1) by Lemmas 7 and 1.

The following corollary shows that in a triangulated graph a monotone transitive ordering can be found such that any given clique is ordered last.

COROLLARY. Let $G = (X, E)$ be triangulated with clique $C \subseteq X$. Then there exists a monotone transitive ordering $\alpha$ such that $\alpha(j) \in C$ for $j = k + 1, k + 2, \ldots, |X|$ where $k = |X| - |C|$.

Proof. The proof follows from Lemma 6 and the induction argument of Lemma 7.
The next theorem states that minimal $a, b$ separators of a triangulated graph are generated in the elimination process.

**Theorem 3.** Let $G = (X, E)$ be triangulated and $\alpha$ be a monotone transitive ordering. If $S$ is a minimal $a, b$ separation clique of $G$ then $S = M\adj(x_i)$ for some $x_i \in X$. Conversely for any $x_i \in X$ such that the vertices of the elimination graph $G_{i-1}$ are not a clique, $M\adj(x_i)$ is a separation clique of $G$.

**Proof.** To prove the first assertion let $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$ be the components of $G$ with respect to $S$ containing $a$ and $b$ respectively. For each $V_i$ let $v_i^*$ be the vertex such that

$$\alpha^{-1}(v_i^*) = \max_{v \in V_i} \alpha^{-1}(v).$$

Choose $\vartheta \in \{v_1^*, v_2^*\}$ such that

$$\alpha^{-1}(\vartheta) = \min(\alpha^{-1}(v_1^*), \alpha^{-1}(v_2^*)).$$

Because $S$ is minimal each $s \in S$ is adjacent to some vertex in $C_a$ and $C_b$. Hence if $j = \alpha^{-1}(\vartheta)$ by monotone transitivity and the connectivity of $C_a$ and $C_b$ we have $S = M\adj(x_j)$.

To prove the second assertion note first that $M\adj(x_i)$ is a separation clique of $G$ unless $X$ is a clique. Also the elimination graph $G_1$ is a leaf of $G$ with respect to $M\adj(x_i)$ and $G_1 = (X_1, E_1)$ has $|X_1| = |X| - 1$. The assertion then follows by induction on $|X|$ and Lemma 5.

We begin the proof of Theorem 2 with the following:

**Lemma 8.** Let $G = (X, F)$ be triangulated with a subgraph $\hat{G} = (X, E)$, $E \subseteq F$. Suppose $S$ is a separation clique of $\hat{G}$ such that for each edge $e = \{x, y\} \in F - E$ and $y$ are in different components. Then $\hat{G}$ is triangulated.

**Proof.** Let $\mu$ be any cycle in $\hat{G}$ with $\ell \geq 4$. If $\mu$ is entirely with some leaf of $\hat{G}$, then $\mu$ contains a chord because $\mu$ is also a cycle in $G$. If $\mu$ has vertices in more than one component, then $\mu$ must contain at least two distinct vertices of $S$; these vertices are adjacent, hence $\mu$ has a chord.

**Proof of Theorem 2.** The "if" part of the theorem follows by successive applications of Lemma 8. Given some $S_e$ discard all edges in $F - E$ with incident vertices in different components. $S_e$ is then a separation clique of this new graph, $\hat{G}$, and by Lemma 8, $\hat{G}$ is triangulated. Continue for each edge in $F - E$ not already discarded. The converse is clear by Lemma 2 because for each $e = \{a, b\} \in F - E$ there exists a minimal $a, b$ separator $S_e$ in $\hat{G}$ and $S_e$ is a clique.
5. Criterion Functions

Let $G = (X, E, \alpha)$ be an ordered graph associated with $G = (X, E)$ and $d(\alpha(i))$ be the degree of the vertex $\alpha(i)$ in the elimination graph $G_{i-1}$ (i.e., $d(\alpha(i)) = |\text{Adj}(\alpha(i))|$ in $G_{i-1}$). If $G = (X, E, \alpha)$ is monotone transitive with $|X| = n$ then

$$\sum_{i=1}^{n-1} d(\alpha(i)) = |E|$$

because each edge in $E$ is counted once and only once in some $d(\alpha(i))$. In this case the $(n - 1)$ integers $d(\alpha(i))$ form a partition (or degree partition) of $|E|$.

For two ordered monotone transitive graphs $G_a = (X, E, \alpha)$ and $G_\beta = (X, F, \beta)$ with $|X| = n$, the partitions of $|E|$ and $|F|$ generated by the $d(\alpha(i))$ and the $d(\beta(i))$, respectively, will be called equal if there exists a permutation $\pi$ on the integers $1, 2, \ldots, n - 1$ such that

$$d(\alpha(i)) = d(\beta(\pi(i))), \quad i = 1, 2, \ldots, n - 1.$$ 

Similarly, the partition generated by the $d(\alpha(i))$ dominates the partition generated by the $d(\beta(i))$ if

$$d(\alpha(i)) \geq d(\beta(\pi(i))), \quad i = 1, 2, \ldots, n - 1.$$ 

In this section we consider a class of functions defined on the quantities $d(\alpha(i))$ each of which may represent a "cost of elimination"; or if the graph $G = (X, E)$ is not triangulated these functions can be considered as "criterion" functions for choosing an "optimal" ordering.

As criterion functions for the graph $G = (X, E)$ ($|X| = n$) we choose the class of symmetric isotone functions; i.e., real valued functions

$$F(a_1, a_2, \ldots, a_{n-1}), \quad a_i \text{ integer}$$

such that:

(i) $F(a_1, a_2, \ldots, a_{n-1}) = F(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n-1)})$ for any permutation on $\{1, 2, \ldots, n - 1\}$;

(ii) $F(a_1, a_2, \ldots, a_{n-1}) \geq F(b_1, b_2, \ldots, b_{n-1})$ when $a_i \geq b_i, i = 1, \ldots, n - 1$.

We now show (Theorem 4) that if $F$ is a criterion function for a triangulated graph $G = (X, E)$ with monotone transitive orderings $\alpha$ and $\beta$, then

$$F(d(\alpha(1)), d(\alpha(2)), \ldots, d(\alpha(n - 1))) = F(d(\beta(1)), d(\beta(2)), \ldots, d(\beta(n - 1))).$$
Furthermore (Theorem 5) if $\gamma$ is any non monotone transitive ordering of $X$, then $G(x) = (X, E, \alpha)$ is a subgraph of

$$\text{MTE}(G; \gamma) = \left( X, E \bigcup_{i=1}^{n-1} \text{MD}(\gamma(i)) \right)$$

and we show

$$F(d(\gamma(1)), d(\gamma(2)), \ldots, d(\gamma(n - 1))) \geq F(d(\alpha(1)), d(\alpha(2)), \ldots, d(\alpha(n - 1))).$$

Thus, with respect to criterion functions on triangulated graphs, monotone transitive orderings may be regarded as “optimal.”

We begin with

**Theorem 4.** Let $G = (X, E)$ be triangulated and let $\alpha$ and $\beta$ be two distinct monotone transitive orderings of $X$. Then the two partitions of $|E|$ generated by the $d(\alpha(i))$ and the $d(\beta(i))$ are equal.

**Proof.** We use induction on $|X|$; the case $|X| = 2$ is clear. Supposing the case $|X| = k - 1$ we consider $G$ with $|X| = k$. If $\alpha(1) = \beta(1) = x$ (say) the result follows immediately from the induction hypothesis on the elimination graph $G_x$.

Suppose, then, that $\alpha(1) = y$ and $\beta(1) = z$; note that if $y \in \text{Adj}(z)$ then $\text{Adj}(y) - \{z\} = \text{Adj}(z) - \{y\}$ by monotone transitivity and $|\text{Adj}(y)| = |\text{Adj}(z)|$. Consider the new monotone transitive orderings $\alpha'$ and $\beta'$ defined by:

$$\alpha'(1) = y; \quad \beta'(1) = z$$

$$\alpha'(2) = z; \quad \beta'(2) = y$$

$$\alpha'(i) = \beta'(i) = \gamma(i - 2) \quad i = 3, \ldots, n$$

where $\gamma$ is any monotone transitive ordering of the triangulated section graph $G(X - \{x, y\})$. By the first part of the proof, the partitions generated by $d(\alpha'(i))$ and $d(\alpha(i))$ are equal as are those generated by $d(\beta'(i))$ and $d(\beta(i))$. It remains to show that the partitions generated by $d(\alpha'(i))$ and $d(\beta'(i))$ are equal. Now $\alpha'$ yields the partition $|\text{Adj}(y)|, |\text{Adj}(z)|, d(\gamma(i - 2)), i = 3, \ldots, n$, if $y \notin \text{Adj}(z)$ and $|\text{Adj}(y)|, |\text{Adj}(z)| - 1, d(\gamma(i - 2)), i = 3, \ldots, n$, if $y \in \text{Adj}(z)$. But $\beta'$ gives an equal partition in each case because $|\text{Adj}(y)| = |\text{Adj}(z)|$ if $y \in \text{Adj}(z)$.

**Lemma 1.** Let $G = (X, F)$ be triangulated with triangulated subgraph $\hat{G} = (X, E), E \subseteq F$. If $\alpha$ any monotone transitive ordering for both $G$ and $\hat{G}$, then the degree partition of $F$ dominates the degree partition of $E$. 
Proof. Clearly \( d(x_i) \leq d(x_i') \), and the elimination graphs

\[
\hat{G}_1 = (X - \{x_i\}, E_1) \quad \text{and} \quad G_1 = (X - \{x_i\}, F_1)
\]

are triangulated with \( E_1 \subseteq F_1 \). The proof then follows by induction on \( |X| \).

**Lemma 2.** Let \( G = (X, E) \) be triangulated and \( x \in X \). Then \( \hat{G} = (X, E \cup D(x)) \) is triangulated.

Proof. Assuming \( D(x) \neq \phi \) we need only show that cycles in \( \hat{G} \) of the form

\[
\mu = [x_1, y_1, p_1, \ldots, p_n, x_1] \quad (n \geq 2)
\]

with \( \{x_1, y_1\} \in D(x) \) have a chord. There are two cases:

(i) if some \( p_i \in \text{Adj}(x) \) then there is a chord \( \{x_1, p_i\} \) (or \( \{y_1, p_n\} \) if \( i = n \)) in \( E \cup D(x) \);

(ii) if no \( p_i \in \text{Adj}(x) \) the cycle

\[
\mu' = [x_1, x, y_1, p_1, \ldots, p_n, x_1]
\]

in \( G \) has a chord \( \{x_1, p_1\}, \{y_1, p_1\} \), or \( \{p_i, p_j\} \) in \( E \) which is also a chord in \( \hat{G} \).

**Lemma 3.** Let \( \hat{G} = (X, E) \) and \( G = (X, F) \) be triangulated with strict inclusion \( E \subset F \). Then there exists a monotone transitive ordering \( \alpha \) for \( G \) such that in \( \hat{G} \), \( \text{MTE}(\hat{G}; \alpha) = (X, E \cup T(\alpha)) \) with strict inclusion \( (E \cup T(\alpha)) \subset F \).

Proof. If \( X \) is a clique in \( G \), the lemma is true for any \( \alpha \) which is a monotone transitive ordering for \( \hat{G} \). Hence, we assume \( X \) is not a clique in \( G \) and prove the assertion by induction on \( |X| \). One easily verifies the cases when \( |X| = 4 \), and, assuming the case \( |X| = k - 1 \), we consider such graphs \( G \) and \( \hat{G} \) with \( |X| = k \).

Let \( S = \{y \in X \mid D(y) = \phi \text{ in } G\} \). We first dispense with two special cases.

(i) If for some \( y \in S \), \( [\text{Adj}(y)]_G \subset [\text{Adj}(y)]_C \) (i.e., there is an edge \( e = \{y, x\} \in F - E \)), then by choosing any monotone transitive ordering for \( G \) with \( \alpha(1) = y \) we have \( (E \cup T(\alpha)) \subset F \).

(ii) If some \( y \in S \) with \( [\text{Adj}(y)]_G = [\text{Adj}(y)]_C \) has \( D(y) = \phi \) in \( \hat{G} \) also, then by choosing \( \alpha(1) = y \), the lemma follows by the induction hypothesis on the elimination graph \( \hat{G}_y \).

These cases being thus dismissed we may assume that for each \( y \in S \) \( [\text{Adj}(y)]_G = [\text{Adj}(y)]_C \) and that the clique \( \text{Adj}(y) \) in \( G \) contains (at least) one pair of vertices \( e_y = \{v_1, v_2\} \in F - E \). By Corollary 1, Lemma 6, since
X in G is not a clique, there exists y, z ∈ S with y \notin \text{Adj}(z). For such vertices \( e_y \neq e_z \) because if \( e_y = e_z = \{v_1, v_2\} \) the cycle \( \mu = [y, v_1, z, v_2, y] \) has no chord in E so \( G \) could not be triangulated.

Hence for some \( y \in S \) choose \( \alpha(1) = y \) and consider the \( y \)-elimination graphs \( G_y = (X - \{y\}, E_1) \) and \( G_y = (X - \{y\}, F_1) \). It is clear from above that strict inclusion \( E_1 \subset F_1 \) holds because there exists a \( z \in S \) with \( y \notin \text{Adj}(z) \) and such that \( e_z \in F_1 \) but \( e_z \notin E_1 \). Also by Lemma 2 \( G_y \) is triangulated (as is \( G_y \)). Hence the Lemma follows by using induction on the graphs \( G_y \) and \( G_y \).

These lemmas give us

**Theorem 5.** Let \( G = (X, E) \) and \( G = (X, F) \) be triangulated with \( E \subset F \). Let \( \alpha \) and \( \beta \) be monotone transitive orderings of \( G \) and \( G \), respectively. Then the degree partition of \( |F| \) dominates the degree partition of \( |E| \).

**Proof.** We use induction of \( |F| \); if \( |F| = |X| - 1 \) (i.e., \( G \) is a tree), then \( E = F \) (because both \( G \) and \( \hat{G} \) are assumed connected) and the conclusion follows from Theorem 4. Suppose the theorem is true whenever \( |X| - 1 < |F| \leq k - 1 \) and let \( G \) and \( \hat{G} \) be as above with \( |F| = k \). If the subgraph \( \hat{G} = (X, E) \) has \( E = F \), then again the conclusion follows from Theorem 4. Assume then \( E \subset F \) (strict); by Lemma 3 there exists a monotone transitive ordering \( \hat{\alpha} \) for \( G \) such that \( \text{MTE}(\hat{G}; \hat{\alpha}) = (X, E \cup T(\hat{\alpha})) \) and \( (E \cup T(\hat{\alpha})) \subset F \) (strict). By the induction hypothesis the degree partition of \( |E \cup T(\hat{\alpha})| \) (generated by \( \hat{\alpha} \) in the triangulated graph \( \text{MTE}(\hat{G}; \hat{\alpha}) \)) dominates the degree partition of \( |E| \), and by Lemma 1 the degree partition of \( |F| \) generated by \( \hat{\alpha} \) dominates the degree partition of \( |E \cup T(\hat{\alpha})| \). By Theorem 4 the degree partitions of \( |F| \) generated by \( \beta \) and \( \hat{\alpha} \) are equal.

**A Conjecture**

Let \( G = (X, E) \) be a graph and \( \text{MTE}(G; \alpha) = (X, E \cup T(\alpha)) \); that is \( T(\alpha) \) is a triangulation of \( G \) generated by the ordering \( \alpha \). If \( d(\alpha(i)) \) is the degree partition of \( |E \cup T(\alpha)| \) we have (\( |X| = n \))

\[
\sum_{i=1}^{n-1} d(\alpha(i)) = |E| + |T(\alpha)|.
\]

Thus minimizing the criterion function \( \sum d(\alpha(i)) \) over all orderings \( \alpha \) gives a minimum triangulation; i.e., the least number of edges necessary to triangulate a graph. How do we obtain a minimum triangulation?

Note that by Theorem 2 no minimum triangulation, \( \hat{T} \), of a graph \( G = (X, E) \) will have edges joining vertices in different components with respect to some separation clique. Hence we may regard a triangulation of \( G \) as a choice of which separator is to become a clique since we may then proceed to triangulate the leaves.
Let $G = (X, E)$ and $A \subseteq X$. The clique deficiency, $D(A)$, of $A$ is defined as

$$D(A) = \{\{x, y\} \mid x, y \in A \text{ and } x \notin \text{Adj}(y)\}.$$  

Denoting $\Sigma(G) = \{S \mid S \text{ a separator of } G\}$ we present the following

**Conjecture.** Let $S_0$ be a separator of a graph $G = (X, E)$ such that

$$|D(S_0)| = \min_{S \in \Sigma(G)} |D(S)|.$$

Then there exists a minimum triangulation, $\hat{T}$, of $G$ such that $S_0$ is a separation clique in $\hat{G} = (X, E \cup \hat{T})$.

### 6. An Example: $k$ Trees

A $k$-tree is a graph defined recursively as follows:

A $k$-tree on $k$ vertices is a clique on $k$ vertices; and given a $k$ tree, $T_n$, on $n$ vertices, a $k$-tree on $n + 1$ vertices is obtained when the $(n + 1)$-st vertex is adjacent to the vertices of a clique on $k$ vertices in $T_n$. If $x_i, i = 1, 2, ..., n$, are the vertices of a $k$-tree on $n$ vertices constructed as above, then clearly this graph is monotone transitive with ordering $\alpha(i) = x_{n+1-i} \, i = 1, 2, ..., n$.

$k$-trees also have the interesting property that every minimal separator, $S$, (which is a clique) has $|S| = k$. This is easily seen by induction; it is clear for a $k$-tree on $k + 2$ vertices. Suppose it is true for a $k$ tree on $n > k + 2$ vertices and let $S$ be any minimal separator in an $n + 1$ vertex $k$-tree, $T_{n+1}$. By Lemma 6 (also by the definition of a $k$-tree) there are vertices $\hat{x}$ in $(T_{n+1} - S)$ with $D(\hat{x}) = \phi$. If $S = \text{Adj}(\hat{x})$ for some such $\hat{x}$ the assertion is clear; otherwise $S$ is minimal separator of the elimination graph $(T_{n+1})_{\hat{x}}$, and the assertion follows by induction.

**Note added in proof.** Triangulated graphs are also known as rigid circuit graphs and have been studied earlier in other contexts. For statements of parts of Theorem 1 see [4] and [5].

### References