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Cross-diffusion induced Turing instability for a three species food chain model

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ABSTRACT

In this paper, we study a strongly coupled reaction–diffusion system describing three interacting species in a food chain model, where the third species preys on the second one and simultaneously the second species preys on the first one. We first show that the unique positive equilibrium solution is globally asymptotically stable for the corresponding ODE system. The positive equilibrium solution remains linearly stable for the reaction–diffusion system without cross-diffusion, hence it does not belong to the classical Turing instability scheme. We further proved that the positive equilibrium solution is globally asymptotically stable for the reaction–diffusion system without cross-diffusion by constructing a Lyapunov function. But it becomes linearly unstable only when cross-diffusion also plays a role in the reaction–diffusion system, hence the instability is driven solely from the effect of cross-diffusion. Our results also exhibit some interesting combining effects of cross-diffusion, intra-species competitions and inter-species interactions.

1. Introduction

Denote by u_1, u_2 and u_3 the population densities of three interacting species. Let $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{K}(\mathbf{u}) = (K_{ij}(\mathbf{u}))_{3\times 3}$. The strongly coupled reaction–diffusion system describing three interacting species can be written as

$$\mathbf{u}_t = \Delta\big[\mathbf{K}(\mathbf{u})\big] + \mathbf{G}(\mathbf{u}). \tag{1}$$

The authors of [21] studied the system for $\mathbf{K}(\mathbf{u}) = \operatorname{diag}(d_1 + \frac{k}{\epsilon + u_2^2}, d_2, d_3)\mathbf{u}$ and

$$\mathbf{G}(\mathbf{u}) = \left(u_1\left(-1 + \frac{u_2 u_3}{u_1 + u_2}\right), u_2\left(-\alpha + \frac{\beta u_1 u_3}{u_1 + u_2}\right), u_3\left(r - u_3 - \frac{(1+\beta)u_1 u_2}{u_1 + u_2}\right)\right).$$

Then the system of Eqs. (1) is a strongly coupled system of partial differential equations which models the dynamics of a two-predator–one-prey ecosystem in which the prey exercises a defense switching mechanism and the predators collaboratively take advantage of the prey's strategy. They demonstrated the emergence of stationary patterns due to the cross-diffusion that arises naturally in the model. They proved that the system (1) has no non-constant positive steady states if $k = 0$ (without cross-diffusion) and it possesses non-constant steady states with appropriate conditions and large k. In [25], the author studied an elliptic system which models the dynamics of a two-preys–one-predator ecosystem and he showed that the cross-diffusions can create the stationary patterns where cross-diffusions are included in such a way that the predator chases the prey and the prey runs away from the predator. In this paper, we study the ecosystem which models

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the dynamics of simple food chain in three species, i.e., the third species preys on the second one and simultaneously the second species preys on the first one.

$$\begin{cases} u_{1t} = \Delta\big[(k_{11} + k_{12}u_2)u_1\big] + u_1(d_1 - b_{11}u_1 - b_{12}u_2), \\ u_{2t} = \Delta\big[(k_{21}u_1 + k_{22} + k_{23}u_3)u_2\big] + u_2(d_2 + b_{21}u_1 - b_{22}u_2 - b_{23}u_3), \\ u_{3t} = \Delta\big[(k_{32}u_2 + k_{33})u_3\big] + u_3(d_3 + b_{32}u_2 - b_{33}u_3) \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \dfrac{\partial u_3}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u_i(x,0) = u_{i0}(x) \quad \text{in } \Omega \text{ for } i = 1,2,3, \end{cases} \tag{2}$$

where n is the unit outward normal to $\partial\Omega$. k_{ij}, b_{ij}, d_i, $1 \leqslant i, j \leqslant 3$ are all positive constants. d_i denotes the intrinsic growth rate of i-th species. k_{ii} is the diffusion rate of i-th species and k_{ij} $(i \neq j)$ is the cross-diffusion rate of i-th species due to the pressure of the presence of j-th species. b_{ii}s are for intra-species competitions and b_{ij} $(i \neq j)$ are for inter-species interactions (see Murray [19] and Okubo [20] for a detailed discussion on biological models). Here k_{13} and k_{31} are missing in (2), which means that we do not care the "cross-diffusion" between third and first species in the simple food chain.

The role of diffusion and cross-diffusion in the modeling of many physical, chemical and biological processes has been extensively studied. A pure diffusion process usually leads to a stabilizing effect so that the system tends to a constant equilibrium state. However the combined effect of diffusion and chemical reaction may result in destabilizing the constant equilibrium. In 1952, Alan Turing published a paper "The chemical basis of morphogenesis" [24] which is now regarded as the foundation of basic chemical theory or reaction–diffusion theory of morphogenesis. Turing suggested that, under certain conditions, chemicals can react and diffuse in such a way as to produce non-constant equilibrium solutions, which represent spatial patterns of chemical or morphogen concentration.

Turing's idea is a simple but profound one. He considered a reaction–diffusion system

$$\begin{cases} u_t = D_u\Delta u + f(u,v), \quad t > 0, \\ v_t = D_v\Delta v + g(u,v), \quad t > 0, \end{cases} \tag{3}$$

and its corresponding kinetic equation

$$\begin{cases} u' = f(u,v), \quad t > 0, \\ v' = g(u,v), \quad t > 0. \end{cases} \tag{4}$$

He said that if, in the absence of the diffusion (considering (4)), u and v tend to a linearly stable uniform steady state, then, with the presence of diffusion and under certain conditions, the uniform steady state can become unstable, and spatially inhomogeneous patterns can evolve through bifurcations. In other words, a constant equilibrium can be asymptotically stable with respect to (4), but it is unstable with respect to (3). Therefore this constant equilibrium solution becomes unstable because of the diffusion, which is called a *diffusion-driven instability*.

In [23], the authors presented a general instability analysis on cross-diffusion system with two species. They showed that cross-diffusion can destabilize a uniform equilibrium which is stable for the kinetic and self-diffusion–reaction systems; on the other hand, cross-diffusion can also stabilize a uniform equilibrium which is stable for the kinetic system but unstable for the self-diffusion–reaction system. Zeng [29] studied a prey–predator system with the Holling type-I functional response involving cross-diffusions. The author conducted Turing instability analysis and established the existence and non-existence of its non-constant positive solutions. Fu, Wen and Cui [7] established the existence and the uniform boundedness of global solutions for a similar strongly coupled three species food chain model with cross-diffusion by using the energy estimate and Gagliardo–Nirenberg-type inequalities.

Over the years, Turing's idea has attracted the attention of a great number of investigators and was successfully developed on the theoretical backgrounds. Not only it has been studied in biological and chemical fields, some investigations range as far as economics, semiconductor physics, and star formation. These include the predator–prey model [1–12,15–17, 22,25–28], the vegetation pattern formation [13,14,23], the chemotactic diffusion model [18,26] and the references therein.

The main purpose of this paper is to give an interesting example of the Turing instability which is driven solely from the effect of cross-diffusion. In Section 2 we first show that the unique positive equilibrium of (2) is globally asymptotically stable for the ODE system (5). In Section 3 we show that the positive equilibrium remains linearly stable in the presence of self-diffusion without cross-diffusion. It becomes linearly unstable with the inclusion of some appropriate cross-diffusion influences. The Turing instability occurs only when the cross-diffusion rate k_{21} and k_{32} are large.

Remark 1.1. Since our main attention in this paper is to discuss the non-constant positive solutions of (2), we shall not discuss the well-posedness of the initial and boundary value problem of the corresponding time-dependent PDE system.

2. Stability of the positive equilibrium solution of the ODE system

In this section, we consider the ODE system associated with (2):

$$\begin{cases} \dfrac{du_1}{dt} = u_1 g_1(u_1, u_2) := u_1(d_1 - b_{11}u_1 - b_{12}u_2), \\ \dfrac{du_2}{dt} = u_2 g_2(u_1, u_2, u_3) := u_2(d_2 + b_{21}u_1 - b_{22}u_2 - b_{23}u_3), \\ \dfrac{du_3}{dt} = u_3 g_3(u_2, u_3) := u_3(d_3 + b_{32}u_2 - b_{33}u_3), \\ u_i(0) = u_{i0} \quad \text{in } \Omega \text{ for } i = 1, 2, 3. \end{cases} \tag{5}$$

Let $\mathbf{u} = (u_1, u_2, u_3)^T$ be a positive solution of (5), i.e. $u_i > 0$, $i = 1, 2, 3$. It is easy to know that the ODE system (5) has a unique positive equilibrium if

$$\begin{cases} b_{32}d_1b_{23} + d_1b_{22}b_{33} + b_{12}d_3b_{23} > b_{12}b_{33}d_2, \\ b_{33}b_{11}d_2 + b_{33}b_{21}d_1 > d_3b_{11}b_{23}. \end{cases} \tag{6}$$

Denote by $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)^T$ the unique positive equilibrium. Then $\bar{\mathbf{u}}$ is given by

$$\bar{u}_1 = \frac{m_1}{M}, \qquad \bar{u}_2 = \frac{m_2}{M}, \qquad \bar{u}_3 = \frac{m_3}{M}, \tag{7}$$

where

$$\begin{aligned} m_1 &= b_{32}d_1b_{23} + d_1b_{22}b_{33} + b_{12}d_3b_{23} - b_{12}b_{33}d_2, \\ m_2 &= b_{33}b_{11}d_2 + b_{33}b_{21}d_1 - d_3b_{11}b_{23}, \\ m_3 &= b_{11}b_{32}d_2 + b_{21}b_{12}d_3 + b_{11}b_{22}d_3 + b_{21}b_{32}d_1, \\ M &= b_{11}b_{32}b_{23} + b_{11}b_{22}b_{33} + b_{21}b_{12}b_{33}. \end{aligned} \tag{8}$$

Remark 2.1. We give two examples at the end of the paper, for which the conditions (6) are satisfied. Each of them gives a unique positive equilibrium.

Theorem 2.2. *The unique positive equilibrium $\bar{\mathbf{u}}$ given by* (7) *is globally asymptotically stable for the ODE system* (5)*.*

Proof. We are going to construct a Lyapunov function for the system (5) to prove the theorem. Define

$$\begin{aligned} V\big(\mathbf{u}(t)\big) = {} & \left(u_1 - \bar{u}_1 - \bar{u}_1 \ln \frac{u_1}{\bar{u}_1}\right) \\ & + p\left(u_2 - \bar{u}_2 - \bar{u}_2 \ln \frac{u_2}{\bar{u}_2}\right) + q\left(u_3 - \bar{u}_3 - \bar{u}_3 \ln \frac{u_3}{\bar{u}_3}\right), \end{aligned} \tag{9}$$

where

$$p = b_{12}/b_{21}, \qquad q = \frac{b_{12}b_{23}}{b_{21}b_{32}}.$$

Then $V(\bar{\mathbf{u}}) = 0$, $V(\mathbf{u}) > 0$ if $\mathbf{u} \neq \bar{\mathbf{u}}$. By using (5), we compute

$$\begin{aligned} \frac{dV}{dt} &= \left(1 - \frac{\bar{u}_1}{u_1}\right)u_1' + p\left(1 - \frac{\bar{u}_2}{u_2}\right)u_2' + q\left(1 - \frac{\bar{u}_3}{u_3}\right)u_3' \\ &= (u_1 - \bar{u}_1)g_1(u_1, u_2) + p(u_2 - \bar{u}_2)g_2(u_1, u_2, u_3) + q(u_3 - \bar{u}_3)g_3(u_2, u_3). \end{aligned}$$

Note that $g_1(\bar{u}_1, \bar{u}_2) = 0$, i.e. $d_1 = b_{11}\bar{u}_1 + b_{12}\bar{u}_2$ and similarly for g_2 and g_3. Then we have

$$\begin{aligned} \frac{dV}{dt} &= (u_1 - \bar{u}_1)\big(-b_{11}(u_1 - \bar{u}_1) - b_{12}(u_2 - \bar{u}_2)\big) \\ &\quad + p(u_2 - \bar{u}_2)\big(b_{21}(u_1 - \bar{u}_1) - b_{22}(u_2 - \bar{u}_2) - b_{23}(u_3 - \bar{u}_3)\big) \\ &\quad + q(u_3 - \bar{u}_3)\big(b_{32}(u_2 - \bar{u}_2) - b_{33}(u_3 - \bar{u}_3)\big) \\ &= -\big(b_{11}(u_1 - \bar{u}_1)^2 + (b_{12} - pb_{21})(u_1 - \bar{u}_1)(u_2 - \bar{u}_2) + pb_{22}(u_2 - \bar{u}_2)^2 \\ &\quad + (pb_{23} - qb_{32})(u_2 - \bar{u}_2)(u_3 - \bar{u}_3) + qb_{33}(u_3 - \bar{u}_3)^2\big) \\ &= -\left(b_{11}(u_1 - \bar{u}_1)^2 + \frac{b_{12}b_{22}}{b_{21}}(u_2 - \bar{u}_2)^2 + \frac{b_{12}b_{23}b_{33}}{b_{21}b_{32}}(u_3 - \bar{u}_3)^2\right) \\ &\leqslant -\sigma\big((u_1 - \bar{u}_1)^2 + (u_2 - \bar{u}_2)^2 + (u_3 - \bar{u}_3)^2\big), \end{aligned}$$

where $\sigma = \min\{b_{11}, \frac{b_{12}b_{22}}{b_{21}}, \frac{b_{12}b_{23}b_{33}}{b_{21}b_{32}}\} > 0$. Then

$$\frac{dV}{dt} < 0 \quad \text{for all } \mathbf{u} \neq \bar{\mathbf{u}}.$$

By the Lyapunov–LaSalle invariance principle [8], $\bar{\mathbf{u}}$ is globally asymptotically stable for the ODE system (5). □

Theorem 2.3. *Suppose that $k_{ij} = 0$ for $i \neq j$. The unique positive equilibrium $\bar{\mathbf{u}}$ given by* (7) *is globally asymptotically stable for the reaction–diffusion system* (2) *without cross-diffusion.*

Proof. To study the global behavior of system (2), we introduce the following Lyapunov functional

$$W(t) = \int_\Omega V\big(\mathbf{u}(x,t)\big)\,dx, \tag{10}$$

where $V(\mathbf{u}(x,t))$ is given by (9). More methods of constructing Lyapunov function can be found in Hsu's paper [10]. By direct computation, we have

$$\begin{aligned}
\frac{dW}{dt} &= \int_\Omega \operatorname{grad}_{\mathbf{u}} V \cdot \frac{\partial \mathbf{u}}{\partial t}\,dx \\
&= \int_\Omega \left(1 - \frac{\bar{u}_1}{u_1}, p\left(1 - \frac{\bar{u}_2}{u_2}\right), q\left(1 - \frac{\bar{u}_3}{u_3}\right)\right) \cdot (k_{11}\Delta u_1 + u_1 g_1, k_{22}\Delta u_2 + u_2 g_2, k_{33}\Delta u_3 + u_3 g_3)\,dx \\
&= \int_\Omega \left(k_{11}\left(1 - \frac{\bar{u}_1}{u_1}\right)\triangle u_1\right)dx + \int_\Omega p\left(k_{22}\left(1 - \frac{\bar{u}_2}{u_2}\right)\triangle u_2\right)dx + \int_\Omega q\left(k_{33}\left(1 - \frac{\bar{u}_3}{u_3}\right)\triangle u_3\right)dx + \int_\Omega \frac{dV}{dt}\,dx.
\end{aligned}$$

From Green's identity, it follows that

$$\begin{aligned}
\int_\Omega \left(k_{ii}\left(1 - \frac{\bar{u}_i}{u_i}\right)\triangle u_i\right)dx &= \int_{\partial\Omega} k_{ii}\left(1 - \frac{\bar{u}_i}{u_i}\right)\frac{\partial u_i}{\partial n}\,dS - \int_\Omega k_{ii}\nabla_x\left(1 - \frac{\bar{u}_i}{u_i}\right)\cdot \nabla_x u_i\,dx \\
&= -\int_\Omega k_{ii}\bar{u}_i u_i^{-2}|\nabla_x u_i|^2\,dx \leqslant 0.
\end{aligned}$$

Since $\frac{dV}{dt} \leqslant 0$, $\int_\Omega \frac{dV}{dt}\,dx \leqslant 0$. Thus,

$$\frac{dW}{dt} < 0 \quad \text{for all } \mathbf{u} \neq \bar{\mathbf{u}}.$$

By the Lyapunov–LaSalle invariance principle [8], $\bar{\mathbf{u}}$ is globally asymptotically stable for the reaction–diffusion system (2) without cross-diffusion. □

3. Turing instability without or with cross-diffusion

For simplicity we denote

$$\mathbf{K}(\mathbf{u}) = \begin{pmatrix} (k_{11} + k_{12}u_2)u_1 \\ (k_{21}u_1 + k_{22} + k_{23}u_3)u_2 \\ (k_{32}u_2 + k_{33})u_3 \end{pmatrix},$$

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} u_1(d_1 - b_{11}u_1 - b_{12}u_2) \\ u_2(d_2 + b_{21}u_1 - b_{22}u_2 - b_{23}u_3) \\ u_3(d_3 + b_{32}u_2 - b_{33}u_3) \end{pmatrix}.$$

Then the reaction–diffusion system (2) can be rewritten in matrix notation as:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{K}(\mathbf{u}) + \mathbf{G}(\mathbf{u}) \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial \mathbf{u}}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{u}(x,0) = \big(u_{10}(x), u_{20}(x), u_{30}(x)\big)^T \quad \text{in } \Omega. \end{cases} \tag{11}$$

Linearizing the reaction–diffusion system (11) about the positive equilibrium $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$, we have

$$\frac{\partial \boldsymbol{\Psi}}{\partial t} = \mathbf{K_u}(\bar{\mathbf{u}})\Delta\boldsymbol{\Psi} + \mathbf{G_u}(\bar{\mathbf{u}})\boldsymbol{\Psi}, \tag{12}$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)^T$ and

$$\mathbf{K_u}(\bar{\mathbf{u}}) = \begin{pmatrix} k_{11} + k_{12}\bar{u}_2 & k_{12}\bar{u}_1 & 0 \\ k_{21}\bar{u}_2 & k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3 & k_{23}\bar{u}_2 \\ 0 & k_{32}\bar{u}_3 & k_{33} + k_{32}\bar{u}_2 \end{pmatrix},$$

$$\mathbf{G_u}(\bar{\mathbf{u}}) = \begin{pmatrix} -b_{11}\bar{u}_1 & -b_{12}\bar{u}_1 & 0 \\ b_{21}\bar{u}_2 & -b_{22}\bar{u}_2 & -b_{23}\bar{u}_2 \\ 0 & b_{32}\bar{u}_3 & -b_{33}\bar{u}_3 \end{pmatrix}.$$

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^2(\Omega)$. Let $\mathbf{X} = \{\mathbf{u} \in [C^1(\bar{\Omega})]^3 \mid \partial_n \mathbf{u} = 0 \text{ on } \partial\Omega\}$, $\{\phi_{ij}\}_{j=1,2,\ldots,\dim E(\mu_i)}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbf{R}^3\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}.$$

For each $i \geqslant 1$, $\mathbf{X}_i$ is invariant under the operator $\mathbf{K_u}(\bar{\mathbf{u}})\Delta + \mathbf{G_u}(\bar{\mathbf{u}})$. Then problem (12) has a non-trivial solution of the form $\boldsymbol{\Psi} = \mathbf{c}\phi\exp(\lambda t)$ if and only if $(\lambda, \mathbf{c})$ is an eigenpair for the matrix $-\mu_i\mathbf{K_u}(\bar{\mathbf{u}}) + \mathbf{G_u}(\bar{\mathbf{u}})$, where $\mathbf{c}$ is a constant vector. Then the equilibrium $\bar{\mathbf{u}}$ is unstable if at least one eigenvalue λ has a positive real part for some μ_i.

The characteristic polynomial of $-\mu_i\mathbf{K_u}(\bar{\mathbf{u}}) + \mathbf{G_u}(\bar{\mathbf{u}})$ is given by

$$\rho_i(\lambda) = \lambda^3 + B_{2i}\lambda^2 + B_{1i}\lambda + B_{0i}, \tag{13}$$

where

$$B_{2i} = b_{22}\bar{u}_2 + b_{11}\bar{u}_1 + b_{33}\bar{u}_3 + (k_{33} + k_{32}\bar{u}_2 + k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3 + k_{11} + k_{12}\bar{u}_2)\mu_i, \tag{14}$$

$$\begin{aligned} B_{1i} = {} & b_{11}\bar{u}_1 b_{33}\bar{u}_3 + \bar{u}_2 b_{23}\bar{u}_3 b_{32} + \bar{u}_2 b_{21}\bar{u}_1 b_{12} + b_{22}\bar{u}_2 b_{33}\bar{u}_3 + b_{11}\bar{u}_1 b_{22}\bar{u}_2 \\ & + \big[b_{22}\bar{u}_2(k_{33} + k_{32}\bar{u}_2) + (k_{11} + k_{12}\bar{u}_2)b_{33}\bar{u}_3 + (k_{11} + k_{12}\bar{u}_2)b_{22}\bar{u}_2 + k_{12}\bar{u}_1\bar{u}_2 b_{21} \\ & + (k_{33} + k_{32}\bar{u}_2)b_{11}\bar{u}_1 + (k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3)b_{11}\bar{u}_1 + k_{23}\bar{u}_2\bar{u}_3 b_{32} \\ & + (k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3)b_{33}\bar{u}_3 - b_{12}k_{21}\bar{u}_1\bar{u}_2 - b_{23}k_{32}\bar{u}_2\bar{u}_3\big]\mu_i \\ & + \big[k_{23}k_{33}\bar{u}_3 + k_{11}k_{21}\bar{u}_1 + (k_{21}\bar{u}_1 + k_{22})(k_{33} + k_{32}\bar{u}_2) \\ & + (k_{11} + k_{12}\bar{u}_2)(k_{33} + k_{32}\bar{u}_2) + (k_{11} + k_{12}\bar{u}_2)(k_{22} + k_{23}\bar{u}_3)\big]\mu_i^2, \end{aligned} \tag{15}$$

$$\begin{aligned} B_{0i} = {} & [b_{11}\bar{u}_1 b_{22}\bar{u}_2 b_{33}\bar{u}_3 + \bar{u}_2 b_{21}\bar{u}_1 b_{12} b_{33}\bar{u}_3 + b_{11}\bar{u}_1\bar{u}_2 b_{23}\bar{u}_3 b_{32}] \\ & + \big[b_{11}\bar{u}_1 b_{22}\bar{u}_2(k_{33} + k_{32}\bar{u}_2) + \bar{u}_2 b_{21}k_{12}\bar{u}_1 b_{33}\bar{u}_3 \\ & + \bar{u}_2 b_{21}\bar{u}_1 b_{12}(k_{33} + k_{32}\bar{u}_2) + (k_{11} + k_{12}\bar{u}_2)\bar{u}_2 b_{23}\bar{u}_3 b_{32} \\ & + b_{11}\bar{u}_1 k_{23}\bar{u}_2\bar{u}_3 b_{32} + (k_{11} + k_{12}\bar{u}_2)b_{22}\bar{u}_2 b_{33}\bar{u}_3 + b_{11}\bar{u}_1(k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3)b_{33}\bar{u}_3 \\ & - b_{11}\bar{u}_1\bar{u}_2 b_{23}k_{32}\bar{u}_3 - k_{21}\bar{u}_2\bar{u}_1 b_{12}b_{33}\bar{u}_3\big]\mu_i \\ & + \big[\bar{u}_2 b_{21}k_{12}\bar{u}_1(k_{33} + k_{32}\bar{u}_2) + b_{11}\bar{u}_1(k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3)(k_{33} + k_{32}\bar{u}_2) \\ & + (k_{11} + k_{12}\bar{u}_2)k_{23}\bar{u}_2\bar{u}_3 b_{32} + (k_{11} + k_{12}\bar{u}_2)b_{22}\bar{u}_2(k_{33} + k_{32}\bar{u}_2) \\ & + (k_{11} + k_{12}\bar{u}_2)(k_{21}\bar{u}_1 + k_{22} + k_{23}\bar{u}_3)b_{33}\bar{u}_3 - k_{21}\bar{u}_2\bar{u}_1 b_{12}(k_{33} + k_{32}\bar{u}_2) \\ & - k_{21}\bar{u}_2 k_{12}\bar{u}_1 b_{33}\bar{u}_3 - b_{11}\bar{u}_1 k_{23}\bar{u}_2 k_{32}\bar{u}_3 - (k_{11} + k_{12}\bar{u}_2)\bar{u}_2 b_{23}k_{32}\bar{u}_3\big]\mu_i^2 \\ & + \big[k_{11}k_{21}\bar{u}_1 k_{33} + k_{11}k_{21}\bar{u}_1 k_{32}\bar{u}_2 + k_{11}k_{22}k_{33} + k_{11}k_{22}k_{32}\bar{u}_2 + k_{11}k_{23}\bar{u}_3 k_{33} \\ & + k_{12}\bar{u}_2 k_{22}k_{33} + k_{12}\bar{u}_2^2 k_{22}k_{32} + k_{12}\bar{u}_2 k_{23}\bar{u}_3 k_{33}\big]\mu_i^3. \end{aligned} \tag{16}$$

Let λ_{1i}, λ_{2i}, λ_{3i} be the three roots of (13)

$$\rho_i(\lambda) = \lambda^3 + B_{2i}\lambda^2 + B_{1i}\lambda + B_{0i} = 0.$$

In order to obtain the stability of $\bar{\mathbf{u}}$, we need to show that there exists a positive constant δ such that

$$Re\{\lambda_{1i}\}, Re\{\lambda_{2i}\}, Re\{\lambda_{3i}\} < -\delta \quad \text{for all } i \geqslant 1. \tag{17}$$

The aim of the following theorem is to prove that the diffusion alone (without cross-diffusion, i.e. $k_{21} = k_{12} = k_{32} = k_{23} = 0$) cannot drive instability for this food chain model, i.e., Turing instability does not occur in the three species food chain model without cross-diffusion.

Theorem 3.1. *Suppose that* (6) *holds and* $k_{21} = k_{12} = k_{32} = k_{23} = 0$*. Then the positive equilibrium* $\bar{\mathbf{u}}$ *of* (11) *is linearly stable.*

Proof. Substituting $k_{21} = k_{12} = k_{32} = k_{23} = 0$ into (14), (15), and (16), we have

$$\begin{aligned}
B_{2i} &= b_{22}\bar{u}_2 + b_{11}\bar{u}_1 + b_{33}\bar{u}_3 + (k_{33} + k_{22} + k_{11})\mu_i > 0,\\
B_{1i} &= b_{11}\bar{u}_1 b_{33}\bar{u}_3 + \bar{u}_2 b_{23}\bar{u}_3 b_{32} + \bar{u}_2 b_{21}\bar{u}_1 b_{12} + b_{22}\bar{u}_2 b_{33}\bar{u}_3 + b_{11}\bar{u}_1 b_{22}\bar{u}_2\\
&\quad + [b_{22}k_{33}\bar{u}_2 + k_{11}b_{33}\bar{u}_3 + k_{33}b_{11}\bar{u}_1 + k_{22}b_{11}\bar{u}_1 + k_{22}b_{33}\bar{u}_3 + k_{11}b_{22}\bar{u}_2]\mu_i\\
&\quad + [k_{22}k_{33} + k_{11}k_{33} + k_{11}k_{22}]\mu_i^2 > 0,\\
B_{0i} &= b_{11}\bar{u}_1 b_{22}\bar{u}_2 b_{33}\bar{u}_3 + \bar{u}_2 b_{21}\bar{u}_1 b_{12} b_{33}\bar{u}_3 + b_{11}\bar{u}_1\bar{u}_2 b_{23}\bar{u}_3 b_{32}\\
&\quad + [b_{11}\bar{u}_1 b_{22}\bar{u}_2 k_{33} + \bar{u}_2 b_{21}\bar{u}_1 b_{12} k_{33} + k_{11}\bar{u}_2 b_{23}\bar{u}_3 b_{32} + k_{11}b_{22}\bar{u}_2 b_{33}\bar{u}_3 + b_{11}\bar{u}_1 k_{22} b_{33}\bar{u}_3]\mu_i\\
&\quad + [b_{11}\bar{u}_1 k_{22}k_{33} + k_{11}k_{22}b_{33}\bar{u}_3 + k_{11}b_{22}\bar{u}_2 k_{33}]\mu_i^2 + k_{11}k_{22}k_{33}\mu_i^3 > 0.
\end{aligned}$$

A direct calculation shows that $B_{2i}B_{1i} - B_{0i} > 0$ for all $i \geqslant 1$. It follows from Routh–Hurwitz criterion that, all the three roots λ_{1i}, λ_{2i}, λ_{3i} of $\rho_i(\lambda) = 0$ have negative real parts for each $i \geqslant 1$.

Let $\lambda = \mu_i\xi$, then

$$\rho_i(\lambda) = \mu_i^3\xi^3 + B_{2i}\mu_i^2\xi^2 + B_{1i}\mu_i\xi + B_3 \equiv \tilde{\rho}_i(\xi).$$

Since $\mu_i \to \infty$ as $i \to \infty$, we have

$$\bar{\rho}(\xi) = \lim_{i\to\infty}\frac{\tilde{\rho}_i(\xi)}{\mu_i^3} = \xi^3 + (k_{33} + k_{22} + k_{11})\xi^2 + (k_{22}k_{33} + k_{11}k_{33} + k_{11}k_{22})\xi + k_{11}k_{22}k_{33}.$$

Applying the Routh–Hurwitz criterion it follows that the three roots ξ_1, ξ_2, ξ_3 of $\bar{\rho}(\xi) = 0$ all have negative real parts. Thus, there exists a positive constant $\bar{\delta}$ such that $Re\{\xi_1\}$, $Re\{\xi_2\}$, $Re\{\xi_3\} \leqslant -2\bar{\delta}$. By continuity, we see that there exists $i_0 \geqslant 1$ such that $\mu_{i_0} > 1$ and the three roots ξ_{i1}, ξ_{i2}, ξ_{i3} of $\tilde{\rho}_i(\xi) = 0$ satisfy $Re\{\xi_{i1}\}$, $Re\{\xi_{i2}\}$, $Re\{\xi_{i3}\} \leqslant -\bar{\delta}$ for any $i \geqslant i_0$. Then $Re\{\lambda_{i1}\}$, $Re\{\lambda_{i2}\}$, $Re\{\lambda_{i3}\} \leqslant -\mu_i\bar{\delta} \leqslant -\mu_{i0}\bar{\delta} \leqslant -\bar{\delta}$ for any $i \geqslant i_0$. Let $-\tilde{\delta} = \max_{1\leqslant i\leqslant i_0}\{Re\{\lambda_{i1}\}, Re\{\lambda_{i2}\}, Re\{\lambda_{i3}\}\}$ and $\delta = \min\{\tilde{\delta}, \bar{\delta}\}$. Then

$$Re\{\lambda_{1i}\}, Re\{\lambda_{2i}\}, Re\{\lambda_{3i}\} < -\delta \quad \text{for all } i \geqslant 1.$$

Consequently the equilibrium $\bar{\mathbf{u}}$ is linearly stable. □

Note that $B_{2i} > 0$ in (14), $B_{1i} > 0$ in (15), $B_{0i} > 0$ in (16), and $B_{2i}B_{1i} - B_{0i} > 0$ if $k_{21} = k_{32} = 0$ since the possible negative terms all involve either k_{21} or k_{32}. By the same arguments as in Theorem 3.1, we have

Theorem 3.2. *Suppose that* (6) *holds and* $k_{21} = k_{32} = 0$*. Then the positive equilibrium* $\bar{\mathbf{u}}$ *of* (11) *is linearly stable.*

The Turing instability [24] refers to "diffusion-driven instability," i.e., the stability of the positive equilibrium $\bar{\mathbf{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ changing from stable, for the ODE dynamics (5), to unstable, for the PDE dynamics (11). Here we are going to give sufficient conditions on cross-diffusion which drives the instability. k_{21} and k_{32} are chosen as variation parameters, whereas the other constants are fixed.

Theorem 3.3.

(1) *Suppose that* $b_{11}\bar{u}_1 - b_{12}\bar{u}_2 < 0$*. Consider* k_{21} *as the variation parameter. Then there exists a positive constant* δ_{21} *such that when* $k_{21} > \delta_{21}$*, the equilibrium* $\mathbf{u}(x,t) = \bar{\mathbf{u}}$ *is linearly unstable for some domain* Ω*.*
(2) *Suppose that* $(b_{21}b_{12}\bar{u}_2 + b_{11}b_{22}\bar{u}_2 - b_{11}b_{23}\bar{u}_3) < 0$*. Consider* k_{32} *as the variation parameter. Then there exists a positive constant* δ_{32} *such that when* $k_{32} > \delta_{32}$*, the equilibrium* $\mathbf{u}(x,t) = \bar{\mathbf{u}}$ *is linearly unstable for some domain* Ω*.*

Remark 3.4. (A) The conditions in Theorem 3.3 $b_{11}\bar{u}_1 - b_{12}\bar{u}_2 < 0$ and $(b_{21}b_{12}\bar{u}_2 + b_{11}b_{22}\bar{u}_2 - b_{11}b_{23}\bar{u}_3) < 0$ are compatible with the condition (6).

(B) k_{21} and k_{32} can be chosen as variation parameters because the number of sign of change for the polynomial (18) could be bigger than one for large values of k_{21} or k_{32}. By Descartes' rule, the polynomial (18) could have positive roots which lead to linear instability.

(C) **Biological interpretation:** In our model, the third species preys on the second one and simultaneously the second species preys on the first one (prey). The positive steady state of the three species food chain can be broken by the reaction–diffusion among two species on the chain. Case one: In this case, the first species (the prey) are assumed to reproduce exponentially unless subject to intra-species competitions and predation. This exponential growth is represented in the equation by the term (d_1u_1). The level of intra-species competitions among first species is assumed to be proportional to the population density of first species by the term $(b_{11}u_1)$. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet by term $(b_{12}u_1u_2)$. When the effects on first species due the fact that the second species preys on the first one $(b_{12}\bar{u}_2)$ are larger than the effects on first species due to the intra-species competitions among first species $(b_{11}\bar{u}_1)$, the large cross-diffusion of the second species due to the first species (k_{21}) can break the stability of the positive steady state. In other words, if the predator (second species) has a dominate effect on the decreasing of the prey (first species) such as predation rate is larger than the rate of intra-species competitions, then the predator (the second species) with large cross-diffusion can destabilize the constant steady state. Case two: In this case, the third species (predator) shall have a dominate effect on the decreasing of the second species. Because $(b_{21}b_{12}\bar{u}_2 + b_{11}b_{22}\bar{u}_2 - b_{11}b_{23}\bar{u}_3) < 0$ implies $b_{22}\bar{u}_2 < b_{23}\bar{u}_3$, predation rate of third species on the second species is larger than the rate of intra-species competitions in second species. The similar situation as in case one happens in the case two: the predator (the third species) with large cross-diffusion can destabilize the constant steady state.

Proof. Denote $A(\mu) = -\mu\mathbf{K_u}(\bar{\mathbf{u}}) + \mathbf{G_u}(\bar{\mathbf{u}})$. By the direct computations we have

$$\det\bigl(A(\mu)\bigr) = -\bigl(C_3\mu^3 + C_2\mu^2 + C_1\mu + C_0\bigr), \tag{18}$$

where

$$\begin{aligned}
C_3 &= [k_{11}k_{21}\bar{u}_1k_{33} + k_{11}k_{21}\bar{u}_1k_{32}\bar{u}_2 + k_{11}k_{22}k_{33} + k_{11}k_{22}k_{32}\bar{u}_2 + k_{11}k_{23}\bar{u}_3k_{33} \\
&\quad + k_{12}\bar{u}_2k_{22}k_{33} + k_{12}\bar{u}_2^2k_{22}k_{32} + k_{12}\bar{u}_2k_{23}\bar{u}_3k_{33}] > 0, \\
C_2 &= -k_{21}\bar{u}_2^2\bar{u}_1b_{12}k_{32} - k_{11}\bar{u}_2b_{23}k_{32}\bar{u}_3 - k_{21}\bar{u}_2\bar{u}_1b_{12}k_{33} - k_{12}\bar{u}_2^2b_{23}k_{32}\bar{u}_3 \\
&\quad + \bar{u}_2b_{21}k_{12}\bar{u}_1k_{33} + k_{12}\bar{u}_2k_{23}\bar{u}_3^2b_{33} + k_{11}k_{21}\bar{u}_1b_{33}\bar{u}_3 + k_{11}b_{22}\bar{u}_2^2k_{32} + b_{11}\bar{u}_1k_{22}k_{33} \\
&\quad + k_{12}\bar{u}_2k_{22}b_{33}\bar{u}_3 + b_{11}\bar{u}_1^2k_{21}k_{32}\bar{u}_2 + b_{11}\bar{u}_1k_{23}\bar{u}_3k_{33} + k_{12}\bar{u}_2^2b_{22}k_{33} + b_{11}\bar{u}_1k_{22}k_{32}\bar{u}_2 \\
&\quad + \bar{u}_2^2b_{21}k_{12}\bar{u}_1k_{32} + k_{12}\bar{u}_2^2k_{23}\bar{u}_3b_{32} + k_{12}\bar{u}_2^3b_{22}k_{32} + k_{11}b_{22}\bar{u}_2k_{33} \\
&\quad + k_{11}k_{23}\bar{u}_2\bar{u}_3b_{32} + b_{11}\bar{u}_1^2k_{21}k_{33} + k_{11}k_{23}\bar{u}_3^2b_{33} + k_{11}k_{22}b_{33}\bar{u}_3, \\
C_1 &= b_{33}\bar{u}_1\bar{u}_3(b_{11}\bar{u}_1 - b_{12}\bar{u}_2)k_{21} + \bar{u}_1\bar{u}_2(\bar{u}_2b_{21}b_{12} + \bar{u}_2b_{11}b_{22} - b_{11}b_{23}\bar{u}_3)k_{32} \\
&\quad + k_{11}\bar{u}_2b_{23}\bar{u}_3b_{32} + k_{11}b_{22}\bar{u}_2b_{33}\bar{u}_3 + k_{12}\bar{u}_2^2b_{23}\bar{u}_3b_{32} + \bar{u}_2b_{21}k_{12}\bar{u}_1b_{33}\bar{u}_3 + \bar{u}_2b_{21}\bar{u}_1b_{12}k_{33} \\
&\quad + b_{11}\bar{u}_1k_{23}\bar{u}_3^2b_{33} + k_{12}\bar{u}_2^2b_{22}b_{33}\bar{u}_3 + b_{11}\bar{u}_1k_{22}b_{33}\bar{u}_3 + b_{11}\bar{u}_1k_{23}\bar{u}_2\bar{u}_3b_{32} + b_{11}\bar{u}_1b_{22}\bar{u}_2k_{33}, \\
C_0 &= \bar{u}_1\bar{u}_2\bar{u}_3(b_{11}b_{23}b_{32} + b_{11}b_{22}b_{33} + b_{21}b_{12}b_{33}) > 0.
\end{aligned}$$

Case 1: k_{21} is the variation parameter.

We assume that $b_{11}\bar{u}_1 - b_{12}\bar{u}_2 < 0$. The following arguments by continuation are based on the fact that each root of the algebraic equation (18) is a continuous function of the variation parameter k_{21}. It is easy to prove that Eq. (18) has three real roots $\mu_1^{(i)} = \mu_1^{(i)}(k_{21})$, $i = 1, 2, 3$ when k_{21} goes to infinity and they are $\lim_{k_{21}\to\infty}\mu_1^{(1)}(k_{21}) = -\frac{b_{33}\bar{u}_3}{k_{33}+k_{32}\bar{u}_2} < 0$ and $\lim_{k_{21}\to\infty}\mu_1^{(2)}(k_{21}) = 0$ and

$$\lim_{k_{21}\to\infty}\mu_1^{(3)}(k_{21}) = -\frac{b_{11}\bar{u}_1 - b_{12}\bar{u}_2}{k_{11}} > 0.$$

By continuation, there exists a positive constant δ_{21} such that when $k_{21} > \delta_{21}$, $C_1 < 0$ and $\det(A(\mu)) = 0$ has three real roots. Because $C_3 > 0$ and $C_0 > 0$, the number of sign changes of (18) is exactly two. Therefore by Descartes' rule, the three real roots have the following properties:

(i) $-\infty < \mu_1^{(1)} < 0 < \mu_1^{(2)} < \mu_1^{(3)} < \infty$;
(ii) $\det(A(\mu)) > 0$ if $\mu \in (-\infty, \mu_1^{(1)}) \cup (\mu_1^{(2)}, \mu_1^{(3)})$;
(iii) $\det(A(\mu)) < 0$ if $\mu \in (\mu_1^{(1)}, \mu_1^{(2)}) \cup (\mu_1^{(3)}, \infty)$.

If $\mu_i \in (\mu_1^{(2)}, \mu_1^{(3)})$ for some i, then $\det(A(\mu_i)) > 0$ by (ii), and consequently $B_{0i} = -\det(A(\mu_i)) < 0$. The number of sign changes of the characteristic polynomial (13) $\rho(\lambda) = \lambda^3 + B_{2i}\lambda^2 + B_{1i}\lambda + B_{0i}$ is either one or three. By Descartes' rule, the characteristic polynomial (13) has at least one positive eigenvalue. Hence, the equilibrium $\bar{\mathbf{u}}$ of (11) is linearly unstable for any domain Ω on which at least one eigenvalue μ_i of $-\Delta$ is in the interval $(\mu_1^{(2)}, \mu_1^{(3)})$.

Case 2: k_{32} is the variation parameter.

We assume that $(b_{21}b_{12}\bar{u}_2 + b_{11}b_{22}\bar{u}_2 - b_{11}b_{23}\bar{u}_3) < 0$. The following arguments by continuation are based on the fact that each root of the algebraic equation (18) is a continuous function of the variation parameter k_{32}. It is easy to prove that Eq. (18) has three real roots $\mu_2^{(i)} = \mu_2^{(i)}(k_{32})$, $i = 1, 2, 3$ when k_{32} goes to infinity and they are $\lim_{k_{32}\to\infty} \mu_2^{(1)}(k_{32}) < 0$ and $\lim_{k_{32}\to\infty} \mu_2^{(2)}(k_{32}) = 0$ and $\lim_{k_{32}\to\infty} \mu_2^{(3)}(k_{32}) > 0$.

By continuation, there exists a positive constant δ_{32} such that when $k_{32} > \delta_{32}$, $C_1 < 0$ and $\det(A(\mu)) = 0$ has three real roots. Because $C_3 > 0$ and $C_0 > 0$, the number of sign changes of (18) is exactly two. Therefore by Descartes' rule, the three real roots have the following properties:

(i) $-\infty < \mu_2^{(1)} < 0 < \mu_2^{(2)} < \mu_2^{(3)} < \infty$;
(ii) $\det(A(\mu)) > 0$ if $\mu \in (-\infty, \mu_2^{(1)}) \cup (\mu_2^{(2)}, \mu_2^{(3)})$;
(iii) $\det(A(\mu)) < 0$ if $\mu \in (\mu_2^{(1)}, \mu_2^{(2)}) \cup (\mu_2^{(3)}, \infty)$.

If $\mu_i \in (\mu_2^{(2)}, \mu_2^{(3)})$ for some i, then $\det(A(\mu_i)) > 0$, and consequently $B_{0i} = -\det(A(\mu_i)) < 0$ and $\rho_i(0) = B_{0i} < 0$. By similar argument as in case 1, the characteristic polynomial (13) has at least one positive eigenvalue. Hence, the equilibrium $\bar{\mathbf{u}}$ of (11) is linearly unstable for any domain Ω on which at least one eigenvalue μ_i of $-\Delta$ is in the interval $(\mu_2^{(2)}, \mu_2^{(3)})$. □

We end our paper by the following three examples. They demonstrate the reason why K_{21} and K_{32} can be chosen as variation parameters and the existence of the parameters with which Theorem 3.3 holds.

Example 3.5. This example is to show the reason why k_{21} and k_{32} can be chosen as variation parameters. Let $d_1 = 3$; $d_2 = 2$; $d_3 = 1$; $b_{11} = 1$; $b_{22} = 1$; $b_{33} = 1$; $b_{12} = 20$; $b_{21} = 2$; $b_{23} = 3$; $b_{32} = 3$. Then $\bar{\mathbf{u}} = (1, \frac{1}{10}, \frac{13}{10})$, $(b_{11}\bar{u}_1 - b_{12}\bar{u}_2) = -1 < 0$ and the coefficients of the polynomial (18) are: $C_0 = \frac{13}{2}$;

$$C_1 = -\frac{13}{10}k_{21} + \frac{1}{50}k_{32} + \frac{13}{10}k_{11} + \frac{39}{100}k_{12} + \frac{41}{10}k_{33} + \frac{52}{25}k_{23} + \frac{13}{10}k_{22};$$
$$C_2 = -\frac{19}{50}k_{11}k_{32} - \frac{9}{500}k_{12}k_{32} - k_{21}k_{33} - \frac{1}{10}k_{21}k_{32} + \frac{21}{100}k_{12}k_{33} + \frac{26}{125}k_{12}k_{23} + \frac{13}{10}k_{11}k_{21}$$
$$+ k_{22}k_{33} + \frac{13}{100}k_{12}k_{22} + \frac{1}{10}k_{22}k_{32} + \frac{13}{10}k_{23}k_{33} + \frac{1}{10}k_{11}k_{33} + \frac{52}{25}k_{11}k_{23} + \frac{13}{10}k_{11}k_{22};$$
$$C_3 = \frac{1}{100}k_{12}k_{22}k_{32} + \frac{1}{10}k_{12}k_{22}k_{33} + \frac{1}{10}k_{11}k_{21}k_{32} + \frac{1}{10}k_{11}k_{22}k_{32}$$
$$+ k_{11}k_{21}k_{33} + \frac{13}{10}k_{11}k_{23}k_{33} + \frac{13}{100}k_{12}k_{23}k_{33} + k_{11}k_{22}k_{33}.$$

It is easy to see that the only negative term in C_1 is k_{21}. For the large value of k_{21}, the number of change of the signs in polynomial (18) is two. Then there are some possible positive roots which are shown in the next examples.

Example 3.6. This example is for case 1 in Theorem 3.3. Let $d_1 = 3$; $d_2 = 2$; $d_3 = 1$; $b_{11} = 1$; $b_{22} = 1$; $b_{33} = 1$; $b_{12} = 20$; $b_{21} = 2$; $b_{23} = 3$; $b_{32} = 3$; $k_{11} = 1$; $k_{12} = 2$; $k_{22} = 1$; $k_{23} = 2$; $k_{33} = 1$. We choose $k_{21} = 200$ (large k_{21}), $k_{32} = 2$. Then $\bar{\mathbf{u}} = (1, \frac{1}{10}, \frac{13}{10})$, $(b_{11}\bar{u}_1 - b_{12}\bar{u}_2) = -1 < 0$ and $C_1 = -\frac{6208}{25} < 0$. The three real roots of Eq. (18) are $\mu_1^{(1)} = -1.083026588$, $\mu_1^{(2)} = 0.0262773031$, $\mu_1^{(3)} = 0.933916442$.

Example 3.7. This example is for case 2 in Theorem 3.3. We choose $b_{12} = 2$, $k_{21} = 2$; $k_{32} = 200$ (large k_{32}) whereas other constants remain the same as in Example 3.6. Then $\bar{\mathbf{u}} = (\frac{16}{7}, \frac{5}{14}, \frac{29}{14})$, $(b_{21}b_{12}\bar{u}_2 + b_{11}b_{22}\bar{u}_2 - b_{11}b_{23}\bar{u}_3) = -\frac{31}{7} < 0$ and $C_1 = -\frac{222994}{343} < 0$. The three real roots of Eq. (18) are $\mu_2^{(1)} = -1.501263441$, $\mu_2^{(2)} = 0.369764704$, $\mu_2^{(3)} = 0.922342861$.

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References

[1] X. Chen, Y. Qi, M. Wang, A strongly coupled predator–prey system with non-monotonic functional response, Nonlinear Anal. 67 (2007) 1966–1979.
[2] L. Chen, A. Jungel, Analysis of a parabolic cross-diffusion population model without self-diffusion, J. Differential Equations 224 (1) (2006) 39–59.
[3] Y.H. Du, Y. Lou, Qualitative behaviour of positive solutions of a predator–prey model: effects of saturation, Proc. Roy. Soc. Edinburgh Sect. A 131 (2) (2001) 321–349.
[4] Y.H. Du, J.P. Shi, Some recent results on diffusive predator–prey models in spatially heterogeneous environment, in: Nonlinear Dynamics and Evolution Equations, in: Fields Inst. Commun., vol. 48, Amer. Math. Soc., Providence, RI, 2006, pp. 95–135.
[5] Y.H. Du, J.P. Shi, A diffusive predator–prey model with a protection zone, J. Differential Equations 229 (1) (2006) 63–91.
[6] Y.H. Du, J.P. Shi, Allee effect and bistability in a spatially heterogeneous predator–prey model, Trans. Amer. Math. Soc. 359 (9) (2007) 4557–4593.
[7] S.M. Fu, Z.J. Wen, S.B. Cui, On global solutions for the three-species food-chain model with cross-diffusion, Acta Math. Sinica (Chin. Ser.) 50 (1) (2007) 75–88.
[8] J.K. Hale, Ordinary Differential Equations, Krieger, Malabar, FL, 1980.
[9] L. Hei, Global bifurcation of co-existence states for a predator–prey–mutualist model with diffusion, Nonlinear Anal. Real World Appl. 8 (2007) 619–635.
[10] S.B. Hsu, A survey of constructing Lyapunov functions for mathematical models in population biology, Taiwanese J. Math. 9 (2) (2005) 151–173.
[11] S.B. Hsu, S.P. Hubbell, P. Waltman, Competing predators, SIAM J. Appl. Math. 35 (4) (1978) 617–625.
[12] S.B. Hsu, J.P. Shi, Relaxation oscillator profile of limit cycle in predator–prey system, Discrete Contin. Dyn. Syst. Ser. B 11 (4) (2009) 893–911.
[13] J. Von Hardenberg, E. Meron, M. Shachak, Y. Zarmi, Diversity of vegetation patterns and desertification, Phys. Rev. Lett. 87 (2001) 198101.
[14] E. Meron, E. Gilad, J. von Hardenberg, M. Shachak, Y. Zarmi, Vegetation patterns along a rainfall gradient, Chaos Solitons Fractals 19 (2004) 367–376.
[15] K. Ik Kim, Z. Lin, Coexistence of three species in a strongly coupled elliptic system, Nonlinear Anal. 55 (2003) 313–333.
[16] T. Kadota, K. Kuto, Positive steady states for a prey–predator model with some nonlinear diffusion terms, J. Math. Anal. Appl. 323 (2006) 1387–1401.
[17] K. Kuto, Y. Yamada, Multiple coexistence states for a prey–predator system with cross-diffusion, J. Differential Equations 197 (2004) 315–348.
[18] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, J. Differential Equations 72 (1988) 1–27.
[19] J.D. Murray, Mathematical Biology. I. An Introduction, third ed., Interdiscip. Appl. Math., vol. 17, Springer-Verlag, New York, 2002; II. Spatial Models and Biomedical Applications, Interdiscip. Appl. Math., vol. 18, Springer-Verlag, New York, 2003.
[20] A. Okubo, Diffusion and Ecological Problems: Mathematical Models, Springer-Verlag, Berlin, 1980.
[21] P.Y.H. Pang, M. Wang, Strategy and stationary pattern in a three-species predator–prey model, J. Differential Equations 200 (2004) 245–273.
[22] W.H. Ruan, Positive steady-state solutions of a competing reaction–diffusion system with large cross-diffusion, J. Math. Anal. Appl. 197 (1996) 558–578.
[23] J. Shi, Z. Xie, K. Little, Cross-diffusion induced instability and stability in reaction–diffusion systems, J. Appl. Anal. Comput. 1 (1) (2011) 95–119.
[24] A.M. Turing, The chemical basis of morphogenesis, Philos. Trans. R. Soc. Lond. Ser. B 237 (1952) 37–72.
[25] M. Wang, Stationary patterns of strongly coupled prey–predator models, J. Math. Anal. Appl. 292 (2004) 484–505.
[26] X. Wang, Qualitative behavior of solutions of chemotactic diffusion systems: effects of motility and chemotaxis and dynamics, SIAM J. Math. Anal. 31 (3) (2000) 535–560.
[27] Z.F. Xie, Turing instability in a coupled predator–prey model with different Holling type functional responses, Discrete Contin. Dyn. Syst. Ser. S 4 (2011) 1621–1628.
[28] F.Q. Yi, J.J. Wei, J.P. Shi, Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator–prey system, J. Differential Equations 246 (5) (2009) 1944–1977.
[29] X. Zeng, Non-constant positive steady states of a prey–predator system with cross-diffusions, J. Math. Anal. Appl. 332 (2007) 989–1009.