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On invariant tori of full dimension for 1D periodic NLS

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Abstract

Consider the NLS with periodic boundary conditions in 1D

$$iu_t + \Delta u + Mu \pm \varepsilon u|u|^4 = 0, \tag{0.1}$$

where M is a random Fourier multiplier defined by

$$\widehat{Mu}(n) = V_n \hat{u}(n) \tag{0.2}$$

and $(V_n)_{n\in\mathbb{Z}}$ are independently chosen in $[-1, 1]$.

The quintic nonlinearity in (0.1) is unimportant and may be replaced by $u|u|^{p-2}$, $p \in 2\mathbb{Z}$, $p \geqslant 4$.

We give a proof of the following fact.

Theorem. *For appropriate* M, (0.1) *has an invariant tori* $\mathscr{T}$ (*of full dimension*) *satisfying*

$$\frac{1}{2}e^{-r\sqrt{|n|}} < |q_n| < 2e^{-r\sqrt{|n|}} \qquad (n \in \mathbb{Z}, q \in \mathscr{T})$$

($r > 0$ *is arbitrary*).

Remark. The statement holds in fact for most $(V_n)_{n\in\mathbb{Z}} \in [-1, 1]^{\mathbb{Z}}$, although not explicitly proven here.

Written in Fourier modes $(q_n)_{n\in\mathbb{Z}}$, the Hamiltonian corresponding to (0.1) is given by

$$H(q, \bar{q}) = \sum (n^2 + V_n)|q_n|^2 + \varepsilon \sum_{n_1-n_2+n_3-n_4+n_5-n_6=0} q_{n_1}\bar{q}_{n_2}q_{n_3}\bar{q}_{n_4}q_{n_5}\bar{q}_{n_6}. \tag{0.3}$$

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The proof of Theorem 1 will proceed along the 'usual' KAM scheme where the perturbation is eventually removed by consecutive canonical transformations of phase space. The most relevant literature in the present context of an infinite dimensional phase space are the papers of Fröhlich et al. [Fröhlich, Spencer, Wayne, Localization in disordered, nonlinear dynamical systems, J. Statist. Phys. 42 (1986) 247–274] and especially Pöschel [Pöschel, Small divisors with spatial structure in infinite dimensional Hamiltonian systems, CMP 127 (1990) 351–393] on disordered systems.

Both [Fröhlich, Spencer, Wayne, Localization in disordered, nonlinear dynamical systems, J. Statist. Phys. 42 (1986) 247–274, Pöschel, Small divisors with spatial structure in infinite dimensional Hamiltonian systems, CMP 127 (1990) 351–393] consider Hamiltonians with short-range interactions and hence these results do not apply to our problem. It turns out, however that the scheme, as elaborated on in great detail in [Pöschel, Small divisors with spatial structure in infinite dimensional Hamiltonian systems, CMP 127 (1990) 351–393], is still applicable to (0.3), due to special arithmetical features as will be explained in the next section. Roughly speaking, the key point is the following observation. Let (n_i) be a finite set of modes, $|n_1| \geqslant |n_2| \geqslant \cdots$ and

$$n_1 - n_2 + n_3 - \cdots = 0. \tag{0.4}$$

In the case of a 'near' resonance, there is also a relation

$$n_1^2 - n_2^2 + n_3^2 - \cdots = o(1). \tag{0.5}$$

Unless $n_1 = n_2$, one may then control $|n_1| + |n_2|$ from (0.4), (0.5) by $\sum_{j \geqslant 3} |n_j|$. This feature is specifically 1-dimensional and we do not know at this time how to prove a 2D-analogue of Theorem 1, considering for instance the cubic NLS $iu_t + \Delta u \pm u|u|^2 = 0$ on $\mathbb{T}^2$.

It should also be pointed out that almost periodic solutions on a full set of frequencies for NLS and NLW in 1D were constructed in earlier works (see [Bourgain, Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations, GAFA 6 (2) (1996) 201–230] and [Pöschel, On the construction of almost periodic solutions for nonlinear Schrödinger equations, Ergodic Theory Dynamical Systems 22 (5) (2002) 1537–1559]). These invariant tori (of full dimension) were obtained by successive small perturbations of finite-dimensional tori, resulting in very strong compactness properties and in fact a nonexplicit decay rate of the action variables I_n for $n \to \infty$. On the other hand, the construction in this paper (similarly to [Pöschel, Small divisors with spatial structure in infinite dimensional Hamiltonian systems, CMP 127 (1990) 351–393]) treats all Fourier modes at once and requires explicit and realistic decay conditions.

The multiplier $M = (V_n)$ in (0.3) is to be considered as a parameter and (0.1) a parameter-dependent equation. The role of this parameter is essential to ensure appropriate nonresonance properties of the (modulated) frequencies along the iteration. In the absence of exterior parameters, these conditions need to be realized from amplitude–frequency modulation and suitable restriction of the action-variables. This problem is harder. Indeed, a fast decay of the action-variables (enhancing convergence of the process) allows less frequency modulation and worse small divisors (cf. [Bourgain, On diffusion in high-dimensional Hamiltonian systems and PDE, J. Anal. Math. 80 (2000) 1–35]).

Keywords: Nonlinear Schrodinger equation; Invariant tori; Normal forms

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1. Representation of the Hamiltonians

Our analysis will be performed in complex conjugate variables $(q_n, \bar{q}_n)$ without passing to action-angle variables. The Hamiltonian expressions may involve $I_n = |q_n|^2$

and $J_n = I_n - I_n(0)$ as notation but not as new variables. The invariant torus in Theorem 1 will be the pull-back under the resulting symplectic transformation C of the torus $[I_n = I_n(0) | n \in \mathbb{Z}]$, where $I_n(0)$ are fixed positive numbers with a certain decay rate (to be specified later). At every stage of the iteration, our Hamiltonian H will be expanded in monomials $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ ($\bar{a}, \bar{k}, \bar{k}'$ are multi-indices) of the following form:

$$\prod_n I_n(0)^{a_n} \; q_n^{k_n} \bar{q}_n^{k'_n} = \mathcal{M}_{\bar{a},\bar{k},\bar{k}'}, \tag{1.1}$$

$$a_n, k_n, k'_n \in \mathbb{Z}_+ \cup \{0\}; \sum k_n = \sum k'_n; \sum n k_n = \sum n k'_n,$$

$\operatorname{supp} \mathcal{M}_{\bar{a},\bar{k},\bar{k}'} = \{n | a_n + k_n + k'_n \neq 0\}$ and ‘degree’ of $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'} = \sum_n [2a_n + k_n + k'_n] < \infty$.

With this notation, H has the form

$$H = \sum B_{\bar{a},\bar{k},\bar{k}'} \mathcal{M}_{\bar{a},\bar{k},\bar{k}'} + \sum (n^2 + \tilde{V}_n) |q_n|^2 \tag{1.2}$$

(where $\sum (2a_n + k_n + k'_n) \geqslant 6$) with coefficients $B_{\bar{a},\bar{k},\bar{k}'}$. They will satisfy an estimate

$$B_{\bar{a},\bar{k},\bar{k}'} \leqslant e^{\rho \sum_n (2a_n + k_n + k'_n) \sqrt{|n|} - 2\rho \sqrt{n_1^*}} \tag{1.3}$$

denoting

$$|n| = \max\{1, n, -n\} \text{ and } n_1^* = \max\{|n| \big| a_n + k_n + k'_n \neq 0\}$$

and where $\rho > 0$ is a parameter which will vary slightly along the iteration (as usual in the KAM scheme).

In order to justify (1.3), it has to be pointed out that the expressions $\sum_n (2a_n + k_n + k'_n) \sqrt{|n|} - 2\sqrt{n_1^*}$ are positive. In fact

Lemma 1.1. *Denote* $(n_i^*)_{i \geqslant 1}$ *the decreasing rearrangement of*

$$\{|n| \text{ where } n \text{ is repeated } 2a_n + k_n + k'_n \text{ times}\}.$$

Then

$$\sum_n (2a_n + k_n + k'_n) \sqrt{|n|} \geqslant 2\sqrt{n_1^*} + \frac{1}{4} \sum_{i \geqslant 3} \sqrt{n_i^*}. \tag{1.4}$$

Proof. Denote (n_i), $|n_1| \geqslant |n_2| \geqslant \cdots$, the system $\{n$ repeated $2a_n + k_n + k'_n$ times$\}$. From the definition of the monomials $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ in (1.1), clearly $\sum \varepsilon_i n_i = 0$ for some signs $\varepsilon_i = \pm 1$ (as a consequence of the relation $\sum_n (k_n - k'_n) n = 0$). Therefore n_1^*

$= |n_1| \leqslant \sum_{i \geqslant 2} |n_i|$ and $\sqrt{n_1^*} \leqslant (\sum_{i \geqslant 2} |n_i|)^{1/2}$, so that (1.4) will follow from the inequality

$$\sum_{i \geqslant 2} \sqrt{|n_i|} \geqslant \left(\sum_{i \geqslant 2} |n_i| \right)^{1/2} + \frac{1}{4} \sum_{i \geqslant 3} \sqrt{|n_i|}. \tag{1.5}$$

To justify (1.5), we need to show that if $n_1 \geqslant n_2 \geqslant \cdots \geqslant 1$, then

$$\sum_{i \geqslant 1} \sqrt{n_i} \geqslant \sqrt{\sum_{i \geqslant 1} n_i} + \frac{1}{4} \sum_{i \geqslant 2} \sqrt{n_i}. \tag{1.6}$$

Assume $\sqrt{n_1} \leqslant \frac{1}{2} \sum_{i \geqslant 1} \sqrt{n_i}$.

Then, writing $n_i \leqslant \sqrt{n_1} \cdot \sqrt{n_i}$

$$\sum_{i \geqslant 1} n_i \leqslant \frac{1}{2} \left(\sum_{i \geqslant 1} \sqrt{n_i} \right)^2 \Rightarrow \sqrt{\sum_{i \geqslant 1} n_i} \leqslant \frac{1}{\sqrt{2}} \left(\sum_{i \geqslant 1} \sqrt{n_i} \right),$$

$$\sum_{i \geqslant 1} \sqrt{n_i} \geqslant \sqrt{\sum_{i \geqslant 1} n_i} + \left(1 - \frac{1}{\sqrt{2}} \right) \left(\sum_{i \geqslant 1} \sqrt{n_i} \right).$$

Assume next $\sqrt{n_1} > \frac{1}{2} \sum_{i \geqslant 1} \sqrt{n_i}$.

We need to verify that

$$\left(\sum_{i \geqslant 1} \sqrt{n_i} \right)^2 > \sum_{i \geqslant 1} n_i + \frac{1}{16} \left(\sum_{i \geqslant 2} \sqrt{n_i} \right)^2 + \frac{1}{2} \sqrt{\sum_{i \geqslant 1} n_i} \left(\sum_{i \geqslant 2} \sqrt{n_i} \right)$$

and this follows from

$$\begin{aligned} \sum_{i \geqslant 1} n_i + 2\sqrt{n_1} \left(\sum_{i \geqslant 2} \sqrt{n_i} \right) &> \sum_{i \geqslant 1} n_i + \left(\sum_{i \geqslant 1} \sqrt{n_i} \right) \left(\sum_{i \geqslant 2} \sqrt{n_i} \right) \\ &> \sum_{i \geqslant 1} n_i + \sqrt{n_1} \left(\sum_{i \geqslant 2} \sqrt{n_i} \right) + \left(\sum_{i \geqslant 2} \sqrt{n_i} \right)^2 \\ &> \sum_{i \geqslant 1} n_i + \frac{1}{2} \sqrt{\sum_{i \geqslant 1} n_i} \left(\sum_{i \geqslant 2} \sqrt{n_i} \right) + \left(\sum_{i \geqslant 2} \sqrt{n_i} \right)^2. \end{aligned}$$

This proves Lemma 1.1. $\square$

Remark. 1. Assuming $|q_n| < e^{-\delta\sqrt{|n|}}$, it follows from (1.4) that

$$|\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}| < e^{-2\delta\sqrt{n_1^*}-\frac{\delta}{4}\sum_{i\geqslant 3}\sqrt{n_i^*}}. \tag{1.7}$$

For the result of [Po1] to be applicable, we would need a bound

$$|\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}| < e^{-(2+\mu)\delta\sqrt{n_1^*}} \tag{1.8}$$

for some $\mu > 0$, which is in general not implied by (1.7). It turns out, however that (1.7) does suffice to carry out the analysis. The specific (arithmetic) structure is of importance here. Assume, say, that the monomial $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ creates a small divisor, hence

$$\sum (k_n - k_n')(n^2 + \tilde{V}_n) = o(1) \tag{1.9}$$

implying

$$\left|\sum (k_n - k_n')n^2\right| < \sum (k_n + k_n') + o(1). \tag{1.10}$$

Let $(m_i), |m_1| \geqslant |m_2| \geqslant \cdots$, denote the system $\{n$ repeated $k_n + k_n'$ times$\}$. Since $\sum (k_n - k_n')n = 0$,

$$|m_1 \pm m_2| \leqslant \sum_{i\geqslant 3} |m_i| \tag{1.11}$$

while (1.10) implies

$$|m_1^2 \pm m_2^2| \leqslant \sum_{i\geqslant 3} (1 + m_i^2) + o(1) \tag{1.12}$$

(with sign correspondence in (1.11), (1.12)).

In case of '−' sign. We may assume $m_1 \neq m_2$ since otherwise m_1, m_2 cancel in the small divisor. From (1.11), (1.12)

$$|m_1 - m_2| + |m_1 + m_2| \leqslant 5 \sum_{i\geqslant 3} m_i^2$$

hence

$$|m_1|^{1/4} + |m_2|^{1/4} \leqslant 3 \sum_{i\geqslant 3} \sqrt{|m_i|}.$$

In the case of '+' sign, obviously

$$m_1^2 + m_2^2 \leqslant 3 \sum_{i \geqslant 3} m_i^2,$$

$$|m_1|^{1/4} + |m_2|^{1/4} \leqslant 3 \sum_{i \geqslant 3} \sqrt{|m_i|}.$$

In both cases, assuming (1.9) or (1.10)

$$\frac{1}{30} \sum_n |k_n - k_n'| \ |n|^{1/4} \leqslant \frac{1}{4} \sum_{i \geqslant 3} \sqrt{n_i^*} \leqslant \sum_n (2a_n + k_n + k_n') \sqrt{|n|} - 2\sqrt{n_1^*} \tag{1.13}$$

and in particular, small divisor effects may be taken care of using only the modes $\{n_i | i \geqslant 3\}$.

2. The weight function $\sum_n (2a_n + k_n + k_n')\sqrt{|n|}$ may have been replaced by any expression $\sum (2a_n + k_n + k_n')|n|^\theta$ for some $0 < \theta < 1$. Possibly slower growing weights (as considered in [Po1]) may work as well but will not be explored here. If on the other hand, we want to construct invariant tori in the real analytic category, replace $\sqrt{|n|}$ by $\sqrt{|n|} + c|n|$ (for some $c > 0$). The presence of the $\sqrt{|n|}$ in the weight and inequality (1.4) remains essential in our analysis.

3. Returning to (1.2), the $\tilde{V}_n$ are modulated frequencies. Suitably adjustment of the V_n in (0.3) will enable us to freeze $\tilde{V}_n = \omega_n$ along the process, where $\omega = (\omega_n)_{n \in \mathbb{Z}}$ is a fixed frequency vector with good diophantine properties.

Definition.

$$\|H\|_\rho = \max_{\bar{a}, \bar{k}, \bar{k}'} \frac{|B_{\bar{a}, \bar{k}, \bar{k}'}|}{e^{\rho \sum_n \sqrt{n}(2a_n + k_n + k_n') - 2\rho \sqrt{n_1^*}}}. \tag{1.14}$$

At every stage of the process, H will be controlled in a norm (1.14), where ρ will increase slightly from one step to the next.

The remainder of the paper consists mainly in doing the bookkeeping of the $B_{\bar{a}, \bar{k}, \bar{k}'}$-coefficients when performing the consecutive symplectic transformations of phase space. The procedure and issues are 'standard' and may be found in [3]. We already pointed out the main conceptual novelty. Remains the usual tedious technicalities.

2. Estimation of the Poisson brackets

Let

$$H_1 = \sum b_{\bar{a}, \bar{k}, \bar{k}'} \mathcal{M}_{\bar{a}, \bar{k}, \bar{k}'} \quad \text{and} \quad H_2 = \sum B_{\bar{A}, \bar{K}, \bar{K}'} \mathcal{M}_{\bar{A}, \bar{K}, \bar{K}'}$$

hence

$$\{H_1, H_2\} = \sum b_{\bar{a},\bar{k},\bar{k}'} B_{\bar{A},\bar{k},\bar{k}'} \{\mathcal{M}_{a,k,k'}, \mathcal{M}_{A,K,K'}\}.$$

Check coefficient of $\prod_n I_n(0)^{\alpha_n} q_n^{\kappa_n} \bar{q}_n^{\kappa'_n} = \mathcal{M}_{\alpha,\kappa,\kappa'}$. Write

$$\begin{aligned}
&\{\mathcal{M}_{a,k,k'}, \mathcal{M}_{A,K,K'}\} \\
&\quad = \frac{1}{2i} \sum_n \left(\frac{\partial \mathcal{M}_{a,k,k'}}{\partial q_n} \frac{\partial \mathcal{M}_{A,K,K'}}{\partial \bar{q}_n} - \frac{\partial \mathcal{M}_{a,k,k'}}{\partial \bar{q}_n} \frac{\partial \mathcal{M}_{A,K,K'}}{\partial q_n} \right) \\
&\quad \sim \left(\prod_n I_n(0)^{a_n + A_n} \right) \sum_n (k_n K'_n - k'_n K_n) q_n^{k_n + K_n - 1} \bar{q}_n^{k'_n + K'_n - 1} \prod_{m \neq n} q_m^{k_m + K_m} \bar{q}_m^{k'_m + K'_m}.
\end{aligned}$$

Thus according to (1.14)

$$\begin{aligned}
&\|\{H_1, H_2\}\|_\rho = \max_{\alpha,\kappa,\kappa'} \\
&\quad \exp\left[2\rho\sqrt{v_1^*} - \rho \sum_n (2\alpha_n + \kappa_n + \kappa'_n)\sqrt{|n|} \right] \sum_{*(\bar{a}+\bar{A}=\bar{\alpha})} \sum_n \\
&\quad \times \sum_{* \left\{ \begin{array}{l} \left\{ \begin{array}{l} k_n + K_n - 1 = \kappa_n \\ k'_n + K'_n - 1 = \kappa'_n \end{array} \right. \\ \quad \left\{ \begin{array}{l} k_m + K_m = \kappa_m \\ k'_m + K'_m = \kappa'_m \end{array} \right. \quad (m \neq n) \end{array} \right.} |k_n K'_n - k'_n K_n| \, |b_{a,k,k'}| \, |B_{A,K,K'}|.
\end{aligned} \tag{2.1}$$

Estimate

$$|b_{a,k,k'}| \leqslant \|\mathcal{H}_1\|_{\rho_1} e^{\rho_1 \sum (2a_n + k_n + k'_n)\sqrt{n} - 2\rho_1\sqrt{n_1^*}}, \tag{2.2}$$

$$|B_{A,K,K'}| \leqslant \|\mathcal{H}_2\|_{\rho_2} e^{\rho_2 \sum (2A_n + K_n + K'_n)\sqrt{n} - 2\rho_2\sqrt{N_1^*}}. \tag{2.3}$$

Here we let

$$\rho_1, \rho_2 < \rho; \quad \rho = \rho_1 + \varepsilon_1 = \rho_2 + \varepsilon_2.$$

From (1.4), we get

$$(2.2) \leqslant \|\mathcal{H}_1\|_{\rho_1} \; e^{\rho\sum(2a_n+k_n+k'_n)\sqrt{n}-2\rho\sqrt{n_1^*}} e^{\frac{-\varepsilon_1}{4}\sum_{i\geqslant 3}\sqrt{n_i^*}}, \tag{2.4}$$

$$(2.3) \leqslant \|\mathcal{H}_2\|_{\rho_2} \; e^{\rho\sum(2A_n+K_n+K'_n)\sqrt{n}-2\rho\sqrt{N_1^*}} \; e^{-\frac{\varepsilon_2}{4}\sum_{i\geqslant 3}\sqrt{N_i^*}}. \tag{2.5}$$

Substitution of (2.4), (2.5) in (2.1) gives

$$\|\mathcal{H}_1\|_{\rho_1} \cdot \|\mathcal{H}_2\|_{\rho_2}.$$

$$\sum_n e^{2\rho\sqrt{n}} \sum_{\substack{* \\ n_1^*, N_1^* \geqslant |n|}} e^{2\rho(\sqrt{v_1^*}-\sqrt{n_1^*}-\sqrt{N_1^*})} |k_n K'_n - k'_n K_n|$$
$$\times e^{-\frac{\varepsilon_1}{4}\sum_{i\geqslant 3}\sqrt{n_i^*}} \; e^{-\frac{\varepsilon_2}{4}\sum_{i\geqslant 3}\sqrt{N_i^*}}. \tag{2.6}$$

(i) Assume $v_1^* \leqslant N_1^*$
Case (i_1): $|n| \leqslant n_3^*$.
Since

$$e^{2\rho(\sqrt{n}-\sqrt{n_1^*})} \leqslant e^{\frac{\varepsilon_1}{4}(\sqrt{n_3^*}-\sqrt{n_1^*})}$$

we get for (2.6) the bound

$$\sum_n \sum_* (k_n + k'_n)(K_n + K'_n) e^{-\frac{\varepsilon_1}{4}(\sqrt{n_1^*}+\sum_{i\geqslant 4}\sqrt{n_i^*})} e^{-\frac{\varepsilon_2}{4}\sum_{i\geqslant 3}\sqrt{N_i^*}}$$
$$\leqslant \sum_n \sum_* (k_n + k'_n)(K_n + K'_n) e^{-\frac{\varepsilon_1}{12}\sum_{i\geqslant 1}\sqrt{n_i^*}} e^{-\frac{\varepsilon_2}{4}\sum_{i\geqslant 3}\sqrt{N_i^*}}. \tag{2.7}$$

Concerning ($*$), if n is specified and $\bar{a}, \bar{k}, \bar{k}'$, then $\bar{A}, \bar{K}, \bar{K}'$ are uniquely determined. Also

$$\sum_{i\geqslant 1}\sqrt{n_i^*} = \sum (2a_n + k_n + k'_n)\sqrt{n} > \sum (2a_n + k_n + k'_n),$$

$$\sum_{i\geqslant 3}\sqrt{N_i^*} > \sum (2A_n + K_n + K'_n) - 2 \geqslant \frac{1}{2}\sum (2A_n + K_n + K'_n) \geqslant \frac{1}{2}(K_n + K'_n).$$

Thus

$$(2.7) < \frac{C}{\varepsilon_2} \sum_{\bar{a},\bar{k},\bar{k}'} \left(\sum_n (k_n + k_n') \right) e^{-\frac{\varepsilon_1}{12} \sum (2a_n + k_n + k_n')\sqrt{|n|}}$$

$$< \frac{C}{\varepsilon_1 \varepsilon_2} \sum_{\bar{a},\bar{k},\bar{k}'} e^{-\frac{\varepsilon_1}{20} \sum (2a_n + k_n + k_n')\sqrt{|n|}}$$

$$< \frac{C}{\varepsilon_1 \varepsilon_2} \prod_{n \geqslant 1} (1 - e^{-\frac{\varepsilon_1}{10}\sqrt{n}})^{-1} (1 - e^{-\frac{\varepsilon_1}{20}\sqrt{n}})^{-2}.$$

Estimate

$$\prod_n (1 - e^{-\frac{\varepsilon_1}{20}\sqrt{n}})^{-1} \leqslant \prod_{n \lesssim \frac{1}{\varepsilon_1^2}} \prod_{n \gtrsim \frac{1}{\varepsilon_1^2}} \leqslant \left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}} e^{[\sum_{n \gtrsim \frac{1}{\varepsilon_1^2}} e^{-\frac{\varepsilon_1}{30}\sqrt{n}}]} < \left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}.$$

Hence we get the bound

$$(2.7) < \left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}} \frac{1}{\varepsilon_2}. \tag{2.8}$$

Case (i_2): $n \in \{n_1, n_2\}$ where $|n_1| = n_1^*$, $|n_2| = n_2^*$.
$\underline{2a_n + k_n + k_n' > 2}$: Then $|n| \leqslant n_3^*$ and we are in Case (i_1)
$\underline{2a_n + k_n + k_n' \leqslant 2}$
Thus

$$(2.6) \lesssim \sum_{\bar{a},\bar{k},\bar{k}'} e^{-\frac{\varepsilon_1}{4} \sum_{i \geqslant 3} \sqrt{n_i^*}} (K_{n_1} + K_{n_1}' + K_{n_2} + K_{n_2}') e^{-\frac{\varepsilon_2}{4} \sum_{i \geqslant 3} \sqrt{N_i^*}}. \tag{2.9}$$

Since

$$\sum_m (2\alpha_m + \kappa_m + \kappa_m') \leqslant \sum_m (2a_m + k_m + k_m') + \sum_m (2A_m + K_m + K_m')$$

$$\leqslant 2 \sum_{i \geqslant 3} \sqrt{n_i^*} + 2 \sum_{i \geqslant 3} \sqrt{N_i^*}$$

and

$$
\begin{aligned}
K_m + K'_m &\leqslant \kappa_m + \kappa'_m - k_m - k'_m + 2 \leqslant \kappa_m + \kappa'_m + 2 \\
(2.9) &\lesssim \sum_{\bar{a},\bar{k},\bar{k}'} e^{-\frac{\varepsilon_1}{8}\sum_{i\geqslant 3}\sqrt{n_i}} (\kappa_{n_1} + \kappa'_{n_1} + \kappa_{n_2} + \kappa'_{n_2} + 1) \\
&\quad \times e^{-\frac{\varepsilon_1\wedge\varepsilon_2}{16}\sum_m(\alpha_m+\kappa_m+\kappa_{m'})}.
\end{aligned} \tag{2.10}
$$

Also, clearly $\{n_1, n_2\} \cap \operatorname{supp} \mathcal{M}_{\bar{\alpha},\bar{\kappa},\bar{\kappa}'} \neq \phi$. Given n_i $(i \geqslant 3)$, n_1 (resp., n_2) is determined by n_2 (resp., n_1) and hence $\{n_1, n_2\}$ range in a set of size $|\operatorname{supp} \mathcal{M}_{\bar{\alpha},\bar{\kappa},\bar{\kappa}'}| \leqslant \sum(\alpha_m + \kappa_m + \kappa'_m)$.

Finally, if (n_i) is given, then $(2a_m + k_m + k'_m)_m$ is specified and hence $(\bar{a}, \bar{k}, \bar{k}')$ up to a factor $\prod_m(1+\ell_m^2)$, denoting $\ell_m = \#\{i | n_i = m\}$. Hence

$$
\begin{aligned}
(2.10) \leqslant& \sum_{n_i (i\geqslant 3)} \prod_m (1+\ell_m^2) e^{-\frac{\varepsilon_1}{8}\sum_{i\geqslant 3}\sqrt{n_i}} \left[\sum_m (\kappa_m + \kappa'_m) + |\operatorname{supp}\mathcal{M}_{\bar{\alpha},\bar{\kappa},\bar{\kappa}'}|\right] \\
&\times e^{-\frac{\varepsilon_1\wedge\varepsilon_2}{16}\sum_m(\alpha_m+\kappa_m+\kappa'_m)} \sum_{n_i (i\geqslant 3)} \prod_m (1+\ell_m^2) e^{-\frac{\varepsilon_1}{8}\sum_{i\geqslant 3}\sqrt{n_i}} \\
&\times \left[\sum_m (\alpha_m + \kappa_m + \kappa'_m)\right] e^{-\frac{\varepsilon_1\wedge\varepsilon_2}{16}\sum_m(\alpha_m+\kappa_m+\kappa'_m)}
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
(2.11) &< \frac{C}{\varepsilon_1\wedge\varepsilon_2} \sum_{n_3,n_4\cdots} e^{-\frac{\varepsilon_1}{8}\sum_{i\geqslant 3}\sqrt{n_i}} \prod_m (1+\ell_m^2) < \frac{C}{\varepsilon_1\varepsilon_2}\sum_{\bar{\ell}} e^{-\frac{\varepsilon_1}{8}\sum_m^{\ell_m}\sqrt{m}} \prod_m (1+\ell_m^2) \\
&< \frac{1}{\varepsilon_2}\left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}} \prod_m \left(1 - e^{-\frac{\varepsilon_1}{10}\sqrt{m}}\right)^{-1} \\
&< \left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}} \frac{1}{\varepsilon_2}.
\end{aligned} \tag{2.12}
$$

(ii) Assume $v_1^* = n_1^* > N_1^*$

Thus n_1 is specified and n_2 determined from n_1 and $\{n_i\}_{i\geqslant 3}$.

We get again, with $\ell_m = \#\{i \,|\, n_i = m\}$

$$(2.6) < \sum_{n_i\,(i\geqslant 3)} \prod (1+\ell_m^2) \left[\sum_n (k_n + k_n') \right] \frac{C}{\varepsilon_2}\ e^{-\frac{\varepsilon_1}{4}\sum_{i\geqslant 3}\sqrt{n_i}}$$

$$< \frac{C}{\varepsilon_2}\left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}. \tag{2.13}$$

In conclusion, we proved the following inequality:

Lemma 2.1.

$$\|\{\mathcal{H}_1,\mathcal{H}_2\}\|_\rho \leqslant \left(\frac{1}{\varepsilon_1}\right)^{C\varepsilon_1^{-2}} \frac{1}{\varepsilon_2}\|\mathcal{H}_1\|_{\rho-\varepsilon_1}\|\mathcal{H}_2\|_{\rho-\varepsilon_2}. \tag{2.14}$$

The reason we need that type of asymmetric estimate will be clear in the next section.

3. Estimating the symplectic transformation

Denote $\mathcal{C}_\mathcal{F}$ the symplectic transformation induced by the Hamiltonian $\mathcal{F}$.
It follows from Taylor's formula that

$$\mathcal{H}\circ C_\mathcal{F} = \sum \frac{1}{r!}\mathcal{H}^{(r)} \text{ where } \mathcal{H}^{(r)} = \{\mathcal{H}^{(r-1)},\mathcal{F}\}. \tag{3.1}$$

Estimate from (2.14), replacing ε_2 by $\frac{\varepsilon_2}{r}$.

$$\begin{aligned}\|\mathcal{H}^{(r)}\|_\rho &\leqslant \left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}\|\mathcal{F}\|_{\rho-\varepsilon_1}\frac{r}{\varepsilon_2}\|\mathcal{H}^{(r-1)}\|_{\rho-\frac{\varepsilon_2}{r}}\\ &\leqslant \left[\left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}\|\mathcal{F}\|_{\rho-\varepsilon_1}\right]^2\left(\frac{r}{\varepsilon_2}\right)^2\|\mathcal{H}^{(r-2)}\|_{\rho-\frac{2\varepsilon_2}{r}}\\ &\leqslant \left[\left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}\|\mathcal{F}\|_{\rho-\varepsilon_1}\right]^r\left(\frac{r}{\varepsilon_2}\right)^r\|\mathcal{H}\|_{\rho-\varepsilon_2}.\end{aligned} \tag{3.2}$$

Assume

$$\frac{1}{\varepsilon_2}\left(\frac{1}{\varepsilon_1}\right)^{\frac{C}{\varepsilon_1^2}}\|\mathcal{F}\|_{\rho-\varepsilon_1} \ll 1. \tag{3.3}$$

It follows thus that

$$\|\mathcal{H} \circ C_{\mathcal{F}}\|_\rho < \big(1 + (3.3)\big)\|\mathcal{H}\|_{\rho-\varepsilon_2}. \tag{3.4}$$

4. Small divisor effects

Take, for simplicity, the V_n to be random in $[-1, 1]$.

Denote $\|x\| =$ dist $(x, \mathbb{Z})$. The following statement addresses the resonance issues:

Lemma 4.1. *Let* (V_n) *be as above. Then, except on a set of small measure in* $[-1, 1]^{\mathbb{Z}}$, *the following holds*

$$\Big\|{\sum}' \ell_n V_n\Big\| \geqslant \delta \prod_n (1 + \ell^2 n^4)^{-1} \tag{4.1}$$

whenever $0 \neq \bar{\ell} = (\ell_n)_{n\in\mathbb{Z}}$ *is a* (*finitely supported*) *sequence of integers.*

Proof. Letting $\delta > 0$ be a small number, we get clearly

$$\text{mes}\left[\bigcup_{\bar{\ell}\neq 0}\left[\Big\|\sum \ell_n V_n\Big\| < \delta \prod_n \frac{1}{(1+\ell_n^2 n^4)}\right]\right]$$

$$\lesssim \delta \sum_{s\geqslant 1}\left[\sum_{\substack{\ell_s,\ell_{s+1},\cdots\\ \ell_s\neq 0}} \prod_{n\geqslant s} \frac{1}{(1+\ell_n^2 n^4)}\right] \leqslant \delta \sum_s s^{-2}\left(\sum_{r=1}^{\infty}\frac{1}{r^2}\right)\left(\prod_{n>s}\right)\left(\sum_{r=0}^{\infty}\frac{1}{1+r^2 n^4}\right)$$

$$\lesssim \delta \sum_s s^{-2} \prod_n \left(1+\frac{C}{n^4}\right) \lesssim \delta$$

proving the claim.

Following the usual KAM scheme, resonant monomials $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ give a 'small divisor' $\sum_n (k_n - k_n')(\tilde{V}_n + n^2)$, where $\tilde{V}_n$ denote the modulated frequencies and

$$\mathcal{H}_{nr} = \sum_{nr} B_{a,k,k'}\mathcal{M}_{a,k,k'} \to \mathcal{F} = \sum_{nr} \frac{B_{a,k,k'}}{\sum_n (k_n - k_n')(\tilde{V}_n + n^2)} \mathcal{M}_{a,k,k'}$$

(nr denoting 'nonresonant').

In our approach, we will readjust the multiplier (V_n) in (0.3) to ensure that at each stage $\tilde{V}_n = \omega_n$, with $\omega = (\omega_n)$ a fixed frequency vector satisfying Lemma 4.1.

We estimate $\|\mathcal{F}\|_\rho$

$$\|\mathcal{F}\|_\rho = \max_{a,k,k'(nr)} \left\{ \frac{|B_{a,k,k'}|}{|\sum(k_n - k_n')(\tilde{V}_n + n^2)|} e^{2\rho\sqrt{n_1^*} - \rho\sum(2a_n + k_n + k_n')\sqrt{n}} \right\}. \tag{4.2}$$

Distinguish two cases

(i) $|\sum(k_n - k_n')n^2| > 10\sum|k_n - k_n'|$
Since $|\tilde{V}_n| \leqslant 1$, $\left|\sum(k_n - k_n')(\tilde{V}_n + n^2)\right| > 10\sum(k_n + k_n') - \sum|k_n - k_n'| \geqslant 1$ and there is no small divisor issue.
(ii) $\left|\sum(k_n - k_n')n^2\right| \leqslant 10\sum|k_n - k_n'|$
From (1.13)

$$\sum|k_n - k_n'|n^{1/4} \lesssim \sum_n (2a_n + k_n + k_n')\sqrt{n} - 2\sqrt{n_1^*}. \tag{4.3}$$

Since $\tilde{V}_n$ satisfies (4.1)

$$\left|\sum(k_n - k_n')(\tilde{V}_n + n^2)\right| \gtrsim \prod_n \frac{1}{1 + (k_n - k_n')^2 n^4}.$$

Hence (4.2) becomes

$$|B_{a,k,k'}| \prod_n [1 + (k_n - k_n')^2 n^4]\, e^{2\rho\sqrt{n_1^*} - \rho\sum(2a_n + k_n + k_n')\sqrt{n}}$$

$$\underset{<}{\text{by (4.3)}} \quad e^{c\sum\log(n|k_n - k_n'|+1)} \quad e^{-\varepsilon\sum|k_n - k_n'|n^{1/4}} \quad \|\mathcal{H}\|_{\rho-\varepsilon}. \tag{4.4}$$

Assume $k_n \neq k_n'$. Then

$$\log n|k_n - k_n'| > \varepsilon|k_n - k_n'|n^{1/4} \Rightarrow |n| < \frac{1}{\varepsilon^5}, |k_n - k_n'| < \frac{1}{\varepsilon^2}. \tag{4.5}$$

Hence

$$(4.4) \lesssim e^{\frac{C}{\varepsilon^6}} \|\mathcal{H}\|_{\rho-\varepsilon}. \qquad \square \tag{4.6}$$

In conclusion, we showed the following:

Lemma 4.2. *Let $\mathcal{F}$ be defined from $\mathcal{H}_{nr}$ as above. Then*

$$\|\mathcal{F}\|_\rho < e^{C\varepsilon^{-6}} \|\mathcal{H}\|_{\rho-\varepsilon}. \tag{4.7}$$

5. Normal forms

Hamiltonians will be of the form

$$\mathcal{H} = \sum_n (n^2 + \tilde{V}_n)|q_n|^2 + \sum_{\substack{\operatorname{supp}\bar{k}\cap\operatorname{supp}\bar{k}'=\phi \\ |\bar{k}|+|k'|\geqslant 2}} B_{a,k,k'}\mathcal{M}_{a,k,k'}$$

$$+ \sum_n J_n \left(\sum_{\substack{\operatorname{supp} k\cap\operatorname{supp} k'=\phi \\ |k|+|k'|\geqslant 2}} B^{(n)}_{a,k,k'}\mathcal{M}_{a,k,k'} \right)$$

$$+ \sum_{n_1,n_2} J_{n_1} J_{n_2} \left(\sum_{\substack{\uparrow \\ \text{no assumption}}} B^{(n_1,n_2)}_{a,k,k'}\mathcal{M}_{a,k,k'} \right), \tag{5.1}$$

where $J_n = I_n - I_n(0)$, $I_n = |q_n|^2$.

As pointed out earlier, use of the symbols I_n, J_n is only notational and does not indicate a change of variable.

Rewrite according to (5.1)

$$\mathcal{H} = H_0 + \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2,$$

which is the Hamiltonian obtained at a given stage.

Next step involves conversion $\mathcal{H} \to \mathcal{H}' = \mathcal{H} \circ C_{\mathcal{F}}$, $C_{\mathcal{F}}$ = symplectic transformation with generating function $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1$, as to remove $\mathcal{H}_0 + \mathcal{H}_1$. Thus

$$\mathcal{F}_0 \sim \sum_{\operatorname{supp}\bar{k}\cap\operatorname{supp} k'=\phi} \frac{B_{a,k,k'}}{\sum(k_n - k'_n)(n^2 + \tilde{V}_n)} \mathcal{M}_{a,k,k'},$$

$$\mathcal{F}_1 \sim \sum_n J_n \left(\sum_{\dots} \frac{B^{(n)}_{a,k,k'}}{\sum\cdots} \mathcal{M}_{a,k,k'} \right).$$

Hence

$$\mathcal{H}' = H_0 + \mathcal{H}_2 + \sum_{r\geqslant 2} \frac{1}{r!} \underbrace{\{\{H_0,\mathcal{F}\},\dots,\mathcal{F}\}}_{r-\text{fold}} + \sum_{r\geqslant 1} \frac{1}{r!} \underbrace{\{\{\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2, \mathcal{F}\}\cdots\mathcal{F}\}}_{r-\text{fold}}$$

$$= H_0 + \mathcal{H}_2 + \sum_{r\geqslant 1} 0\left(\frac{1}{r!}\right) \underbrace{\left\{ \cdots \left\{ \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_1, \ \mathcal{F} \\ \mathcal{H}_2 \end{matrix} \right\}, \cdots \mathcal{F} \right\}}_{r-\text{fold}}. \tag{5.2}$$

Last term of (5.2) is then again converted to the format (5.1).

We first discuss how the coefficients in representations (1.2) and (5.1) relate.

5.1. Coefficient estimates in convention (1.2)–(5.1)

Write $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ in the form $\mathcal{M}_{\bar{a},\bar{b},\bar{\ell},\bar{\ell}'} = \prod_n I_n(0)^{a_n} I_n^{b_n} q_n^{\ell_n} \bar{q}_n^{\ell'_n}$ where $I_n = |q_n|^2$, $b_n = k_n \wedge k'_n$ and $\ell_n = k_n - b_n$, $\ell'_n = k'_n - b_n$ satisfying $\ell_n \ell'_n = 0$, $\forall n$.

List the I-factors in natural order $I.$, $I.$, $I.$, $\ldots$, $I.$ and express $\prod_n I_n^{b_n}$ by monomials of the form

$$\prod_n I_n(0)^{b_n}, \tag{5.3}$$

$$\sum_{m|b_m \geqslant 1} \prod_{n \neq m} I_n(0)^{b_n} \big(I_m(0)^{b_m - 1} J_m\big), \tag{5.4}$$

$$\sum_{\substack{m|b_m \geqslant 2 \\ r \leqslant b_m - 2}} \prod_{n<m} I_n(0)^{b_n} \prod_{n>m} I_n^{b_n} \big(I_m(0)^r J_m^2 I_m^{b_m - r - 2}\big), \tag{5.5}$$

$$\sum_{\substack{m<m' (b_m, b_{m'} \geqslant 1) \\ r \leqslant b_{m'} - 1}} \left(\prod_{n<m} I_n(0)^{b_n}\right) \left(I_m(0)^{b_m - 1} J_m\right) \times \left(\prod_{m<n<m'} I_n(0)^{b_n}\right) I_{m'}(0)^r J_{m'} I_{m'}^{b_{m'} - r - 1} \left(\prod_{n>m'} I_n^{b_n}\right). \tag{5.6}$$

This gives the following bounds for the coefficients in (5.1), as easily verified

$$|B_{a,k,k'}| < \prod_n (1 + a_n) e^{\rho(\sum_n (2a_n + k_n + k'_n)\sqrt{n} - 2\sqrt{n_1^*})} \|\mathcal{H}\|_\rho, \tag{5.7}$$

$$|B^{(m)}_{a,k,k'}| \leqslant \prod_{n \neq m} (1 + a_n)(1 + a_m)^2 e^{\rho(\sum_n (2a_n + k_n + k'_n)\sqrt{n} + 2\sqrt{m} - 2\sqrt{n_1^*})} \|\mathcal{H}\|_\rho, \tag{5.8}$$

$$|B^{(m,m)}_{a,k,k'}| \leqslant \prod_{n<m} (1 + a_n)(1 + a_m)^3 e^{\rho(\sum_n \sqrt{n}(2a_n + k_n + k'_n) + 4\sqrt{m} - 2\sqrt{n_1^*})} \|\mathcal{H}\|_\rho, \tag{5.9}$$

$$|B^{(m,m')}_{a,k,k'}| \leqslant \prod_{n<m} (1 + a_n) \prod_{m<n<m'} (1 + a_n)(1 + a_m)^2 (1 + a_{m'})^2 \times e^{\rho(\sum_n \sqrt{n}(2a_n + k_n + k'_n) + 2\sqrt{m} + 2\sqrt{m'} - 2\sqrt{n_1^*})} \|\mathcal{H}\|_\rho. \tag{5.10}$$

Hence for representation (5.1)

$$\|\mathcal{H}\|_{\rho+\varepsilon}^{(5.1)} \leqslant \left(\frac{1}{\varepsilon}\right)^{\frac{C}{\varepsilon^2}} \|\mathcal{H}\|_{\rho}^{(1.2)}. \tag{5.11}$$

5.2. Coefficient estimates in conversion (5.1)–(1.2)

The coefficient of $\mathcal{M}_{\bar{a},\bar{b},\bar{k},\bar{k}'}$ increases by at most a factor $(\sum a_n + b_n)^2$.
Hence

$$\|\mathcal{H}\|_{\rho+\varepsilon}^{(1.2)} < \left(\frac{\log\frac{1}{\varepsilon}}{\varepsilon}\right)^2 \|\mathcal{H}\|_{\rho}^{(5.1)}. \tag{5.12}$$

Return to (5.2). We evaluate the last term.
Express $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F}_0, \mathcal{F}_1$ in the form (1.2).
From (4.7) and (5.12)

$$\|\mathcal{H}_{(i)}\|_{\rho+\varepsilon}^{(1.2)} < \frac{1}{\varepsilon^3} e^{\frac{1}{\varepsilon^6}} \|\mathcal{H}_{(i)}\|_{\rho}^{(5.1)} \quad (i = 0, 1, 2), \tag{5.13}$$

$$\|\mathcal{F}_{(i)}\|_{\rho+\varepsilon}^{(1.2)} < \frac{1}{\varepsilon^3} e^{\frac{1}{\varepsilon^6}} \|\mathcal{H}_{(i)}\|_{\rho}^{(5.1)} \qquad (i = 0, 1). \tag{5.14}$$

Consider the expression $\sum_{r\geqslant 1} 0\left(\frac{1}{r!}\right) \left\{ \cdots \left\{ \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_1, \ \mathcal{F} \\ \mathcal{H}_2 \end{matrix} \right\}, \ldots, \mathcal{F} \right\}$ in (5.2) which we evaluate by means of (2.14), (3.3), (3.4).
To satisfy (3.3), assume

$$e^{\frac{2}{\varepsilon^6}} (\|\mathcal{H}_0\|_{\rho}^{(5.1)} + \|\mathcal{H}_1\|_{\rho}^{(5.1)}) \ll 1 \tag{5.15}$$

so that by (4.7)

$$\|\mathcal{F}_0\|_{\rho+\varepsilon}^{(1.2)}, \quad \|\mathcal{F}_1\|_{\rho+\varepsilon}^{(1.2)} < \varepsilon^{\frac{C}{\varepsilon^2}}. \tag{5.16}$$

(In (3.3), (3.4), $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and ρ replaced by $\rho + 2\varepsilon$).

Hence, by the estimate in Section 3 and (5.13), (5.14)

$$\left\| \sum_{r\geqslant 1} 0\left(\frac{1}{r!}\right)\{\cdots\{\mathcal{H}_0,\mathcal{F}\},\ldots,\mathcal{F}\}\right\|^{(1.2)}_{\rho+2\varepsilon} < \left(\frac{1}{\varepsilon}\right)^{\frac{C}{\varepsilon^2}} \|\mathcal{F}\|^{(1.2)}_{\rho+\varepsilon}\|\mathcal{H}_0\|^{(1.2)}_{\rho+\varepsilon}$$

$$< e^{\frac{2}{\varepsilon^6}}(\|\mathcal{H}_0\|^{(5.1)}_\rho + \|\mathcal{H}_1\|^{(5.1)}_\rho) \times\|\mathcal{H}_0\|^{(5.1)}_\rho \tag{5.17}$$

and thus

$$\left\| \sum_{r\geqslant 1} 0\left(\frac{1}{r!}\right)\{\cdots\{\mathcal{H}_0,\mathcal{F}\},\cdots\mathcal{F}\}\right\|^{(5.1)}_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}(\|\mathcal{H}_0\|^{(5.1)}_\rho + \|\mathcal{H}_1\|^{(5.1)}_\rho)\|\mathcal{H}_0\|^{(5.1)}_\rho. \tag{5.18}$$

Similarly

$$\left\| \sum_{r\geqslant 1} 0\left(\frac{1}{r!}\right)\left\{\cdots\{\mathcal{H}_1,\mathcal{F}_0\},\mathcal{F},\cdots,\mathcal{F}\right\}\right\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}\|\mathcal{H}_0\|_\rho\|\mathcal{H}_1\|_\rho, \tag{5.19}$$

$$\|\{\mathcal{H}_1,\mathcal{F}_1\}\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}\|\mathcal{H}_1\|_\rho\|\mathcal{H}_1\|_\rho, \tag{5.20}$$

$$\left\| \sum_{r\geqslant 2} 0\left(\frac{1}{r!}\right)\left\{\cdots\{\mathcal{H}_1,\mathcal{F}_1\},\mathcal{F},\ldots,\mathcal{F}\right\}\right\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}\|\mathcal{H}_1\|_\rho\|\mathcal{H}_1\|_\rho \times(\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho), \tag{5.21}$$

$$\|\{\mathcal{H}_2,\mathcal{F}_0\}\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}\|\mathcal{H}_0\|_\rho\|\mathcal{H}_2\|_\rho, \tag{5.22}$$

$$\left\| \sum_{r\geqslant 2} 0\left(\frac{1}{r!}\right)\left\{\cdots\{\mathcal{H}_2,\mathcal{F}_0\},\mathcal{F},\ldots\mathcal{F}\right\}\right\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}}\|\mathcal{H}_2\|_\rho\|\mathcal{H}_0\|_\rho \times(\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho), \tag{5.23}$$

$$\|\{\mathcal{H}_2, \mathcal{F}_1\}\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}} \|\mathcal{H}_2\|_\rho \|\mathcal{H}_1\|_\rho, \tag{5.24}$$

$$\left\|\left\{\{\mathcal{H}_2, \mathcal{F}_1\}, \mathcal{F}\right\}\right\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}} \|\mathcal{H}_2\|_\rho \|\mathcal{H}_1\|_\rho (\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho), \tag{5.25}$$

$$\left\| \sum_{r \geqslant 3} \| 0 \left(\frac{1}{r!}\right) \left\{ \cdots \{\mathcal{H}_2, \mathcal{F}_1\}, \mathcal{F}, \ldots, \mathcal{F}\right\} \right\|_{\rho+3\varepsilon}$$
$$< e^{\frac{3}{\varepsilon^6}} \|\mathcal{H}_2\|_\rho \|\mathcal{H}_1\|_\rho (\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho)^2. \tag{5.26}$$

Notice that the terms in (5.20) are at least linear in J.
Therefore $\{\mathcal{H}_1, \mathcal{F}_1\}$ will only contribute to $\mathcal{H}_1'$ and $\mathcal{H}_2'$.
We use here the fact that the decomposition in monomials

$$\prod_n J_n^{b_n} \prod_n q_n^{k_n} \bar{q}_n^{k_n'}, \quad k_n \cdot k_n' = 0 \tag{5.27}$$

is unique.
Similarly
(5.22) contributes only to $\mathcal{H}_1', \mathcal{H}_2'$.
(5.24) contributes only to $\mathcal{H}_2'$
(5.25) contributes only to $\mathcal{H}_1', \mathcal{H}_2'$.
Consequently

$$\begin{cases} \|\mathcal{H}_0'\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}} \quad (\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho)(\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho^2), & (5.28) \\ \|\mathcal{H}_1'\|_{\rho+3\varepsilon} < e^{\frac{3}{\varepsilon^6}} (\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho^2), & (5.29) \\ \|\mathcal{H}_2'\|_{\rho+2\varepsilon} \leqslant \|\mathcal{H}_2\|_\rho + e^{\frac{3}{\varepsilon^6}} (\|\mathcal{H}_0\|_\rho + \|\mathcal{H}_1\|_\rho). & (5.30) \end{cases}$$

At stage s of the iteration $\rho = \rho_s$ and we take $\varepsilon = \varepsilon_s = \frac{\tau}{s^2}$ (τ, s small constant). From (recursive) inequalities (5.28)-(5.30), we verify inductively that

$$\begin{cases} \|\mathcal{H}_0^{(s)}\|_{\rho_s} < \varepsilon_0^{(\frac{3}{2})^s}, & (5.31) \\ \|\mathcal{H}_1^{(s)}\|_{\rho_s} < \varepsilon_0^{0,9(\frac{3}{2})^{s-1}}, & (5.32) \\ \|\mathcal{H}_2^{(s)}\|_{\rho_s} < \varepsilon_0, & (5.33) \end{cases}$$

(ε_0 small enough).

Indeed

$$(5.28) \Rightarrow \|H_0^{(s+1)}\|_{\rho_{s+1}} < e^{3\frac{s^{12}}{\tau^6}}\left(\varepsilon_0^{(\frac{3}{2})^s+(0.9)(\frac{3}{2})^{s-1}} + \varepsilon_0^{3(\frac{3}{2})^{s-1}(0.9)}\right) < \varepsilon_0^{(\frac{3}{2})^{s+1}}$$

$$\|\mathcal{H}_1^{(s+1)}\|_{\rho_{s+1}} < e^{3\frac{s^{12}}{\tau^6}}\left(\varepsilon_0^{(\frac{3}{2})^s} + \varepsilon_0^{1.8(\frac{3}{2})^{s-1}}\right) < \varepsilon_0^{0.9(\frac{3}{2})^s}$$

(in particular (5.15) is satisfied).

Obviously

$$\rho_{s+1} = \rho_s + \frac{3\tau}{s^2} < \rho + \sum_{s'} \frac{3\tau}{(s')^2} < \rho + 10\tau. \tag{5.34}$$

Remark. In $\mathcal{H}'$, $J' = I' - I(0)$, where $I' = I \circ C_{\mathcal{F}}$. We did not replace $I(0)$ by $I'(0)$ (which we could do with some additional work). Thus $I(0) = \big(I_n(0)\big)_{n\in\mathbb{Z}}$ will be the action-variable of the invariant torus in the new coordinates (after applying the final symplectic transformation).

From $\mathcal{H}_1'$, we need to remove the quadratic terms $\sum\limits_n \big(\sum\limits_{\bar{a}} B^{(n)}_{a,0,0}\mathcal{M}_{\bar{a},0,0}\big)J_n$, where $\mathcal{M}_{\bar{a},0,0} = \prod\limits_n I_n(0)^{a_n}$. It is added to the first term in (6.1) and the new modulated frequencies in $\mathcal{H}'$ are

$$\tilde{V}_n' = \tilde{V}_n + \sum_{\bar{a}} B^{(n)}_{\bar{a},0,0}\mathcal{M}_{\bar{a},0,0}. \tag{5.35}$$

6. Modulated frequencies

In (5.35), we get by (5.32) (at stage $s+1$) that

$$|B^{(n)}_{\bar{a},0,0}| < \|\mathcal{H}_1^{(s+1)}\|_{\rho_{s+1}}.e^{2\rho_{s+1}(\sum_m \sqrt{m}a_m + \sqrt{n} - \sqrt{m_1^*})}$$
$$< \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s}\ e^{2\rho_{s+1}(\sum_m \sqrt{m}\,a_m + \sqrt{n} - \sqrt{m_1^*})}.$$

Consequently

$$\left|\sum_a B^{(n)}_{a,0,0}\mathcal{M}_{\bar{a},0,0}\right| \leqslant \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} \sum_a e^{2\rho_{s+1}(\sum_m \sqrt{m}\bar{a}_m + \sqrt{n} - \sqrt{m_1^*})} \prod_m I_m(0)^{a_m}$$
$$< \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} \sum_a e^{2\rho_{s+1}\sum_m \sqrt{m}a_m} \prod_m I_m(0)^{a_m}. \tag{6.1}$$

Assuming

$$I_m(0) < e^{-2\rho\sqrt{m}}$$

and insuring that $\rho_s < \frac{1}{2}\rho$ at any stage s, we get

$$(6.1) < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} \sum_a e^{-\rho \sum_m \sqrt{m} a_m} < \prod_m (1 - e^{-\rho\sqrt{m}})^{-1} \quad \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s}$$

hence

$$\left| \sum_a B^{(n)}_{a,0,0} \mathcal{M}_{\bar{a},0,0} \right| \lesssim \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s}. \tag{6.2}$$

However, since the $\mathcal{H}^{(s)}$-coefficients depend on V, we need also to make derivative bounds. This is achieved the ‘standard’ way.

(i) Truncation of the Hamiltonians.
(ii) Complexification of the frequency parameter V.

(i) In the step $s \to s+1$, there is saving of a factor

$$e^{-\varepsilon_s(\sum_n \sqrt{n}(k_n + k'_n + 2a_n) - 2\sqrt{n_1^*})} \leqslant e^{-\varepsilon_s(\sqrt{n_3^*} + \sqrt{n_4^*} + \cdots)}, \tag{6.3}$$

where $\varepsilon_s \sim \frac{1}{s^2}$ (from definition of the ρ_s). Denote

$$\kappa = \varepsilon_0^{(\frac{3}{2})^{s+1}}.$$

In the normal forms reduction, we may thus dismiss all monomials $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ for which (6.3) $< \kappa$. Thus we only remove monomials satisfying

$$\sqrt{n_3^*} + \sqrt{n_4^*} + \cdots < Cs^2 \log \frac{1}{\kappa}. \tag{6.4}$$

Returning to the small divisors analysis in Section 5, we only need to impose conditions on the divisor $\sum (k_n - k'_n)(V_n + n^2)$ when (1.15) holds, thus

$$\sum |k_n - k'_n|\, |n|^{1/4} \lesssim \sum_{i \geqslant 3} \sqrt{n_i^*} < Cs^2 \log \frac{1}{\kappa}. \tag{6.5}$$

In particular, all conditions relate only to $(V_n)_{n \leqslant n_*}$, where

$$n_* < Cs^8 \left(\log \frac{1}{\kappa}\right)^4. \tag{6.6}$$

These conditions are of the form (cf. (4.1))

$$\left\| \sum_{n \leqslant n_*} (k_n - k_n')\tilde{V}_n \right\| \gtrsim \left(\prod_{n \leqslant n_*} \frac{1}{(1 + (k_n - k_n')^2 n^4)} \right). \tag{6.7}$$

Assumption (6.5) permits us to get a lower bound on the $\left(\prod_{n \leqslant n_*} \frac{1}{1+(k_n - k_n')^2 n^4}\right)$-factor in (6.7).

Claim. *Assume* $\sum \ell_n n^{1/4} < B$. *Then*

$$\prod (1 + \ell_n^2 n^4) < e^{CB^{6/7}}. \tag{6.8}$$

Proof. Write

$$\prod (1 + \ell_n^2 n^4) = \prod_{n \leqslant N} \cdot \prod_{n \gtrsim N} < e^{CN \cdot \log B}\, e^{C \sum_{n>N} \ell_n \cdot \log n}$$

$$< e^{CN \cdot \log B + \frac{B}{N^{1/5}}}. \tag{6.9}$$

Optimizing in N clearly implies (6.8).

From (6.5), (6.8), it follows that the left-hand side of (6.7) is at least

$$e^{-cs^2 (\log \frac{1}{\kappa})^{6/7}} > e^{-C(\log \frac{1}{\kappa})^{7/8}}. \tag{6.10}$$

Clearly conditions (6.7) will therefore essentially remain preserved if $(\tilde{V}_n)$ are perturbed by $< [Cs^2 (\log \frac{1}{\kappa})\, e^{C(\log \frac{1}{\kappa})^{7/8}}]^{-1}$ (again from (6.5)).

Thus $(\tilde{V}_n)$ is subject to a restriction of $(\tilde{V}_n)_{n \leqslant n_*}$ to cubes of size

$$\gamma = e^{-C(\log \frac{1}{\kappa})^{7/8}}. \tag{6.11}$$

Moreover, all estimates remain clearly preserved if we complexify each $\tilde{V}_n$ to a γ-size neighborhood. Furthermore, there is analyticity on that neighborhood.

We now proceed as follows.

Fix a strongly nonresonant $\omega = (\omega_n)$, i.e. satisfying (4.1)

$$\left\| \sum \ell_n \omega_n \right\| \geqslant \prod_n (1 + \ell_n^2 n^4)^{-1} \quad \forall \bar{\ell} = (\ell_n) \in \prod_n \mathbb{Z}, \quad \bar{\ell} \neq 0. \tag{6.12}$$

We assume at stage s, $\mathcal{H}$ and $\tilde{V}$ in (5.1) extend to analytic functions in V on a set $O_s = \prod\limits_n D(V_n, \eta_s) \subset \prod\limits_n \mathbb{C}$ where $\tilde{V}\big(V = (V_n)\big) = \omega$ (V will depend on s). Call this property $(*)$. We will specify η_s later.

Consider the transformation $\mathcal{H} \to \mathcal{H}' = \mathcal{H}^{(s+1)}$.

Since ω satisfies the desired nonresonance conditions (6.12), it follows from the preceding that it suffices to impose on $\tilde{V}$ the condition

$$\tilde{V}_n \in D(\omega_n, \gamma) \quad \forall n \tag{6.13}$$

where $\gamma = \gamma_s$ is given by (6.11).

Since by assumption, on $\prod_n D(V_n, \frac{1}{2}\eta_s)$

$$\sum_n \left| \frac{\partial \tilde{V}_m}{\partial V_n} \right| \leqslant 10 \eta_s^{-1} \quad \forall m \tag{6.14}$$

clearly

$$\tilde{V} \left(\prod_n D(V_n, \frac{\gamma}{10}\eta_s) \right) \subset \prod_n D(\omega_{n,\gamma}). \tag{6.15}$$

Consequently $\mathcal{H}'$ expands to an analytic function in $V \in \prod_n D(V_n, \frac{\gamma}{10}\eta_s)$.

Returning to (5.35), the perturbation $\tilde{V}' - \tilde{V}$ is an analytic function on $\prod_n D(V_n, \frac{\gamma}{10}\eta_s)$ satisfying the bound (6.2), i.e.

$$|\tilde{V}_m' - \tilde{V}_m| < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} < \kappa^{1/3}. \tag{6.16}$$

It follows that on $\prod_n D(V_n, \gamma\eta_s), \forall m$

$$\sum_n \left| \frac{\partial \tilde{V}_m'}{\partial V_n} - \frac{\partial \tilde{V}_m}{\partial V_n} \right| \lesssim \frac{\kappa^{1/3}}{\gamma \eta_s} < \frac{1}{\eta_s} \kappa^{1/4} = \frac{1}{\eta_s} \varepsilon_0^{\frac{1}{4}(\frac{3}{2})^{s+1}}. \tag{6.17}$$

Assume

$$\eta_s > \varepsilon_0^{\frac{1}{100}(\frac{3}{2})^s}. \tag{6.18}$$

(6.17) gives then on $\prod_n D(V_n, \frac{1}{10}\gamma\eta_s)$, $\forall m$ (by induction)

$$\sum_n \left| \frac{\partial \tilde{V}'_m}{\partial V_n} - \delta_{mn} \right| < \sum_{s' \leqslant s} \frac{1}{\eta_{s'}} \varepsilon_0^{\frac{1}{4}(\frac{3}{2})^{s'+1}} < \varepsilon_0^{\frac{1}{10}} = o(1)$$

or equivalently

$$\left\| \frac{\partial \tilde{V}'}{\partial V} - I \right\|_{\ell^\infty_{\mathbb{Z}} \to \ell^\infty_{\mathbb{Z}}} < \varepsilon_0^{\frac{1}{10}}. \tag{6.19}$$

Recall that $\tilde{V}(V) = \omega$. We invoke an inverse function argument to obtain V' satisfying

$$\tilde{V}'(V') = \omega. \tag{6.20}$$

We consider the map $\tilde{V}' : \ell^\infty_{\mathbb{C}} \supset \prod_n D(V_n, \frac{1}{10}\gamma\eta_s) \to \ell^\infty_{\mathbb{C}}$ satisfying (6.19). Rewriting (6.20) as

$$V' - V = (I - \tilde{V}')(V') - (I - \tilde{V}')(V) + (\tilde{V} - \tilde{V}')(V)$$

(6.19), (6.16) imply

$$\|V - V'\|_\infty \leqslant \varepsilon_0^{\frac{1}{10}} \|V - V'\|_\infty + \kappa^{1/3},$$
$$\|V - V'\|_\infty < 2\kappa^{1/3} \lll \gamma\eta_s. \tag{6.21}$$

Define then

$$\eta_{s+1} = \frac{1}{20}\gamma\eta_s,\ O_{s+1} = \prod_n D(V'_n, \eta_{s+1}) \subset \prod_n D\left(V_n, \frac{1}{15}\gamma\eta_s\right) \subset O_s. \tag{6.22}$$

Then $\mathcal{H}'$, $\tilde{V}'$ extend to analytic functions on O_{s+1} and (6.20) holds.

From (6.11)

$$\eta_{s+1} > \eta_s \varepsilon_0^{C(\frac{3}{2})^{\frac{9}{10}s}} \tag{6.23}$$

hence, iterating

$$\eta_s > \varepsilon_0^{C(\frac{3}{2})^{\frac{9}{10}s}}. \tag{6.24}$$

Clearly (6.18) holds.

This establishes $(*)$ for $\mathcal{H}^{(s+1)}$ and completes the inductive argument. $\square$

7. Mapping properties of the symplectic transformations

Define

$$\Omega_r = \left\{ q = (q_n)_{n\in\mathbb{Z}} \;\middle|\; |q_n| \leqslant e^{-r\sqrt{|n|}}\ ,\ \forall n \right\}$$

and let $|\ |_r$ be the corresponding norm.

Denote $\mathcal{C}_{\mathcal{F}}$ the symplectic transformation at stage $s \to s+1$.

We specify $\mathcal{C}_{\mathcal{F}}(\Omega_r)$. Thus we need to estimate on Ω_r

$$(I_n \circ C_{\mathcal{F}})^{1/2}.$$

Recall that at stage $s \to s+1$, by (4.8), (5.31), (5.32)

$$\|\mathcal{F}\|_{\rho_s + \frac{\tau}{s^2}} \leqslant e^{(\frac{s}{\tau})^{20}} (\|\mathcal{H}_0^{(s)}\|_{\rho_s} + \|\mathcal{H}_1^{(s)}\|_{\rho_s})$$

$$< e^{(\frac{s}{\tau})^{20}} \left(\varepsilon_0^{(\frac{3}{2})^s} + \varepsilon_0^{0.9(\frac{3}{2})^{s-1}} \right) < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}}$$

and hence by (3.3), (3.4)

$$\|I_n - I_n \circ C_{\mathcal{F}}\|_{\rho_s + \frac{2\tau}{s^2}} < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}} \tag{7.1}$$

This means that

$$I_n \circ C_{\mathcal{F}} - I_n = \sum_{\bar{a},\bar{k},\bar{k}'} B_{a,k,k'} \mathcal{M}_{a,k,k'} \tag{7.2}$$

where

$$|B_{a,k,k'}| < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}} \; e^{\rho_{s+1}(\sum_m (2a_m + k_m + k'_m)\sqrt{m} - 2\sqrt{m_1^*})}. \tag{7.3}$$

Observe that in (7.2) we only get monomials $\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}$ satisfying

$$\sum |m|(k_m + k'_m) \geqslant 2|n|. \tag{7.4}$$

For $q \in \Omega_r$,

$$|\mathcal{M}_{\bar{a},\bar{k},\bar{k}'}| \leqslant \prod I_m(0)^{a_m} e^{-r \sum_m \sqrt{m}(k_m + k'_m)}. \tag{7.5}$$

Assume also

$$I_m(0) < e^{-r\sqrt{m}}, \quad \forall m \tag{7.6}$$

so that

$$|\mathcal{M}_{a,k,k'}| < e^{-r\sum_m \sqrt{m}(2a_m+k_m+k'_m)}. \tag{7.7}$$

Letting $m_1^* \geqslant m_2^* \geqslant \cdots$ be the decreasing rearrangement of $\{|m|(2a_m + k_m + k'_m)\text{-times}\}$, we have

$$|B_{a,k,k'}|\ |\mathcal{M}_{a,k,k'}| < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}}\ e^{\rho_{s+1}(\sqrt{m_3^*}+\sqrt{m_4^*}+\cdots)-r(\sqrt{m_1^*}+\sqrt{m_2^*}+\cdots)}. \tag{7.8}$$

We distinguish 2 cases.

(A) $\underline{m_1^* \geqslant |n|}$

By (1.5), $\sum_{i\geqslant 1}\sqrt{m_i^*} \geqslant 2\sqrt{m_1^*} + \frac{1}{4}\sum_{i\geqslant 3}\sqrt{m_i^*} \geqslant 2\sqrt{n} + \frac{1}{4}\sum_{i\geqslant 3}\sqrt{m_i^*} + |\sqrt{m_1^*} - \sqrt{|n|}|$. Assume (cf. (5.34)).

$$r > 10\rho_{s+1}. \tag{7.9}$$

Thus (7.8) <

$$\varepsilon_0^{0.8(\frac{3}{2})^{s-1}}\ e^{-2r\sqrt{n}}\ e^{-\frac{r}{8}\sum_{i\geqslant 3}\sqrt{m_i^*} - r|\sqrt{m_1^*}-\sqrt{|n|}|}. \tag{7.10}$$

Summing over $(\bar{a}, \bar{k}, \bar{k}')$ gives therefore the bound (m_2^* determined by m_1^* and $m_i^*(i \geqslant 3)$)

$$\begin{aligned} &\varepsilon_0^{0.8(\frac{3}{2})^{s-1}} e^{-2r\sqrt{n}} \left(\prod_m \frac{1}{1-e^{-\frac{r}{8}\sqrt{m}}}\right)^3 \sum_{m_1\geqslant |n|} e^{-r|\sqrt{m}_1 - \sqrt{|n|}|} \\ &\lesssim \varepsilon_0^{0.8(\frac{3}{2})^{s-1}} e^{-2r\sqrt{n}} n^{1/2}. \end{aligned} \tag{7.11}$$

(B) $\underline{m_1^* < |n|}$

Recalling (7.4)

$$2|n| \leqslant \sum_{i\geqslant 1} m_i^* \leqslant \sqrt{m_1^*}\sum_{i\geqslant 1}\sqrt{m_i^*},$$

$$|n| \leqslant \sum_{i\geqslant 2} m_i^* \leqslant \sqrt{m_2^*}\sum_{i\geqslant 2}\sqrt{m_i^*}.$$

Hence

$$3\sum_{i\geqslant 1}\sqrt{m_i^*} \geqslant \frac{2|n|}{\sqrt{m_1^*}} + \sqrt{m_1^*} + \frac{n}{\sqrt{m_2^*}} + \sqrt{m_1^*} + \sqrt{m_2^*} + \sum_{i\geqslant 3}\sqrt{m_i^*}$$

$$\geqslant 4\sqrt{n} + 2\sqrt{n} + \sum_{i\geqslant 3}\sqrt{m_i^*} + 2\left(\frac{\sqrt{n}}{\sqrt[4]{m_1^*}} - \sqrt[4]{m_1^*}\right)^2$$

$$\sum_{i\geqslant 1}\sqrt{m_i^*} \geqslant 2\sqrt{n} + \frac{1}{3}\sum_{i\geqslant 3}\sqrt{m_i^*} + \frac{(\sqrt{n}-\sqrt{m_1^*})^2}{3\sqrt{m_1^*}} \tag{7.12}$$

and we get the bound $\varepsilon_0^{0.8(\frac{3}{2})^{s-1}}\, e^{-2r\sqrt{n}} n^{3/4}$.

Hence

$$|(7.2)| < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}}\, e^{-2r\sqrt{n}} n^{3/4}. \tag{7.13}$$

The factor $n^{3/4}$ may be removed by more careful analysis. Recall that

$$\mathcal{F} \sim \sum_{\bar{a},\bar{k},\bar{k}'} \frac{B_{\bar{a},\bar{k},\bar{k}'}}{\sum (k_m - k_m')(m^2 + \tilde{V}_m)} \mathcal{M}_{\bar{a},\bar{k},\bar{k}'},$$

where

$$|B_{a,k,k'}| < \varepsilon_0^{0.8(\frac{3}{2})^{s-1}}\, e^{\rho_s (\sum_m (2a_m + k_m + k_m')\sqrt{m} - 2\sqrt{m_1^*})}. \tag{7.14}$$

Consider the expansion

$$\sum_{r\geqslant 1} \frac{1}{r!}\{\cdots\{I_m, \mathcal{F}\}, \mathcal{F}, \ldots, \mathcal{F}\}$$

and the first Poisson bracket $\{I_n, \mathcal{F}\}$. From (7.14)

$$|B_{a,k,k'}| < e^{\rho_s \sum_{i>2}\sqrt{m_i^*}}. \tag{7.15}$$

We distinguish the following cases ($|n|$ is assumed large).

(I) $\sum_{i>2}\sqrt{m_i^*} > (\log|n|)^2$.

One may then obviously save an $\frac{1}{|n|}$-factor by increasing slightly ρ_s.

(II) $\sum_{i>2}\sqrt{m_i^*} \leqslant (\log|n|)^2$.

Then $\{I_n, \mathcal{M}_{\bar a,\bar k,\bar k'}\} = 0$ unless $n \in \{m_1^*, m_2^*\}$ and $|k_n| + |k_n'| \neq 0 (m_1^* = |m_1|, m_2^* = |m_2|)$.

If $m_1^* \neq m_2^*$, then clearly $|\sum (k_m - k_m')(m^2 + \tilde V_m)| > [(m_1^2 - m_2^2) \wedge n^2] - (\log |n|)^{10} > |n|$.

If $m_1^* = m_2^* = |n|$, then the preceding still holds, unless $k_n = k_n' = 1$, in which case again $\{I_n, \mathcal{M}_{\bar a,\bar k,\bar k'}\} = 0$.

Hence, also

$$|(7.2)| < \varepsilon_0^{0.8(\frac{3}{2})^s} e^{-2r\sqrt{|n|}}. \tag{7.16}$$

Consequently

$$\forall n : |(I_n \circ \mathcal{C}_{\mathcal{F}}) - I_n| < \varepsilon_0^{0.8(\frac{3}{2})^s} e^{-2r\sqrt{|n|}} \tag{7.17}$$

which implies in particular that $\mathcal{C}_{\mathcal{F}}$ maps Ω_r into $(1 + \varepsilon_0^{0.8(\frac{3}{2})^s})\Omega_r$.

Considering the resulting symplectic transformation $\cdots \mathcal{C}_{\mathcal{F}^{(s)}} \circ \mathcal{C}_{\mathcal{F}^{(s-1)}} \circ \cdots \circ \mathcal{C}_{\mathcal{F}^{(1)}} = \mathcal{C}$, iteration of (7.17) gives

$$\forall n : |(I_n \circ \mathcal{C}) - I_n| < e^{-2r\sqrt{|n|}} \sum_{s \geqslant 1} \varepsilon_0^{0.8(\frac{3}{2})^s} < \sqrt{\varepsilon_0}\, e^{-2r\sqrt{|n|}} \tag{7.18}$$

and $\mathcal{C}$ maps Ω_r to $(1 + \sqrt{\varepsilon_0})\Omega_r$.

Recalling (7.6), (7.9), we assume $q|_{t=0} \in \Omega_{r_0}$, thus $I_n(0) < e^{-2r_0\sqrt{n}}$, $\forall n$. The symplectic transformation $\mathcal{C}$ will perturb the action variables $I_n = |q_n|^2$ by at most $\sqrt{\varepsilon_0} e^{-2r_0\sqrt{n}}$.

8. Conclusion of the argument

We consider first the case of finite dimensional phase space $(q_n)_{|n| \leqslant N}$ truncating the original NLS-Hamiltonian $H(q, \bar q)$ at order N the usual way. Consider the corresponding evolution

$$i\dot q_n^{(N)} = \frac{\partial H^{(N)}}{\partial \bar q_n} \quad (|n| \leqslant N). \tag{8.1}$$

Recall (from the wellposedness theory for 1D NLS on $\mathbb{T}$) that if $q = q(t)$ is the NLS solution with $q(0) \in H^1(\mathbb{T})$, thus

$$i\dot q_n = \frac{\partial H}{\partial \bar q_n} \quad (n \in \mathbb{Z}) \quad q\Big|_{t=0} = q(0) \tag{8.2}$$

there is a uniform comparison estimate

$$\max_{|t|\leqslant T} \|q(t) - q^{(N)}(t)\|_{\ell^2} \leqslant \delta_n(T)\|q(0)\|_{H^1} \tag{8.3}$$

where $q^N(0) = P_N q(0)$ and $\delta_N(T) \overset{N\to\infty}{\to} \to 0$ for fixed T.

Starting from $H = H^{(N)}$, perform preceding normal forms reduction up to stage s (chosen large enough depending on N).

All estimates in this process are of course independent of N.

Thus at stage s, we get the Hamiltonian (5.1)

$$\mathcal{H} = \mathcal{H}^{(s)} = \sum_{|n|\leqslant N} (n^2 + \omega_n)|q_n|^2 + \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2, \tag{8.4}$$

where

$$I_n(0) \leqslant e^{-2r_0\sqrt{n}} \tag{8.5}$$

and by (5.31)–(5.33)

$$\|\mathcal{H}_0\|_{\rho_s} < \varepsilon_0^{(\frac{3}{2})^s}, \tag{8.6}$$

$$\|\mathcal{H}_1\|_{\rho_s} < \varepsilon_0^{0.9(\frac{3}{2})^{s-1}}, \tag{8.7}$$

$$\|\mathcal{H}_2\|_{\rho_s} < \varepsilon_0. \tag{8.8}$$

Let $q(0)$ satisfy $|q_n(0)|^2 = I_n(0)$. Consider the solution of

$$i\dot{q}_n = \frac{\partial \mathcal{H}}{\partial \bar{q}_n} \ (|n|\leqslant N), \quad q_n\big|_{t=0} = q_n(0). \tag{8.9}$$

We will show that $q(t)$ remains in $2\Omega_{r_0}$ for $|t|\leqslant T_s \overset{s\to\infty}{\to} \to \infty$ and moreover

$$|q_n(t) - q_n(0)e^{i(n^2+\omega_n)t}| < (\varepsilon_0)^{(\frac{10}{9})^s} e^{-r_0\sqrt{|n|}} \quad (|n|\leqslant N). \tag{8.10}$$

Consider first $I_n(1) - I_n(0)$. Clearly

$$\dot{I} = \{I_n, \mathcal{H}\} \quad (|n|\leqslant N),$$

$$|I_n(1) - I_n(0)| \leqslant \int_0^1 [|\{I_n, \mathcal{H}_0\}| + |\{I_n, \mathcal{H}_1\}| + |\{I_n, \mathcal{H}_2\}|]. \tag{8.11}$$

Assuming $q(t) \in 2\Omega_{r_0}$, it follows from the estimates in Section 7 (cf. (7.13)) and (8.6), (8.7) that

$$
\begin{aligned}
|\{I_n, \mathcal{H}_0\}| + |\{I_n, \mathcal{H}_1\}| &< \varepsilon_0^{0.8(\frac{3}{2})^{s-1}} e^{-2r_0\sqrt{|n|}}\ |n|^{3/4} \\
&< \varepsilon_0^{0.8(\frac{3}{2})^{s-1} e^{-2r_0\sqrt{|n|}}}\ N^{3/4} \\
&< \varepsilon_0^{\frac{79}{100}(\frac{3}{2})^{s-1}} e^{-2r_0\sqrt{|n|}}
\end{aligned}
\tag{8.12}
$$

by the choice of s.

Let

$$\gamma = \sup_{|n|\leqslant N, |t|\leqslant 1} e^{2r_0\sqrt{|n|}} |I_n(t) - I_n(0)|. \tag{8.13}$$

Since $\mathcal{H}_2$ is at least quadratic in J, the estimate (from (8.8))

$$|\{I_n, \mathcal{H}_2\}| < \varepsilon_0 e^{-2r_0\sqrt{n}}\ N^{3/4} \tag{8.14}$$

may be restated as

$$|\{I_n, \mathcal{H}_2\}| < \varepsilon_0 e^{-2r_0\sqrt{n}} N^{3/4}\gamma^2 \tag{8.15}$$

(8.11)–(8.13), (8.15) imply then that

$$\gamma \leqslant \varepsilon_0^{\frac{79}{100}(\frac{3}{2})^{s-1}} + N^{3/4}\gamma^2$$

hence

$$\gamma < 2\varepsilon_0^{\frac{79}{100}(\frac{3}{2})^{s-1}}. \tag{8.16}$$

Consequently

$$|I_n(1) - I_n(0)| < 5\varepsilon_0^{\frac{79}{100}(\frac{3}{2})^{s-1}} e^{-2r_0\sqrt{n}} \quad (|n| \leqslant N) \tag{8.17}$$

and for $0 \leqslant t \leqslant 1$

$$q(t) \in (1 + \varepsilon_0^{0.7(\frac{3}{2})^{s-1}})\Omega_{r_0} \subset 2\Omega_{r_0}. \tag{8.18}$$

Pass then from $t = 1$ to $t = 2$. We redefine $J'_n = I_n - I_n(1)$ and express (5.1) in this form. Thus in $\mathcal{H}_1, \mathcal{H}_2$, replace J_n by $J'_n + (I_n(1) - I_n(0))$, hence

$$\mathcal{H}_1 = \tilde{\mathcal{H}}_1 + \mathcal{H}'_{0,1},$$
$$\mathcal{H}_2 = \tilde{\mathcal{H}}_2 + \mathcal{H}'_{0,2} + \mathcal{H}'_{1,2},$$
$$\mathcal{H} = \mathcal{H}'_0 + \mathcal{H}'_1 + \mathcal{H}'_2 + \sum (n^2 + \omega_n)|q_n|^2,$$

where

$$\begin{cases} \mathcal{H}'_0 = \mathcal{H}_0 + \mathcal{H}'_{0,1} + \mathcal{H}'_{0,2}, \\ \mathcal{H}'_1 = \tilde{\mathcal{H}}_1 + \mathcal{H}'_{1,2}, \\ \mathcal{H}'_2 = \tilde{\mathcal{H}}_2. \end{cases}$$

Clearly

$$\|\mathcal{H}'_{0,1}\|_{\rho_s} \leqslant \|\mathcal{H}_1\|_{\rho_s} \left(\sum_n e^{2\rho_s\sqrt{n}} \varepsilon_0^{0.7(\frac{3}{2})^{s-1}} e^{-2r_0\sqrt{n}} \right) < \varepsilon_0^{(\frac{3}{2})^s},$$

$$\|\mathcal{H}'_{0,2}\|_{\rho_s} \leqslant \|\mathcal{H}_2\|_{\rho_s} \left(\sum_n e^{2\rho_s\sqrt{n}} \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} e^{-2r_0\sqrt{n}} \right)^2 < \varepsilon_0^{(\frac{3}{2})^s},$$

$$\|\mathcal{H}'_{1,2}\|_{\rho_s} \leqslant \|\mathcal{H}_2\|_{\rho_s} \left(\sum_n e^{2\rho_s\sqrt{n}} \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} e^{-2r_0\sqrt{n}} \right) < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s},$$

$$\begin{cases} \|\mathcal{H}'_0\|_{\rho_s} \lesssim \varepsilon_0^{(\frac{3}{2})^s}, & (8.19) \\ \|\mathcal{H}'_1\|_{\rho_s} \lesssim \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s}, & (8.20) \\ \|\mathcal{H}'_2\|_{\rho_s} \leqslant \varepsilon_0 & (8.21) \end{cases}$$

and again

$$|I_n(2) - I_n(1)| \lesssim \varepsilon_0^{\frac{1}{3}(\frac{3}{2})^s} e^{-2r_0\sqrt{n}} \quad (|n| \leqslant N). \tag{8.22}$$

For some $|t| < T_s = s$ say, we will ensure that

$$|I_n(t) - I_n(0)| < e^{-2r_0\sqrt{n}} (\varepsilon_0)^{(\frac{9}{8})^s} < e^{-2r_0\sqrt{n}} \ (|n| \leqslant N) \tag{8.23}$$

hence

$$q(t) \in 2\Omega_{r_0}.$$

Estimate for $0 \leqslant t \leqslant 1$

$$|q_n(t) - q_n(0)e^{i(n^2+\omega_n)t}| \leqslant \int_0^1 \left(\left|\frac{\partial \mathcal{H}_0}{\partial \bar{q}_n}\right| + \left|\frac{\partial \mathcal{H}_1}{\partial \bar{q}_n}\right| + \left|\frac{\partial \mathcal{H}_2}{\partial \bar{q}_n}\right| \right). \tag{8.24}$$

Similarly as we got (8.12), one sees that for $q \in 2\Omega_{r_0}$

$$\left|\frac{\partial \mathcal{H}_0}{\partial \bar{q}_n}\right| + \left|\frac{\partial \mathcal{H}_1}{\bar{q}_n}\right| < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} e^{-r_0\sqrt{n}}. \tag{8.25}$$

Since $\frac{\partial \mathcal{H}_2}{\partial \bar{q}_n}$ contains at least 1 J-factor, (8.17) implies for $q = q(t)$, $0 \leqslant t \leqslant 1$

$$\left|\frac{\partial \mathcal{H}_2}{\partial \bar{q}_n}\right| < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} \, e^{-r_0\sqrt{n}}. \tag{8.26}$$

Substituting (8.25), (8.26) in (8.24) gives for $|t| \leqslant 1$

$$|q_n(t) - q_n(0)e^{i(n^2+\omega_n)t}| < \varepsilon_0^{\frac{1}{2}(\frac{3}{2})^s} \, e^{-r_0\sqrt{n}}. \tag{8.27}$$

Similarly, since $q(t) \in 2\Omega_{r_0}$ up to $t < T_s$ and (8.23)

$$|q_n(t) - q_n(1)e^{i(n^2+\omega_n)(t-1)}| < \varepsilon_0^{(\frac{9}{8})^s} e^{-r_0\sqrt{n}} \quad (1 \leqslant t \leqslant 2)$$

and for $t_0 < t < t_0 + 1 < T_s$

$$|q_n(t) - q_n(t_0)e^{i(n^2+\omega_n)(t-t_0)}| < \varepsilon_0^{(\frac{9}{8})^s} \, e^{-r_0\sqrt{n}}, \tag{8.28}$$

$$|q_n(t) - q_n(0)e^{i(n^2+\omega_n)t}| \leqslant \varepsilon_0^{(\frac{10}{9})^s} e^{-r_0\sqrt{|n|}} \quad (|n| \leqslant N). \tag{8.29}$$

This proves (8.10).

In the limit $(s \to \infty)$ one obtains the almost-periodic motion

$$q_n(t) = q_n(0) \, e^{i(n^2+\omega_n)t} \quad (|n| \leqslant N). \tag{8.30}$$

Returning to (8.1), one needs to pull back the invariant torus $[I_n = I_n(0) \big| \, |n| \leqslant N]$ by the symplectic transformation $\mathcal{C}$. Recalling (7.18), we obtain thus an invariant torus $\mathcal{T}_N$ for (8.1) satisfying

$$\big| \, |q_n|^2 - I_n(0) \big| < \sqrt{\varepsilon_0}\, e^{-2r_0\sqrt{|n|}} \quad (|n| \leqslant N). \tag{8.31}$$

Next, the uniform estimate (8.31) allows us to pass to a limit $\mathcal{T} = \lim_{N\to\infty} \mathcal{T}_N$. Thus $\mathcal{T}$ consists of elements $q = (q_n)_{n\in\mathbb{Z}}$ satisfying (8.31) and such that for all N_0

$$\lim_{N\to\infty} \inf_{q^{(N)}\in\mathcal{T}_N} \max_{|n|\leqslant N_0} |q_n - q_n^{(N)}| = 0. \tag{8.32}$$

Since obviously $\mathcal{T}_N, \mathcal{T}$ are bounded in $H^2(\mathbb{T})$ say, (8.32) also implies

$$\lim_{N\to\infty} \inf_{q^{(N)}\in\mathcal{T}_N} \|P_N q - q^{(N)}\|_{H^1} = 0 \tag{8.33}$$

denoting $\|q\|_{H^1} = (\sum_n n^2 |q_n|^2)^{1/2}$.

Denote $S_N(t)$ the flow map of (9.1) and $S(t)$ the flow map of (8.2). We verify that $\mathcal{T}$ is $S(t)$ invariant. Fix t. Since $S_N(t)$ and $S(t)$ are (with fixed t) Lipschitz on H^1, (8.33) implies for some $q^{(N)} \in \mathcal{T}_N$

$$\|S_N(t)P_N q - S_N(t)q^{(N)}\|_{H^1} \overset{N\to\infty}{\to} 0. \tag{8.34}$$

From (8.3)

$$\|S(t)q - S_N(t)P_N q\|_2 \overset{N\to\infty}{\to} \to 0. \tag{8.35}$$

Since $S_N(t)q^{(N)} \in \mathcal{T}_N$, $\text{dist}_{\ell^2(\mathbb{Z})}\big(S(t)q, \mathcal{T}_N\big) \overset{N\to\infty}{\to} 0$ and hence $S(t)q \in \mathcal{T}$.

Thus $\mathcal{T}$ is an invariant torus for the NLS (8.2) and if $q \in \mathcal{T}$ by (8.31)

$$\big| \, |q_n|^2 - I_n(0) \big| < \sqrt{\varepsilon_0}\, e^{-2r_0\sqrt{|n|}} \quad \text{for all } n \in \mathbb{Z}. \tag{8.36}$$

Taking $I_n(0) = e^{-2r_0\sqrt{n}}$, cf. (8.5), we have for $q \in \mathcal{T}, n \in \mathbb{Z}$

$$(1 - \sqrt{\varepsilon_0})e^{-r_0\sqrt{|n|}} < |q_n| < (1 + \sqrt{\varepsilon_0})e^{-r_0\sqrt{|n|}}. \tag{8.37}$$

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