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Secant varieties of toric varieties

David Cox [a], Jessica Sidman [b,*]

[a] *Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002, United States*
[b] *Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075, United States*

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Abstract

Let X_P be a smooth projective toric variety of dimension n embedded in $\mathbb{P}^r$ using all of the lattice points of the polytope P. We compute the dimension and degree of the secant variety Sec X_P. We also give explicit formulas in dimensions 2 and 3 and obtain partial results for the projective varieties X_A embedded using a set of lattice points $A \subset P \cap \mathbb{Z}^n$ containing the vertices of P and their nearest neighbors.

MSC: 14M25; 14N05; 51N35

1. Introduction

Let $X \subseteq \mathbb{P}^r$ be a reduced and irreducible complex variety of dimension n. Its kth secant variety, $\mathrm{Sec}_k X$, is the closure of the union of all $(k-1)$-planes in $\mathbb{P}^r$ meeting X in at least k points. In this paper, we discuss the dimension and degree of the secant variety $\mathrm{Sec}\, X = \mathrm{Sec}_2 X$ when X is a smooth toric variety equivariantly embedded in projective space.

Secant varieties arise naturally in classical examples (see Examples 1.1–1.3 below) and also in the newly developing fields of algebraic statistics and phylogenetic combinatorics (see, for example, [10]). The basic theory of secant varieties is explained in the books [11,28].

There is a rich collection of ideas relating secant varieties, tangent varieties, dual varieties, and the Gauss map, as discussed in [19] and [11, Section 4.4]. Many authors have studied the problem of classifying the varieties X such that $\mathrm{Sec}_k X$ has the expected dimension, $\min\{r, k(n+1)-1\}$, and what degeneracies may occur [2–5,24]. For the duals of toric varieties, this was done in [9]. Our paper can be regarded as the beginnings of a similar study for the secant varieties of toric varieties.

Currently, there is much activity focused on refining our understanding of secant varieties. A lower bound for the degree of $\mathrm{Sec}_k X$ is given in [6], and the degree of the secant variety of a monomial curve is worked out

* Corresponding author.
E-mail addresses: dac@cs.amherst.edu (D. Cox), jsidman@mtholyoke.edu (J. Sidman).

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in [25]. Questions on the defining equations of secant varieties are discussed in [22,27]. The recent paper [26] uses combinatorial Gröbner methods to study the ideal of $\mathrm{Sec}_k X$ and its degree and dimension.

Many classical varieties whose secant varieties have been studied in the literature are toric—see for example [1–3, 5,8,16,21]. The secant varieties below will play an important role in our paper.

Example 1.1. If X is the image of the Veronese map $\nu_2 : \mathbb{P}^n \to \mathbb{P}^r$ by $\mathcal{O}_{\mathbb{P}^n}(2)$, then its secant variety has the expected dimension only when $n = 1$; otherwise the dimension is $2n < \min\{r, 2n + 1\}$.

The ideal of X is generated by the 2×2 minors of the $(n + 1) \times (n + 1)$ generic symmetric matrix and the ideal of $\mathrm{Sec}\, X$ is generated by the 3×3 minors of the same matrix. (See Example 1.3.6 in [11].)

Example 1.2. If $X = \mathbb{P}^\ell \times \mathbb{P}^{n-\ell}$, $1 \le \ell \le n - 1$, is embedded in $\mathbb{P}^r$ via the Segre map, then $\mathrm{Sec}\, X$ has the expected dimension only when $\ell = 1, n - 1$; otherwise $\dim \mathrm{Sec}\, X = 2n - 1 < \min\{r, 2n + 1\}$.

Here, the ideal of X is also determinantal, generated by the 2×2 minors of the generic $(\ell + 1) \times (n - \ell + 1)$ matrix, and the ideal of $\mathrm{Sec}\, X$ is defined by the 3×3 minors of the same matrix when $\ell + 1, n - \ell + 1 \ge 3$. (See Example 4.5.21 of [11].)

Example 1.3. Given positive integers $d_1, \ldots, d_n$, the rational normal scroll $S_{d_1,\ldots,d_n}$ is the image of

$$\mathbb{P}(\mathcal{E}), \quad \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$$

under the projective embedding given by the ample line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The variety $\mathrm{Sec}\, X$ always has the expected dimension, which is $2n + 1$ except when $\sum_{i=1}^n d_i \le n + 1$.

Furthermore, the ideal of X (resp. $\mathrm{Sec}\, X$) is generated by the 2×2 (resp. 3×3) minors of a matrix built out of Hankel matrices. (See [2] and Section 4.7 of [11].)

A key observation is that Examples 1.1–1.3 each involve a smooth toric variety coming from a particularly simple polytope. It is natural to ask what happens for more general toric varieties. To state our main theorem, we introduce some definitions.

If $P \subset \mathbb{R}^n$ is an m-dimensional lattice polytope, we say that P is *smooth* if it is simplicial and the first lattice vectors along the edges incident at any vertex form part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. (Note that such polytopes are also called *Delzant* in the literature.) The lattice points $\{u_0, \ldots, u_r\} = P \cap \mathbb{Z}^n$ give characters χ^{u_i} of a torus that give an embedding $X \hookrightarrow \mathbb{P}^r$ defined by $x \mapsto [\chi^{u_0}(x) : \cdots : \chi^{u_r}(x)]$, where X is the abstract toric variety associated to the inner normal fan of P. We denote the image of X under this embedding by X_P. Note that X is smooth since P is smooth.

We also observe that X_P is projectively equivalent to X_Q if P is obtained from Q via an element of the group $AGL_n(\mathbb{Z})$ of affine linear isomorphisms of $\mathbb{Z}^n$.

The *standard simplex* of dimension n in $\mathbb{R}^n$ is

$$\Delta_n = \{(p_1, \ldots, p_n) \in \mathbb{R}^n_{\ge 0} \mid p_1 + \cdots + p_n \le 1\}$$

and its multiple by $r \ge 0$ is

$$r\Delta_n = \{(p_1, \ldots, p_n) \in \mathbb{R}^n_{\ge 0} \mid p_1 + \cdots + p_n \le r\}.$$

We let $(2\Delta_n)_k$ denote the convex hull of the lattice points in $2\Delta_n$ minus a k-dimensional face, and let $Bl_k(\mathbb{P}^n)$ denote the blowup of $\mathbb{P}^n$ along a torus-invariant k-dimensional linear subspace.

Here is the main result of the paper.

Theorem 1.4. *Let P be a smooth polytope of dimension n. The dimension and degree of* $\mathrm{Sec}\, X_P$ *are given in the following table:*

P	X_P	*Dimension and degree of* $\operatorname{Sec} X_P$
Δ_n	$\mathbb{P}^n$	$\dim \operatorname{Sec} X_P = n$ $\deg \operatorname{Sec} X_P = 1$
$2\Delta_n$	$\nu_2(\mathbb{P}^n)$	$\dim \operatorname{Sec} X_P = 2n$ $\deg \operatorname{Sec} X_P = \binom{2n-1}{n-1}$
$(2\Delta_n)_k$ $(0 \leq k \leq n-2)$	$Bl_k(\mathbb{P}^n)$	$\dim \operatorname{Sec} X_P = 2n$ $\deg \operatorname{Sec} X_P = \sum_{1 \leq i < j \leq n-k} \left[\binom{n}{n-i}\binom{n-1}{n-j} - \binom{n}{n-j}\binom{n-1}{n-i} \right]$
$\Delta_\ell \times \Delta_{n-\ell}$ $(1 \leq \ell \leq n-1)$	$\mathbb{P}^\ell \times \mathbb{P}^{n-\ell}$	$\dim \operatorname{Sec} X_P = 2n-1$ $\deg \operatorname{Sec} X_P = \prod_{0 \leq i \leq n-\ell-2} \frac{\binom{\ell+1+i}{2}}{\binom{2+i}{2}}$
not $AGL_n(\mathbb{Z})$*–* *equivalent to* *the above*	X_P	$\dim \operatorname{Sec} X_P = 2n+1$ $\deg \operatorname{Sec} X_P = \frac{1}{2}\left((\deg X_P)^2 - \sum_{i=0}^{n} \binom{2n+1}{i} \int_{X_P} c(T_{X_P})^{-1} \cap c_1(\mathcal{L})^i \right)$

In the last row, $\mathcal{L} = \mathcal{O}_{X_P}(1)$ *and* $c(T_{X_P})^{-1}$ *is the inverse of the total Chern class of the tangent bundle of* X_P *in the Chow ring of* X_P*. Furthermore:*

(1) *For* P *in the first four rows of the table, there are infinitely many secant lines of* X_P *through a general point of* $\operatorname{Sec} X_P$.

(2) *For* P *in the last row of the table, there is a unique secant line of* X_P *through a general point of* $\operatorname{Sec} X_P$.

For the remainder of the paper, the table appearing in Theorem 1.4 will be referred to as Table 1.4. In Section 4 we prove that the first four rows of Table 1.4 give all smooth subpolytopes of $2\Delta_n$ up to $AGL_n(\mathbb{Z})$-equivalence. Hence $\dim \operatorname{Sec} X_P = 2n+1$ if and only if P does not fit inside $2\Delta_n$. This also determines the number of secant lines through a general point of $\operatorname{Sec} X_P$.

Theorem 1.4 enables us to decide when the secant variety has the expected dimension.

Corollary 1.5. *When* $X_P \subset \mathbb{P}^r$ *comes from a smooth polytope* P *of dimension* n, $\operatorname{Sec} X_P$ *has the expected dimension* $\min\{r, 2n+1\}$ *unless* P *is* $AGL_n(\mathbb{Z})$*-equivalent to one of*

$$2\Delta_n \ (n \geq 2), \quad (2\Delta_n)_k \quad (0 \leq k \leq n-3), \quad \Delta_\ell \times \Delta_{n-\ell} \quad (2 \leq \ell \leq n-2).$$

We can compute the degree of $\operatorname{Sec} X_P$ explicitly in low dimensions as follows. Throughout, n-dimensional volume is normalized so that the volume of Δ_n is 1.

Corollary 1.6. *Let* P *be a smooth polygon in* $\mathbb{R}^2$*. If* $\dim \operatorname{Sec} X_P = 5$*, then*

$$\deg \operatorname{Sec} X_P = \frac{1}{2}(d^2 - 10d + 5B + 2V - 12),$$

where d *is the area of* P*,* B *is the number of lattice points on the boundary of* P*, and* V *is the number of vertices of* P*.*

Corollary 1.7. *Let* P *be a smooth 3-dimensional polytope in* $\mathbb{R}^3$*. If* $\dim \operatorname{Sec} X_P = 7$*, then*

$$\deg \operatorname{Sec} X_P = \frac{1}{2}(d^2 - 21d + c_1^3 + 8V + 14E - 84I - 132),$$

where d *is the volume of* P*,* E *is the number of lattice points on the edges of* P*,* V *is the number of vertices of* P*, and* I *is the number of interior lattice points. Also,* $c_1 = c_1(T_{X_P})$*.*

Remark 1.8. Computing explicit degree formulas for Sec X_P in terms of the lattice points of P gets more difficult as the dimension increases. However, we would like to stress that given any smooth polytope P, the formula in the last row of Table 1.4 is computable solely from the data of P. Indeed, $c(T_{X_P})^{-1}$ depends only on the inner normal fan of P, and $\mathcal{L}$ is the line bundle given by P. The intersection products may be determined from the combinatorial geometry of P using standard results in [14] or any convenient presentation of the Chow ring of X_P. We give an example of this in Theorem 4.8 when we compute the degree of the secant variety of a Segre–Veronese variety.

We prove Theorem 1.4 in three steps as follows.

First, for the polytopes in the first four rows of Table 1.4, the degrees and dimensions of their secant varieties are computed in Section 2 using determinantal presentations for the ideals of X_P and Sec X_P, together with known results about dimensions and degrees of determinantal varieties. We also relate these polytopes to subpolytopes of $2\Delta_n$.

Second, the bottom row of Table 1.4 uses a well-known formula for the degree of Sec X_P times the degree of the linear projection from the abstract join of X_P with itself to $\mathbb{P}^r$. We discuss this formula in Section 3 and apply it to various examples.

Third, for the polytopes in the bottom row of Table 1.4, we prove in Section 4 that a general point of Sec X_P lies on a unique secant line by showing that P contains configurations of lattice points that are easier to study. Known results about rational normal scrolls will be used in the proof.

From these results, Theorem 1.4 and Corollaries 1.5–1.7 follow easily. The paper concludes with Section 5, where we study the toric varieties X_A embedded using a subset of lattice points $A \subset P \cap \mathbb{Z}^n$ of a smooth polytope P of dimension n. When A contains the vertices of P and their nearest neighbors along the edges, we compute dim Sec X_A and obtain partial results about deg Sec X_A.

2. Smooth subpolytopes of $2\Delta_n$

The purpose of this section is to study the polytopes Δ_n, $2\Delta_n$, $(2\Delta_n)_k$ and $\Delta_\ell \times \Delta_{n-\ell}$ appearing in the first four rows of Table 1.4. The dimensions and degrees of the corresponding secant varieties will be computed using determinantal methods. We will also see that these polytopes give essentially all smooth subpolytopes of $2\Delta_n$.

2.1. Dimension and degree calculations

We begin with the polytope $(2\Delta_n)_k$. As in the introduction, this is defined to be the convex hull of the lattice points of $2\Delta_n \setminus F$, where F is any k-dimensional face of $2\Delta_n$. When $k = -1$, F is the empty face, so that $(2\Delta_n)_{-1} = 2\Delta_n$, and when $k = n-1$, F is a facet, so that $(2\Delta_n)_{n-1} = \Delta_n$. In the discussion that follows, we will usually exclude these cases by requiring that $0 \le k \le n-2$.

The toric variety corresponding to $(2\Delta_n)_k$ is easy to describe.

Proposition 2.1. *Fix k between 0 and $n-2$, and let $Bl_k(\mathbb{P}^n)$ denote the blow-up of $\mathbb{P}^n$ along a torus-invariant k-dimensional subspace. Then, $Bl_k(\mathbb{P}^n)$ is the toric variety $X_{(2\Delta_n)_k}$.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. The standard fan for $\mathbb{P}^n$ has cone generators $v_0, v_1, \ldots, v_n$, where $v_1 = e_1, \ldots, v_n = e_n$ and $v_0 = -\sum_{i=1}^n e_i$. Given k between 0 and $n-2$, the star subdivision of the $(n-k)$-dimensional cone $\sigma = \mathrm{Cone}(v_0, v_{k+2}, \ldots, v_n)$ is obtained by adding the new cone generator

$$v_{n+1} = v_0 + v_{k+2} + \cdots + v_n = -e_1 - \cdots - e_{k+1}$$

and subdividing σ accordingly. As explained in [23, Prop. 1.26], the toric variety of this new fan is the blow up of $\mathbb{P}^n$ along the k-dimensional orbit closure corresponding to σ. Thus we have $Bl_k(\mathbb{P}^n)$. The cone generators $v_0, \ldots, v_{n+1}$ give torus-invariant divisors $D_0, \ldots, D_{n+1}$ on $Bl_k(\mathbb{P}^n)$. One easily checks that $D = 2D_0 + D_{n+1}$ is ample.

Using the description on p. 66 of [14], the polytope P_D determined by D is given by the $n+2$ facet inequalities

$$\begin{aligned} &x_1 + \cdots + x_n \le 2 \\ &x_i \ge 0, \qquad 1 \le i \le n \\ &x_1 + \cdots + x_{k+1} \le 1. \end{aligned}$$

The inequalities on the first two lines define $2\Delta_n$ and the final inequality removes the face corresponding to σ. It follows that $P_D = (2\Delta_n)_k$. $\square$

We will see below that the ideal of $Bl_k(\mathbb{P}^n)$ is determinantal. We now turn our attention to some dimension and degree calculations.

Theorem 2.2. *If P is one of the polytopes*

$$\Delta_n, \quad 2\Delta_n, \quad (2\Delta_n)_k \quad (0 \le k \le n-2), \quad \Delta_\ell \times \Delta_{n-\ell} \ (1 \le \ell \le n-1),$$

then the dimension and degree of Sec X_P *are given by the first four rows of Table 1.4.*

Proof. The case of Δ_n is trivial. Turning our attention to $2\Delta_n$, we let $r = \binom{n+2}{2} - 1$. The lattice points in $2\Delta_n$ correspond to the $r+1$ monomials of degree 2 in $x_0, \ldots, x_n$. Order these monomials lexicographically so that $\nu_2 : \mathbb{P}^n \to \mathbb{P}^r$ is the map

$$[x_0 : \cdots : x_n] \mapsto [x_0^2 : x_0x_1 : \cdots : x_n^2].$$

If $z_0, \ldots, z_r$ are the homogeneous coordinates on $\mathbb{P}^r$, then the 2×2 minors of the $(n+1)\times(n+1)$ symmetric matrix

$$M = \begin{pmatrix} z_0 & z_1 & \cdots & z_n \\ z_1 & z_{n+1} & \cdots & z_{2n} \\ \vdots & \vdots & & \vdots \\ z_n & z_{2n} & \cdots & z_r \end{pmatrix}$$

vanish on $\nu_2(\mathbb{P}^n)$. As noted in Example 1.1, these minors generate the ideal of $\nu_2(\mathbb{P}^n)$ and the 3×3 minors generate the ideal of Sec $\nu_2(\mathbb{P}^n)$. The degree of this determinantal ideal is also classical – see Example 14.4.14 of [13] for a reference. This completes the proof for $2\Delta_n$.

Since $(2\Delta_n)_k$ is constructed from $2\Delta_n$ by removing a face of dimension k, $(2\Delta_n)_k$ has exactly $\binom{k+2}{2}$ fewer lattice points. We will use the convention that these lattice points correspond to the last $\binom{k+2}{2}$ monomials ordered lexicographically. The corresponding variables lie in the last $k+1$ rows and columns of M.

Let M_{k+1} denote the matrix M minus its last $k+1$ rows. The matrix M_{k+1} is the *partially symmetric* $(n-k)\times(n+1)$ matrix of Remark 2.5 (c) in [7]. Since the 2×2 minors of M_{k+1} vanish on $X_{(2\Delta_n)_k}$, it follows that if they generate a prime ideal defining a projective variety of dimension n, then they generate the ideal of $X_{(2\Delta_n)_k}$. Moreover, if the 2×2 minors of M_{k+1} generate the ideal of $X_{(2\Delta_n)_k}$, then the 3×3 minors of M_{k+1} vanish on its secant variety. Thus, if the 3×3 minors generate a prime ideal defining a projective variety of dimension dim Sec $X_{(2\Delta_n)_k}$, then the ideal they generate must be the ideal of the secant variety.

From Example 3.8 of [7] we know that the dimension and degree of the projective variety cut out by the ideal generated by the $t\times t$ minors of M_{k+1} is what we desire for $t = 1, \ldots, n-k$, and from Remark 2.5 of [7] we know that the ideal is prime. Moreover, $X_{(2\Delta_n)_{n-2}}$ is the rational normal scroll $S_{1,\ldots,1,2}$, and by [3], we know that its secant variety fills $\mathbb{P}^{2n}$. Therefore, we just need to show that the secant variety of $X_{(2\Delta_n)_k}$ also has dimension $2n$ for $0 \le k < n-2$. Note that $X_{(2\Delta_n)_k}$ is the image of $X_{(2\Delta_n)_j}$ via a linear projection for all $j < k$. The variety $X_{(2\Delta_n)_{-1}}$ is $\nu_2(\mathbb{P}^n)$. Therefore, applying Lemma 1.12 of [6] twice, tells us that dim Sec $X_{(2\Delta_n)_k} = 2n$ for all $k = 0, \ldots, n-2$.

Finally, $\Delta_\ell \times \Delta_{n-\ell}$ gives the Segre embedding of $\mathbb{P}^\ell \times \mathbb{P}^{n-\ell}$. In this case, the book [11] contains an explicit description of the ideal of the secant variety in Example 1.3.6 (2). The variety Sec $\mathbb{P}^\ell \times \mathbb{P}^{n-\ell}$ is defined by the 3×3 minors of the generic $(\ell+1)\times(n-\ell+1)$ matrix if $\ell+1, n-\ell+1 \ge 3$. The degree of this determinantal ideal was known to Giambelli and the formula appearing in Table 1.4 is given in Example 19.10 of [20]. If either $\ell+1$ or $n+1-\ell$ is strictly less than 3, then $\mathbb{P}^\ell \times \mathbb{P}^{n-\ell}$ embeds into $\mathbb{P}^{2n-1}$, and its secant variety fills the ambient space (see Example 4.5.21 in [11]), so that the degree is 1. This completes the proof of the theorem. □

The determinantal ideals arising from the polytopes $2\Delta_n$, $(2\Delta_n)_k$ and $\Delta_\ell \times \Delta_{n-\ell}$ in the above proof have many beautiful combinatorial and computational properties. Under the lexicographic term ordering (and other sufficiently nice term orderings), the $t\times t$ minors contain a Gröbner basis for the ideal they generate, and this ideal has a square-free initial ideal. Thus, Gröbner basis techniques can be used to show that the minors generate radical ideals, and Stanley–Reisner techniques can be used to compute degrees. This can be proved using [7] and the papers cited in [7].

We also note that for these polytopes P, the ideal of X_P is generated by quadrics (the 2×2 minors) and the ideal of Sec X_P is generated by cubics (the 3×3 minors). Are there other interesting polytopes with these properties? Such questions have been raised in [10, Problem 5.15] in the context of Jukes–Cantor binary models and in [27, Conjecture 3.8], which in the toric case asks whether the ideal of Sec X_{mP} is generated by cubics for $m \gg 0$.

2.2. Classification

The polytopes appearing in Theorem 2.2 fit naturally inside $2\Delta_n$. We will now prove that these are essentially all smooth polytopes with this property.

The proofs given here and in Section 4 require three lemmas about smooth polytopes. We begin with some definitions.

Definition 2.3. If σ is an edge of a lattice polytope P, then its *edge length* is $|\sigma \cap \mathbb{Z}^n| - 1$, or alternatively, the normalized length of σ. We say that P is a polytope of *edge length 1* if its edges all have length 1.

In terms of the corresponding toric variety X_P, P determines an ample divisor D_P and σ determines a curve C_σ. The intersection product $D_P \cdot C_\sigma$ is the edge length of σ.

Definition 2.4. A vertex v of a smooth polytope P is in *standard position* if v is the origin and its nearest lattice neighbors along the edges are the standard basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$.

It is easy to see that any vertex of a smooth polytope can be moved into standard position via an element of $AGL_n(\mathbb{Z})$.

We omit the elementary proof of our first lemma.

Lemma 2.5. *Let P be a smooth polygon with a vertex at the origin in standard position. If one of the edges at the origin has edge length ≥ 2, then P contains the point $e_1 + e_2$.*

Lemma 2.6. *If P is a smooth 3-dimensional polytope of edge length 1, then, up to $AGL_3(\mathbb{Z})$-equivalence, one of the following holds*

$P = \Delta_3$

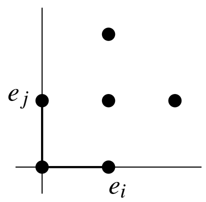
$P = \Delta_1 \times \Delta_2$

P contains Δ_1^3

P contains $P_{1,2,2} = \mathrm{Conv}(0, e_1, e_2, 2e_1 + e_3, 2e_2 + e_3, e_3)$.

Remark 2.7. Even though $P_{1,2,2}$ has two edges of length 2, it can be contained in a smooth 3-dimensional polytope of edge length 1.

Proof. Put a vertex of P in standard position and note that any facet of P containing the origin is one of the following three types: (A) the facet is a triangle Δ_2; (B) it is a square Δ_1^2; or (C) it contains the lattice points



Now consider the three facets that meet at the origin. One of four things can happen:

- Two facets are of type (A). Then $P = \Delta_3$.
- One facet is of type (A) and at least one is of type (B). Then $P = \Delta_1 \times \Delta_2$.
- Two facets are of type (C). Then P contains $P_{1,2,2}$.
- Two facets are of type (B) and the other is not of type (A). Then P contains Δ_1^3.

The proofs of the first three bullets are elementary and are omitted. For the fourth, we can assume that the two type (B) facets meet along the edge connecting 0 to e_1. The third facet meeting at e_1 is of type (B) or (C), which easily implies that P contains Δ_1^3. $\square$

Lemma 2.8. *Let P be a smooth polytope of dimension $n \geq 3$ and edge length 1. If every 3-dimensional face of P is $AGL_3(\mathbb{Z})$-equivalent to Δ_3 or $\Delta_1 \times \Delta_2$, then P is $AGL_n(\mathbb{Z})$-equivalent to $\Delta_\ell \times \Delta_{n-\ell}$, where $0 \leq \ell \leq n$.*

Proof. When we put P into standard position, the edge length 1 hypothesis implies that $0, e_1, \ldots, e_n$ are vertices of P. The vertex e_1 lies in n facets of P, $n-1$ of which lie in coordinate hyperplanes. For the remaining facet Γ, its supporting hyperplane can be defined by

$$x_1 + a_2 x_2 + \cdots + a_n x_n = 1, \quad a_i \in \mathbb{Q}.$$

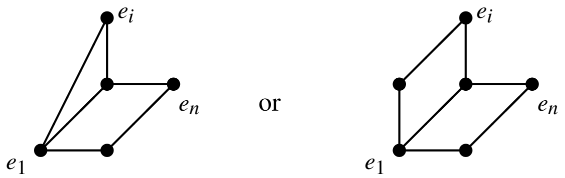
At e_1, the nearest neighbors along the edges consist of 0 and $n-1$ vertices lying in Γ. Our hypothesis on P implies that every 2-dimensional face of P is either Δ_2 or Δ_1^2. By considering the 2-dimensional face contained in $\mathrm{Cone}(e_1, e_i)$ for $i = 2, \ldots, n$, we see that the $n-1$ nearest neighbors in Γ are either e_i or $e_i + e_1$. We can renumber so that the nearest neighbors in Γ are

$$e_2, \ldots, e_\ell, e_{\ell+1} + e_1, \ldots, e_n + e_1$$

for some $1 \leq \ell \leq n$. It follows that the supporting hyperplane of Γ is

$$x_1 + \cdots + x_\ell = 1. \tag{1}$$

If $\ell = n$, then $P = \Delta_n$ follows easily. So now assume that $\ell < n$ and pick i between 2 and $n-1$. Consider the 3-dimensional face F_i of P lying in $\mathrm{Cone}(e_1, e_i, e_n)$. Depending on i, we get the following partial picture of F_i:



$2 \leq i \leq \ell$ $\qquad\qquad$ $\ell + 1 \leq i \leq n - 1$

This picture shows that F_i cannot be Δ_3. Hence by assumption it is $AGL_3(\mathbb{Z})$-equivalent to $\Delta_1 \times \Delta_2$. In terms of the picture, this means that the nearest neighbors of the vertex e_n of F_i, are 0, $e_1 + e_n$, and

$$\begin{array}{ll} e_i + e_n & \text{if } 2 \leq i \leq \ell \\ e_i & \text{if } \ell + 1 \leq i \leq n - 1. \end{array}$$

Thus the nearest neighbors along the edges of P at e_n are

$$0, e_1 + e_n, \ldots, e_\ell + e_n, e_{\ell+1}, \ldots, e_{n-1}.$$

This gives the supporting hyperplane

$$x_{\ell+1} + \cdots + x_n = 1. \tag{2}$$

The combination of (1) and (2) implies easily that $P = \Delta_\ell \times \Delta_{n-\ell}$. $\square$

We now prove a preliminary version of our classification of smooth subpolytopes of $2\Delta_n$.

Theorem 2.9. *Let P be a smooth polytope of dimension n in standard position at the origin. If $P \subset 2\Delta_n$, then P is one of*

$$\Delta_n, \quad 2\Delta_n, \quad (2\Delta_n)_k \ (0 \leq k \leq n-2), \quad \Delta_\ell \times \Delta_{n-\ell} \ (1 \leq \ell \leq n-1)$$

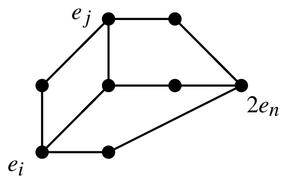
up to $AGL_n(\mathbb{Z})$-equivalence.

Remark 2.10. In Section 4, we will show that the standard position hypothesis in Theorem 2.9 is unnecessary.

Proof. The proof is trivial when $n = 1, 2$. So we will assume that $n \geq 3$. For each $1 \leq i \leq n$, the edge starting from 0 in direction e_i ends at either e_i or $2e_i$. Renumbering if necessary (which can be done by an $AGL_n(\mathbb{Z})$-equivalence), we can assume that

$$0, e_1, \ldots, e_{k+1}, 2e_{k+2}, \ldots, 2e_n \tag{3}$$

are vertices of P. The case $k = -1$ corresponds to $P = 2\Delta_n$. Now suppose that $0 \leq k \leq n-2$, so that $2e_n$ is a vertex of P. Given $1 \leq i < j \leq k+1$, we claim that $e_i + e_j \notin P$. To prove this, assume $e_i + e_j \in P$ and consider the 3-dimensional face F of P determined by the vertices $0, e_i, e_j, 2e_n$:



Note that $e_i + e_n, e_j + e_n \in P$ by Lemma 2.5. Since $P \subset 2\Delta_n$, we have $F \subset 2\Delta_3$, so that the above picture shows all lattice points of F. The convex hull of these points is not smooth, giving the desired contradiction. It follows that

$$P \subset 2\Delta_n \setminus \mathrm{Conv}(2e_1, \ldots, 2e_{k+1}),$$

which easily implies that $P \subset (2\Delta_n)_k$. For the opposite inclusion, note that since (3) consists of lattice points of P, Lemma 2.5 implies that P also contains the lattice point $e_i + e_j$ whenever $1 \leq i \leq k+1$ and $k+2 \leq j \leq n$. Since $(2\Delta_n)_k$ is the convex hull of these points together with (3), we obtain $(2\Delta_n)_k \subset P$.

It remains to consider the case when P contains none of $2e_1, \ldots, 2e_n$. This implies that P has edge length 1. Let F be a 3-dimensional face of P. Of the four possibilities for F listed in Lemma 2.6, note that F cannot contain a configuration $AGL_3(\mathbb{Z})$-equivalent to $P_{1,2,2}$, since $P_{1,2,2}$ contains two parallel segments with 3 lattice points, which is impossible in $2\Delta_n$. Similarly, $2\Delta_n$ cannot contain a configuration $AGL_3(\mathbb{Z})$-equivalent to Δ_1^3. This is because such a "cube" configuration would give an affine relation

$$(v_1 - v_0) + (v_2 - v_0) + (v_3 - v_0) = v_4 - v_0,$$

where v_0 and v_4 are opposite vertices of the cube and v_1, v_2, v_3 are vertices of the cube nearest to v_0. This gives the relation

$$v_1 + v_2 + v_3 = 2v_0 + v_4,$$

which cannot occur among distinct lattice points of $2\Delta_n$ (we omit the elementary argument). It follows from Lemma 2.6 that F is $AGL_3(\mathbb{Z})$-equivalent to Δ_3 or $\Delta_1 \times \Delta_2$. Since F is an arbitrary 3-dimensional face of P, Lemma 2.8 implies that P is $AGL_n(\mathbb{Z})$-equivalent to $\Delta_\ell \times \Delta_{n-\ell}$ for some $0 \leq \ell \leq n$. The proof of the theorem is now complete. □

3. The degree formula for the secant variety

One of the key ingredients in the proof of Theorem 1.4 is a formula relating the degree of $\mathrm{Sec}\, X$ to the Chern classes of the tangent bundle T_X and the line bundle $\mathcal{O}_X(1)$. We review this formula and then interpret it for toric surfaces and 3-folds.

3.1. The abstract join and the secant variety

Given $X \subset \mathbb{P}^r$ of dimension n, consider $\mathbb{P}^{2r+1}$ with coordinates $x_0, \ldots, x_r, y_0, \ldots, y_r$. The *abstract join* $J(X, X)$ of X with itself is the set of all points in $\mathbb{P}^{2r+1}$ of the form $[\lambda x : \mu y]$ with $[x], [y] \in X$ and $[\lambda : \mu] \in \mathbb{P}^1$. Then $J(X, X)$ has dimension $2n + 1$ and degree $(\deg X)^2$ (see [11]).

The linear projection

$$\phi : \mathbb{P}^{2r+1} \dashrightarrow \mathbb{P}^r$$

given by $\phi([x : y]) = [x - y]$ is defined away from the subspace of $\mathbb{P}^{2r+1}$ defined by the vanishing of $x_i - y_i$ for $i = 0, \ldots, r$. This induces a rational map $J(X, X) \dashrightarrow \operatorname{Sec} X$ with base locus the diagonal embedding of X in the abstract join. We have the following well-known result:

Theorem 3.1 (*Theorem 8.2.8 in [11]*). *If X is smooth, then*

$$\deg \operatorname{Sec} X \deg \phi = (\deg X)^2 - \sum_{i=0}^{n} \binom{2n+1}{i} \int_X c(T_X)^{-1} \cap c_1(\mathcal{L})^i,$$

where $\mathcal{L} = \mathcal{O}_X(1)$ and $\deg \phi = 0$ when $\dim \operatorname{Sec} X < 2n + 1$.

This is sometimes called the *double point formula* since a general linear projection $X \to \mathbb{P}^{2n}$ has $\frac{1}{2} \deg \operatorname{Sec} X \deg \phi$ double points (see Corollary 8.2.6 of [11], for example), and was first discovered by Severi. See [11,13] for more general double point formulas and further references.

The following proposition explains the geometric meaning of $\deg \phi$.

Proposition 3.2. *If $X \subset \mathbb{P}^r$ is a variety of dimension $n > 1$, then:*

(1) $\dim \operatorname{Sec} X = 2n + 1$ *if and only if a general point of* $\operatorname{Sec} X$ *lies on at most finitely many secant lines of X.*
(2) $\dim \operatorname{Sec} X = 2n + 1$ *implies that* $\deg \phi$ *is 2 times the number of secant lines of X through a general point of* $\operatorname{Sec} X$.

Proof. The first claim follows immediately from Lemma 2.2 of [12]. The second claim is probably well-known to experts. For completeness, we include a proof using the setup of [11].

If $\dim \operatorname{Sec} X = 2n+1$, then the rational map $\phi : J(X, X) \dashrightarrow \operatorname{Sec} X$ is generically finite. Observe that the trisecant variety of X (the closure of the union of secant lines meeting X in ≥ 3 points) has dimension $\leq 2n$ by Corollary 4.6.17 of [11]. Since $\dim \operatorname{Sec} X = 2n + 1$, it follows that at a general point $z \in \operatorname{Sec} X$, any secant line of X through z meets X in exactly two points, say $p \neq q$. Writing $z = \lambda p + \mu q$ gives distinct points $[\lambda p : -\mu q] \neq [\mu q : -\lambda p]$ in $\phi^{-1}(z)$. Thus the cardinality of $\phi^{-1}(z)$ is twice the number of secant lines through z, as claimed. □

Here is an immediate corollary of Proposition 3.2.

Corollary 3.3. *If $X \subset \mathbb{P}^r$ has dimension $n > 1$, then* $\deg \phi = 2$ *if and only if there is a unique secant line of X through a general point of* $\operatorname{Sec} X$. *Furthermore,* $\dim \operatorname{Sec} X = 2n + 1$ *when either of these conditions is satisfied.*

Theorem 3.1 gives a formula for $\deg \operatorname{Sec} X$ when $\deg \phi = 2$. This explains the $\frac{1}{2}$ appearing in Theorem 1.4 and Corollaries 1.6 and 1.7.

Here is an example of Theorem 3.1 and Corollary 3.3 that will be useful in the proof of Theorem 1.4.

Example 3.4. Consider the rational normal scroll $X = S_{d_1,\ldots,d_n}$, where $d_i \geq 1$ and $d = \sum_{i=1}^{n} d_i$. Since X is a projective bundle over $\mathbb{P}^1$, its Chow ring is well-known, making it easy to compute the formula in Theorem 3.1. This computation appears in [24], with the result

$$\deg \operatorname{Sec} X \deg \phi = d^2 - (2n + 1)d + n(n + 1).$$

Catalano-Johnson proved in [2] that a general point of $\operatorname{Sec} X$ lies on a unique secant line of X when $d \geq n + 2$. Thus $\deg \phi = 2$, so that

$$\deg \operatorname{Sec} X = \frac{1}{2}\Big(d^2 - (2n + 1)d + n(n + 1)\Big), \quad d \geq n + 2.$$

If $d = n, n+1$, this formula gives zero, so that $\dim \operatorname{Sec} X < 2n+1$ by Theorem 3.1. The cases $d = n, n+1$ correspond to the polytopes $\Delta_1 \times \Delta_{n-1}$, and $(2\Delta_n)_{n-2}$, respectively. Since the secant variety of a rational normal scroll always has the expected dimension, this explains why these polytopes don't appear in the statement of Corollary 1.5.

3.2. Dimensions 2 and 3

In low dimensions, the degree formula of Theorem 3.1 can be expressed quite succinctly. The purpose of this section is to prove the following two theorems.

Theorem 3.5. *If P is a smooth lattice polygon in $\mathbb{R}^2$, then*

$$\deg \operatorname{Sec} X_P \deg \phi = d^2 - 10d + 5B + 2V - 12,$$

where d is the area of P, B is the number of lattice points on the boundary of P, and V is the number of vertices of P.

Theorem 3.6. *If P is a smooth 3-dimensional lattice polytope in $\mathbb{R}^3$, then*

$$\deg \operatorname{Sec} X_P \deg \phi = d^2 - 21d + c_1^3 + 8V + 14E - 84I - 132,$$

where d is the volume of P, E is the number of lattice points on the edges of P, V is the number of vertices of P, and I is the number of interior lattice points. Also, $c_1 = c_1(T_{X_P})$.

Both results follow from Theorem 3.1 via applications of Riemann–Roch and the theory of Ehrhart polynomials. We begin with some useful facts about the total Chern class $c(T_X)$ and Todd class $\operatorname{td}(X)$ from pp. 109–112 in [14]. Let $c_i = c_i(T_X)$ and $H = c_1(\mathcal{L})$, where $\mathcal{L}$ is the line bundle coming from P. Then:

$$\begin{aligned} \operatorname{td}(X) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots \\ c(T_X)^{-1} &= 1 - c_1 + (c_1^2 - c_2) + (2c_1c_2 - c_1^3 - c_3) + \cdots. \end{aligned} \tag{4}$$

Furthermore, if $X = X_P$ is the smooth toric variety of the polytope P and $V(F)$ is the orbit closure corresponding to the face F of P, then:

$$\begin{aligned} c(T_X) &= \prod_{\dim F = n-1} (1 + [V(F)]) \\ c_i &= \sum_{\dim F = n-i} [V(F)] \\ H^i \cap [V(F)] &= \operatorname{Vol}_i(F), \end{aligned} \tag{5}$$

where $\operatorname{Vol}_i(F)$ is the normalized volume of the i-dimensional face F.

Proof of Theorem 3.5. Since $\dim X = 2$ and $\deg X = H^2 = d$, the right-hand side of the formula in Theorem 3.1 becomes

$$d^2 - (c_1^2 - c_2) + 5Hc_1 - 10d.$$

Since $c_1^2 + c_2 = 12$ by Noether's formula, this simplifies to

$$d^2 - 12 + 2c_2 + 5Hc_1 - 10d.$$

By (5), c_2 is the sum of the torus-fixed points corresponding to the vertices of P. Thus $c_2 = V$, the number of vertices of P. Furthermore, (5) also implies that Hc_1 is the perimeter of P, which is the number B of lattice points on the boundary of P. The desired formula follows. □

Proof of Theorem 3.6. The formula of Theorem 3.1 reduces to

$$d^2 - (2c_1c_2 - c_1^3 - c_3) - 7H(c_1^2 - c_2) + 21H^2c_1 - 35d,$$

where d again denotes $\deg X$.

Let S denote the surface area of P, $\mathcal{E}$ denote the perimeter (i.e., the sum of the lengths of the edges of P), and V denotes the number of vertices of P. Using (5) as before, one sees that $c_3 = V$, $Hc_2 = \mathcal{E}$, and $H^2c_1 = S$. Also, using (4) together with the fact that $\operatorname{td}_3(X) = [x]$ for any $x \in X$, we obtain $c_1c_2 = 24$. Thus, we have

$$d^2 - 48 + c_1^3 + V - 7Hc_1^2 + 7\mathcal{E} + 21S - 35d.$$

However, the Todd class formula from (4) and Riemann–Roch tell us that the number of lattice points in P is

$$\ell(P) = 1 + \frac{1}{12}(Hc_1^2 + \mathcal{E}) + \frac{1}{4}S + \frac{1}{6}d,$$

as explained in [14, Sec. 5.3]. Solving for Hc_1^2 and substituting into the above expression, we obtain

$$d^2 - 21d + c_1^3 + \underbrace{V + 14\mathcal{E}}_{\alpha} + \underbrace{42S - 84\ell(P) + 36}_{\beta}.$$

It remains to understand the quantities α and β.

Letting E denote the number of lattice points on the edges of P, we can rewrite α as $8V + 14E$ since every vertex of P lies on exactly three edges of P by smoothness.

Next let B (resp. I) denote the number of boundary (resp. interior) lattice points of P, so that $\ell(P) = B + I$. By Ehrhart duality, we have

$$I = (-1)^3\left(1 - \frac{1}{12}(Hc_1^2 + \mathcal{E}) + \frac{1}{4}S - \frac{1}{6}d\right),$$

which allows us to rewrite β as $-84I - 132$. The desired expression follows immediately. $\square$

4. Counting secant lines

To complete the proofs of our main results, we need to study the secant variety of a toric variety coming from the last row of Table 1.4.

4.1. A uniqueness theorem

Here is our result.

Theorem 4.1. *If a smooth n-dimensional polytope P is not $AGL_n(\mathbb{Z})$-equivalent to any of the polytopes*

$$\Delta_n, \quad 2\Delta_n, \quad (2\Delta_n)_k \quad (0 \le k \le n-2), \ \Delta_\ell \times \Delta_{n-\ell} \ (1 \le \ell \le n-1),$$

then a unique secant line of X_P goes through a general point of $\operatorname{Sec} X_P$.

Before beginning the proof of Theorem 4.1, we need some lemmas.

Lemma 4.2. *Let $X \subset \mathbb{P}^r$ be a variety of dimension $n > 1$ and fix a linear projection $\mathbb{P}^r \dashrightarrow \mathbb{P}^s$ such that X is not contained in the center of the projection. Let $Y \subset \mathbb{P}^s$ be the closure of the image of X, so that we have a projection $\pi : X \dashrightarrow Y$. If π is birational and a general point of* $\operatorname{Sec} Y$ *lies on a unique secant line of Y, then* $\dim \operatorname{Sec} X = 2n+1$ *and a general point of* $\operatorname{Sec} X$ *lies on a unique secant line of X.*

Proof. By Proposition 3.2, our hypothesis on Y implies that $\operatorname{Sec} Y$ has dimension $2n+1$, and then $\dim \operatorname{Sec} X = 2n+1$ since $\pi : X \dashrightarrow Y$ induces a dominating map $\operatorname{Sec} X \dashrightarrow \operatorname{Sec} Y$.

Now suppose that a general point $z \in \operatorname{Sec} X$ lies on the secant lines $\overline{pq}$ and $\overline{p'q'}$ of X. These map to secant lines of Y through $\pi(z)$, which coincide by hypothesis. Arguing as in the proof of Proposition 3.2, we can assume that this secant line meets Y at exactly two points. Switching p' and q' if necessary, we get $\pi(p) = \pi(p')$ and $\pi(q) = \pi(q')$. Since π is generically 1-to-1 on X, we conclude that $\overline{pq} = \overline{p'q'}$. Thus a general point of $\operatorname{Sec} X$ lies on a unique secant line of X. $\square$

We will prove Theorem 4.1 by applying Lemma 4.2 to projections constructed from carefully chosen subsets $A = \{u_0, \ldots, u_s\} \subset P \cap \mathbb{Z}^n$. The characters χ^{u_i} give a rational map $X \dashrightarrow \mathbb{P}^s$ defined by $x \mapsto [\chi^{u_0}(x) : \cdots : \chi^{u_s}(x)]$, where X is the abstract toric variety of the normal fan of P. The closure of the image in $\mathbb{P}^s$ is denoted X_A. Note that $X_A = X_P$ when $A = P \cap \mathbb{Z}^n$.

We can view X_A as a projection of X_P as follows. By definition, X_P is the closure of the image of the rational map $X \dashrightarrow \mathbb{P}^r$, $r = |P \cap \mathbb{Z}^n| - 1$. Labeling the coordinates of $\mathbb{P}^r$ using the lattice points of $P \cap \mathbb{Z}^n$, we obtain a projection

$$\pi : \mathbb{P}^r \dashrightarrow \mathbb{P}^s \tag{6}$$

by projecting onto the linear subspace defined by the coordinates corresponding to $A \subset P \cap \mathbb{Z}^n$. This induces the projection $\pi : X_P \dashrightarrow X_A$.

Some of the subsets A that we will use come from the toric interpretation of the rational normal scrolls from the introduction. Let $d_1, \ldots, d_n$ be positive integers and label the vertices of the unit simplex $\Delta_{n-1} \subset \mathbb{R}^{n-1}$ as $v_1 = e_1, \ldots, v_{n-1} = e_{n-1}, v_n = 0$. Define $A_{d_1,\ldots,d_n} \subset \mathbb{Z}^{n-1} \times \mathbb{Z}$ by

$$A_{d_1,\ldots,d_n} = \bigcup_{i=1}^{n} \{v_i + a_i e_n \mid a_i \in \mathbb{Z}, 0 \le a_i \le d_i\},$$

and let $P_{d_1,\ldots,d_n}$ be the convex hull of $A_{d_1,\ldots,d_n}$ in $\mathbb{R}^{n-1} \times \mathbb{R}$. It is straightforward to verify that $P_{d_1,\ldots,d_n}$ is a smooth polytope with lattice points

$$A_{d_1,\ldots,d_n} = P_{d_1,\ldots,d_n} \cap (\mathbb{Z}^{n-1} \times \mathbb{Z}).$$

One can easily show that the toric variety $X_{P_{d_1,\ldots,d_n}}$ is the rational normal scroll $S_{d_1,\ldots,d_n}$ defined in Example 1.3. The result of [2] mentioned in Example 3.4 implies the following lemma.

Lemma 4.3. *If A is one of the two sets*

$$\begin{aligned} &A_{1,\ldots,1,1,3} \subset \mathbb{Z}^n \quad (\textit{there are } n-1\ 1's, n \ge 1)\\ &A_{1,\ldots,1,2,2} \subset \mathbb{Z}^n \quad (\textit{there are } n-2\ 1's, n \ge 2),\end{aligned}$$

then a general point of Sec X_A *lies on a unique secant line of* X_A.

For polytopes of edge length 1 and dimension > 3, we will use the following configurations of lattice points.

Lemma 4.4. *Let $B \subset \mathbb{Z}^3 \times \{0\} \subset \mathbb{Z}^3 \times \mathbb{Z}^{n-3}$ be a set of 8 lattice points such that B affinely generates $\mathbb{Z}^3 \times \{0\}$ and a general point of* Sec X_B *lies on a unique secant line of X_B. For each $i = 4, \ldots, n$, pick $u_i \in \{e_1, e_2\}$ and define*

$$A = B \cup \{e_i, e_i + u_i \mid i = 4, \ldots, n\} \subset \mathbb{Z}^3 \times \mathbb{Z}^{n-3}.$$

Note that A has $2n + 2$ points. If $X_A \subset \mathbb{P}^{2n+1}$ is the corresponding toric variety, then a general point of Sec $X_A = \mathbb{P}^{2n+1}$ *lies on a unique secant line of X_A.*

Proof. Let $t_1, \ldots, t_n$ be torus variables corresponding to $e_1, \ldots, e_n$, and let $m_1, \ldots, m_8$ be the lattice points of B. A point $\mathbf{t} = (t_1, \ldots, t_n)$ on the torus $(\mathbb{C}^*)^n$ maps to

$$\psi(\mathbf{t}) = (\mathbf{t}^{m_1}, \ldots, \mathbf{t}^{m_8}, t_4, t_4\mathbf{t}^{u_4}, \ldots, t_n, t_n\mathbf{t}^{u_n}) \in \mathbb{P}^{2n+1}.$$

We claim that the map $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \times \mathbb{C}^* \dashrightarrow \operatorname{Sec} X_A$ defined by

$$(\mathbf{t}, \mathbf{s}, \gamma) \mapsto \gamma\psi(\mathbf{t}) + (1-\gamma)\psi(\mathbf{s})$$

is generically 2-to-1. To prove this, suppose that

$$\gamma\psi(\mathbf{t}) + (1-\gamma)\psi(\mathbf{s}) = \gamma'\psi(\mathbf{t}') + (1-\gamma')\psi(\mathbf{s}'). \tag{7}$$

We need to show that for $\mathbf{t}, \mathbf{s}, \gamma$ generic, (7) implies that

$$(\mathbf{t}', \mathbf{s}', \gamma') = (\mathbf{t}, \mathbf{s}, \gamma) \text{ or } (\mathbf{s}, \mathbf{t}, 1-\gamma).$$

By projecting onto the first 8 coordinates and using our hypothesis on B and being careful to avoid trisecants and tangents, we obtain

$$(t_1', t_2', t_3', s_1', s_2', s_3', \gamma') = (t_1, t_2, t_3, s_1, s_2, s_3, \gamma) \tag{8}$$

by switching $\mathbf{t}$ and $\mathbf{s}$ if necessary.

Now fix i between 4 and n. Since $u_i \in \{e_1, e_2\}$, (8) implies that $\mathbf{t}^{u_i} = \mathbf{t}'^{u_i}$ and $\mathbf{s}^{u_i} = \mathbf{s}'^{u_i}$. Since we also know that $\gamma = \gamma'$, comparing the coordinates of (7) corresponding to $e_i, e_i + u_i \in B$ gives the equations

$$\begin{aligned}\gamma t_i + (1-\gamma)s_i &= \gamma t_i' + (1-\gamma)s_i'\\ \gamma t_i\mathbf{t}^{u_i} + (1-\gamma)s_i\mathbf{s}^{u_i} &= \gamma t_i'\mathbf{t}^{u_i} + (1-\gamma)s_i'\mathbf{s}^{u_i},\end{aligned}$$

which can be rewritten as

$$\gamma(t_i - t_i') + (1-\gamma)(s_i - s_i') = 0$$
$$\gamma \mathbf{t}^{u_i}(t_i - t_i') + (1-\gamma)\mathbf{s}^{u_i}(s_i - s_i') = 0.$$

The coefficient matrix of this 2×2 system of homogeneous equations has determinant

$$\gamma(1-\gamma)(\mathbf{s}^{u_i} - \mathbf{t}^{u_i}),$$

which is nonzero for generic $(\mathbf{t}, \mathbf{s}, \gamma)$. Thus $t_i' = t_i$ and $s_i' = s_i$ for $i = 4, \ldots, n$. It follows that the map $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \times \mathbb{C}^* \dashrightarrow \operatorname{Sec} X_A$ defined above is generically 2-to-1. $\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. The case of dimension 1 is trivial since $P = d\Delta_1$ gives a rational normal curve of degree d embedded in $\mathbb{P}^d$. It is well-known (see [2] or Proposition 8.2.12 of [11]) that a unique secant line passes through a general point of $\operatorname{Sec} X_P$ when $d \geq 3$.

Now assume that $\dim P \geq 2$. We will consider five cases, depending on the maximum edge length and dimension of P.

Case 1. P has an edge E of length ≥ 3. Put a vertex of E in standard position so that E contains e_n. Now fix i between 1 and $n-1$ and consider the 2-dimensional face F_i of P lying in $\operatorname{Cone}(e_i, e_n)$. Then F_i is a smooth polygon in standard position containing the points

$$0, e_i, e_n, 2e_n, 3e_n.$$

By Lemma 2.5, we conclude that $e_i + e_n \in F$. Hence P contains the $2n+2$ points

$$A = \{0, e_i, e_i + e_n, e_n, 2e_n, 3e_n \mid i = 1, \ldots, n-1\}.$$

This is the set $A = A_{1,\ldots,1,1,3}$ defined earlier. By Lemma 4.3, a general point of $\operatorname{Sec} X_A$ lies in a unique secant line of X_A.

The embeddings of X_P and X_A given by $P \cap \mathbb{Z}^n$ and A respectively share the points $0, e_1, \ldots, e_n$. It follows easily that the linear projection $X_P \dashrightarrow X_A$ given by projecting onto the coordinates corresponding to points of A is the identity on the tori of X_P and X_A and hence is birational. By Lemma 4.2, we conclude that a general point of $\operatorname{Sec} X_P$ lies in a unique secant line of X_P.

Case 2. P has an edge E of length 2 but no edges of length ≥ 3. As in Case 1, put a vertex of E in standard position so that E contains e_n. Now fix i between 1 and $n-1$ and consider the 2-dimensional face F_i of P lying in $\operatorname{Cone}(e_i, e_n)$. Using Lemma 2.5 as in Case 1 shows that F_i contains the points

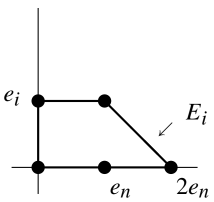
$$0, e_i, e_i + e_n, e_n, 2e_n.$$

If for some i the face F_i also contains $e_i + 2e_n$, then we can relabel so that $i = n-1$. Hence P contains the points

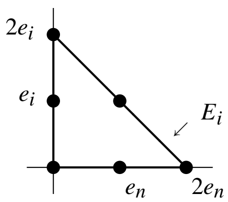
$$A = \{0, e_i, e_i + e_n, e_{n-1}, e_{n-1} + e_n, e_{n-1} + 2e_n, e_n, 2e_n, \mid i = 1, \ldots, n-2\}.$$

This is the set $A = A_{1,\ldots,1,2,2}$ defined earlier in the section. Then using Lemmas 4.2 and 4.3 as in Case 1 implies that a general point of $\operatorname{Sec} X_P$ lies in a unique secant line of X_P.

It remains to show that $e_i + 2e_n \notin F_i$ for all $i = 1, \ldots, n-1$ cannot occur. If this were to happen, it is easy to see that F_i is either



or



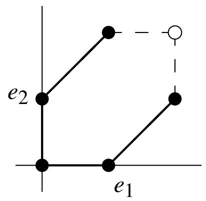
Since P is smooth, n facets of P meet at the vertex $2e_n$, $n-1$ of which lie in coordinate hyperplanes. The remaining facet has a normal vector ν perpendicular to the edge E_i of F_i indicated in the above picture. Thus $\nu \cdot (-e_i + e_n) = 0$

for $i = 1, \ldots, n-1$, which easily implies that the supporting hyperplane of this facet is defined by $x_1 + \cdots + x_n = 2$. This hyperplane and the coordinate hyperplanes bound $2\Delta_n$, and it follows that $P \subset 2\Delta_n$. Since P is in standard position, Theorem 2.9 implies that P is $AGL_n(\mathbb{Z})$-equivalent to one of the polytopes listed in the statement of the theorem, a contradiction.

Case 3. P has only edges of length 1 and dimension 2. Put a vertex of P in standard position and note that e_1 and e_2 are also vertices of P. Since P is smooth, the vertex e_1 of P must lie on an edge containing $ae_1 + e_2$, $a \geq 0$. If $a = 0$ or 1, then one easily sees that P is contained in $2\Delta_2$. On the other hand, if $a \geq 3$, then P contains

$$A = \{0, e_1, e_2, e_1 + e_2, 2e_1 + e_2, 3e_1 + e_2\},$$

which equals $A_{1,3}$ up to $AGL_2(\mathbb{Z})$-equivalence. Using Lemmas 4.2 and 4.3 as usual, we conclude that a general point of $\operatorname{Sec} X_P$ lies in a unique secant line of X_P. Finally, if $a = 2$, then P is not contained in $2\Delta_2$. Applying a similar analysis to the edges emanating from the vertex e_2, one sees that P either contains $A_{1,3}$ up to $AGL_2(\mathbb{Z})$-equivalence or has the four solid edges pictured as follows:



The only way to complete this to a smooth polygon P is to add the vertex $2e_1 + 2e_2$ indicated in the figure. Applying Theorem 3.5 gives

$$\deg \operatorname{Sec} X_P \deg \phi = 6^2 - 10 \cdot 6 + 5 \cdot 6 + 2 \cdot 6 - 12 = 6.$$

Therefore, since $\deg \phi$ must be even, it is either 2 or 6. If it were 6, then $\operatorname{Sec} X_P$ must be a linear space. Since $\dim \operatorname{Sec} X_P \leq 5$, the variety $\operatorname{Sec} X_P$ cannot fill $\mathbb{P}^6$. If $\operatorname{Sec} X_P$ were contained in a nontrivial linear space, then X_P would also be contained in this space which is a contradiction because X_P is easily seen to be nondegenerate. Therefore, we conclude that $\deg \operatorname{Sec} X_P = 3$, and that a general point of $\operatorname{Sec} X_P$ lies on a unique secant line of X_P by Corollary 3.3.

Case 4. P has only edges of length 1 and dimension 3. By Lemma 2.6, P is either Δ_3 or $\Delta_1 \times \Delta_2$, or else P contains $P_{1,2,2}$ or Δ_1^3. When P contains $P_{1,2,2}$, we are done by the usual combination of Lemmas 4.2 and 4.3. When P contains Δ_1^3, note that Theorem 3.6, when applied to $Q = \Delta_1^3$, gives

$$\deg \operatorname{Sec} X_Q \deg \phi = 6^2 - 21 \cdot 6 + 48 + 8 \cdot 8 + 14 \cdot 8 - 84 \cdot 0 - 132 = 2.$$

This implies $\deg \phi = 2$ since $\deg \phi$ is even. Thus a general point of $\operatorname{Sec} X_Q$ lies on a unique secant line of X_Q. Then X_P has the same property by Lemma 4.2.

Case 5. P has only edges of length 1 and dimension > 3. As usual we put a vertex of P into standard position. We will study the 3-dimensional faces of P.

First suppose that P has a 3-dimensional face F that contains Δ_1^3 or $P_{1,2,2}$. We can arrange for this face to lie in $\operatorname{Cone}(e_1, e_2, e_3)$ in such a way that $e_1 + e_2 \in P$. Now fix i between 4 and n and consider the 3-dimensional face F_i of P lying in $\operatorname{Cone}(e_1, e_2, e_i)$. What are the 2-dimensional faces of F_i containing e_i? If both were triangles Δ_2, then the argument of the first bullet in the proof of Lemma 2.6 would imply that $F_i = \Delta_3$, which contradicts $e_1 + e_2 \in F_i$. It follows that one of these faces must contain another lattice point. This shows that there is $u_i \in \{e_1, e_2\}$ such that $e_i + u_i \in F_i \subset P$. If we let B denote the lattice points of Δ_1^3 or $P_{1,2,2}$ contained in our original face F, then P contains the set

$$A = B \cup \{e_i, e_i + u_i \mid i = 4, \ldots, n\}.$$

Furthermore, the proof of Case 4 shows that a general point of $\operatorname{Sec} X_B$ lies on a unique secant line of X_B. By Lemmas 4.2 and 4.4, we conclude that a general point of $\operatorname{Sec} X_P$ lies on a unique secant line of X_P.

By Lemma 2.6, it remains to consider what happens when every 3-dimensional face of P is Δ_3 or $\Delta_1 \times \Delta_2$. Here, Lemma 2.8 implies that $P = \Delta_\ell \times \Delta_{n-\ell}$, $0 \le \ell \le n$, which cannot occur by hypothesis. This completes the proof of the theorem. □

Remark 4.5. In Case 1 above it is possible to see that $\operatorname{Sec} X_A$ has the expected dimension via results in [26]. A lexicographic triangulation of A in which the vertices precede the other lattice points of $\operatorname{conv}(A)$ will include the disjoint simplices at either end of an edge of length ≥ 3. This shows that A satisfies the condition of Theorem 5.4 of [26]. (We thank S. Sullivant for pointing out that a lexicographic triangulation works here.)

4.2. Proofs of the main results

We now prove the four theorems stated in the introduction.

Proof of Theorem 1.4. Theorem 2.2 proves the dimensions and degrees in the first four rows of Table 1.4. Since all of these polytopes satisfy $\dim \operatorname{Sec} X_P < 2n + 1$, part (1) of Theorem 1.4 follows from Proposition 3.2. Also, part (2) follows from Theorem 4.1. Finally, for the last row of Table 1.4, Theorem 4.1 and Corollary 3.3 imply that $\dim \operatorname{Sec} X_P = 2n + 1$ and $\deg \phi = 2$. Theorem 3.1 gives the desired formula for $\deg \operatorname{Sec} X_P$. □

Proof of Corollary 1.5. We always get the expected dimension in the last row of Table 1.4. Since the first four rows have $\dim \operatorname{Sec} X_P < 2n + 1$, the only way to get the expected dimension is when $\dim \operatorname{Sec} X_P = r$, where $r + 1$ is the number of lattice points of P. The lattice points of these polytopes are easy to count, and the result follows. □

Proof of Corollaries 1.6 and 1.7. These follow from Theorem 1.4 and Corollary 3.3, together with the formulas of Theorems 3.5 and 3.6. □

4.3. Subpolytopes of $2\Delta_n$

We can now complete the classification of smooth subpolytopes of $2\Delta_n$ begun in Theorem 2.9.

Theorem 4.6. *Let $P \subset 2\Delta_n$ be a smooth polytope of dimension n. Then P is $AGL_n(\mathbb{Z})$-equivalent to one of*

$$\Delta_n, \quad 2\Delta_n, \quad (2\Delta_n)_k \ (0 \le k \le n-2), \quad \Delta_\ell \times \Delta_{n-\ell} \ (1 \le \ell \le n-1).$$

Proof. $P \subset 2\Delta_n$ gives a dominating map $\operatorname{Sec} \nu_2(\mathbb{P}^n) \dashrightarrow \operatorname{Sec} X_P$, so that $\dim \operatorname{Sec} X_P \le \dim \operatorname{Sec} \nu_2(\mathbb{P}^n) = 2n < 2n + 1$. This excludes the last row of Table 1.4, so that P must be $AGL_n(\mathbb{Z})$-equivalent to a polytope in one of the first four rows, as claimed. □

Remark 4.7. Given a smooth lattice polytope $P \subset \mathbb{R}^n$ of dimension n, it is easy to determine if P is equivalent to a subpolytope of $2\Delta_n$. Let d be the maximum number of length 2 edges incident at any vertex of P and let v be any vertex that witnesses this maximum. Then let Q be the image of P under any element of $AGL_n(\mathbb{Z})$ that places v at the origin in standard position.

We claim that P is equivalent to a subpolytope of $2\Delta_n$ if and only if Q is contained in $2\Delta_n$. For the nontrivial part of the claim, note that if P is equivalent to a subpolytope of $2\Delta_n$, then it must be equivalent to either $\Delta_\ell \times \Delta_{n-\ell}$ with $1 \le \ell \le n-1$ or $(2\Delta_n)_k$ with $-1 \le k \le n-1$ by Theorem 4.6. The automorphism group of $\Delta_\ell \times \Delta_{n-\ell}$ acts transitively on its vertices, and the automorphism group of $(2\Delta_n)_k$ acts transitively on vertices with a maximum number of edges of length 2. Therefore, by symmetry, we can check to see if P is equivalent to a subpolytope of $2\Delta_n$ at any vertex v as specified above.

4.4. Segre–Veronese varieties

The polytope $d_1\Delta_{n_1} \times \cdots \times d_k\Delta_{n_k}$, $d_i \ge 1$, gives an embedding of $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ using $\mathcal{O}_X(d_1, \ldots, d_k)$. The image Y of this embedding is a Segre–Veronese variety of dimension $n = \sum_{i=1}^{k} n_i$. We now compute the degree of $\operatorname{Sec} Y$ using Theorem 1.4.

Theorem 4.8. *Let Y be the Segre–Veronese of $d_1\Delta_{n_1}\times\cdots\times d_k\Delta_{n_k}$, $d_i\geq 1$. If $\sum_{i=1}^k d_i\geq 3$, then* $\dim \operatorname{Sec} Y = 2n+1$ *and*

$$\deg \operatorname{Sec} Y = \frac{1}{2}\Bigg(((n_1,\ldots,n_k)!d_1^{n_1}\cdots d_k^{n_k})^2 - \sum_{\ell=0}^{n}\binom{2n+1}{\ell}(-1)^{n-\ell}\sum_{\sum j_i=n-\ell}(n_1-j_1,\ldots,n_k-j_k)!\prod_{i=1}^{k}\binom{n_i+j_i}{j_i}d_i^{n_i-j_i}\Bigg).$$

where $(m_1,\ldots,m_k)! = \frac{(m_1+\cdots+m_k)!}{m_1!\cdots m_k!}$ is the usual multinomial coefficient.

Proof. One easily sees that $d_1\Delta_{n_1}\times\cdots\times d_k\Delta_{n_k}$ cannot lie in $2\Delta_n$ when $\sum_{i=1}^k d_i\geq 3$. It follows that $\dim\operatorname{Sec} Y = 2n+1$ and $\deg\operatorname{Sec} Y$ is given by the formula of Theorem 1.4. It is well-known that $\deg Y = (n_1,\ldots,n_k)!\,d_1^{n_1}\cdots d_k^{n_k}$.

To evaluate the remaining terms of the formula, we first compute $c(T_Y)^{-1}$. Let $H_i = c_1(\mathcal{O}_X(0,\ldots,1,\ldots,0))$, where the 1 appears in the ith position. Since $c(T_Y)=c(T_X)$, we can use (5) to see that $c(T_Y)=\prod_{i=1}^k(1+H_i)^{n_i+1}$. We compute $c(T_Y)^{-1}$ by inverting each factor and multiplying out the result.

For each i, we have the expansion

$$(1+H_i)^{-(n_i+1)} = \sum_{j=0}^{n_i}(-1)^j\binom{n_i+j}{j}H_i^j \tag{9}$$

since $H_i^{n_i+1}=0$. Then we can expand the product $\prod_{i=1}^k(1+H_i)^{-(n_i+1)}$ and take the degree ℓ piece to obtain

$$c_\ell = (-1)^\ell\sum_{\sum j_i=\ell}\prod\binom{n_i+j_i}{j_i}H_i^{j_i}.$$

The embedding of X is given by $H = d_1H_1+\cdots+d_kH_k$. This makes it easy to complete the computation. For each $\ell = 0,\ldots,n$, we need to compute

$$c_{n-\ell}\cdot(d_1H_1+\cdots+d_kH_k)^\ell = (-1)^{n-\ell}\sum_{\sum j_i=n-\ell}\prod\binom{n_i+j_i}{j_i}H_i^{j_i}\cdot(d_1H_1+\cdots+d_kH_k)^\ell.$$

Since the coefficient of $H_1^{n_1-j_1}\cdots H_k^{n_k-j_k}$ in $(d_1H_1+\cdots+d_kH_k)^\ell$ is $(n_1-j_1,\ldots,n_k-j_k)!\prod_{i=1}^k d_i^{n_i-j_i}$, the result follows. $\square$

Here are two easy corollaries of Theorem 4.8.

Corollary 4.9. *Let $n\geq 2$. If $d\geq 3$ and Y is the d-uple Veronese variety of $\mathbb{P}^n$, then*

$$\deg\operatorname{Sec} Y = \frac{1}{2}\left(d^{2n}-\sum_{j=0}^{n}(-1)^{n-j}d^j\binom{2n+1}{j}\binom{2n-j}{n-j}\right).$$

Corollary 4.10. *If $n\geq 3$ and Y is the n-fold Segre variety $\mathbb{P}^1\times\cdots\times\mathbb{P}^1$, then*

$$\deg\operatorname{Sec} Y = \frac{1}{2}\left((n!)^2-\sum_{j=0}^{n}\binom{2n+1}{j}\binom{n}{n-j}j!(-2)^{n-j}\right).$$

The reader should consult [4] for further results on the secant varieties of Segre–Veronese varieties.

5. Subsets of lattice points

Let P be a smooth polytope of dimension n in $\mathbb{R}^n$. Using all lattice points of P gives the projective variety $X_P \subset \mathbb{P}^r$, $r = |P \cap \mathbb{Z}^n| - 1$, in the usual way. As in the discussion leading up to (6) in Section 4.1, a subset $A \subset P \cap \mathbb{Z}^n$ gives the projective toric variety $X_A \subset \mathbb{P}^s$, $s = |A| - 1$.

Here is our main result concerning the dimension and degree of the secant variety $\operatorname{Sec} X_A$.

Theorem 5.1. *Let $A \subset P \cap \mathbb{Z}^n$, where P is a smooth polytope of dimension n. If A contains the vertices of P and their nearest neighbors along edges of P, then* $\dim \operatorname{Sec} X_A = \dim \operatorname{Sec} X_P$ *and* $\deg \operatorname{Sec} X_A$ *divides* $\deg \operatorname{Sec} X_P$.

When A satisfies the hypothesis of Theorem 5.1, it follows that $\dim \operatorname{Sec} X_A$ is given in Table 1.4. Furthermore, since $\deg \operatorname{Sec} X_A$ divides $\deg \operatorname{Sec} X_P$ and the latter is given in Table 1.4, we get an explicit bound for $\deg \operatorname{Sec} X_A$.

Proof. By (6) of Section 4.1, $A \subset P \cap \mathbb{Z}^n$ gives a projection

$$\pi : \mathbb{P}^r \dashrightarrow \mathbb{P}^s,$$

where $r = |P \cap \mathbb{Z}^n| - 1, s = |A| - 1$. By our hypothesis on A, it is straightforward to show that π induces an isomorphism $\pi : X_P \to X_A$ (see, for example, the proof of the lemma on p. 69 of [14]). This in turn induces a projection

$$\pi : \operatorname{Sec} X_P \dashrightarrow \operatorname{Sec} X_A \tag{10}$$

since projections take lines not meeting the center to lines. Let $\Lambda \subset \mathbb{P}^r$ be the center of the projection. If Λ does not meet $\operatorname{Sec} X_P$, then (10) is finite and surjective, and the theorem follows easily. Hence it remains to show that $\Lambda \cap \operatorname{Sec} X_P = \emptyset$.

Suppose by contradiction that there is $z \in \Lambda \cap \operatorname{Sec} X_P$. Since A contains all vertices of P, it is easy to see that Λ does not meet X_P. Hence $z \notin X_P$. There are now two cases to consider: either z lies on a secant line, i.e., $z = \lambda p + \mu q$, where $p \neq q \in X_P$ and $[\lambda : \mu] \in \mathbb{P}^1$, or z lies on a tangent line to $p \in X_P$.

The toric variety X_P is covered by open affine sets corresponding to the vertices of P. The point p lies in one of these. Assume that we have moved this vertex into standard position at the origin so that local coordinates near p are given by sending $(t_1, \ldots, t_n) \in \mathbb{C}^n$ to $[1 : t_1 : t_2 : \cdots : t_n : \cdots] \in \mathbb{P}^r$, and let $p = [1 : \alpha_1 : \alpha_2 : \cdots : \alpha_n : \cdots]$.

In the first case, we may write $z = p + rq$ where $r = \frac{\mu}{\lambda}$ because we assume that $z \notin X$. Since $z \in \Lambda$, all of the coordinates corresponding to the elements of A must be zero. Therefore, $p + rq$ must be zero in the first $n + 1$ coordinates since A contains each vertex and its nearest neighbors.

Since p is nonzero in the first coordinate and $p + rq$ is zero in that coordinate, q must be in the same open affine chart as p. Therefore, $q = [1 : \beta_1 : \beta_2 : \cdots : \beta_n : \cdots]$, and then $r = -1$ and $\alpha_i = \beta_i$ for all i. This implies $p = q$, a contradiction.

Suppose now that z lies on a tangent line through p. From our local coordinate system near p we see that the tangent space of X_A at p is spanned by p and points $v_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0 : * : \cdots]$. Thus one of the first $n + 1$ coordinates is nonzero, so $z \notin \Lambda$. $\square$

We can also determine when $\operatorname{Sec} X_A$ has the expected dimension.

Theorem 5.2. *Let $X_A \subset \mathbb{P}^s$ come from $A = \{u_0, \ldots, u_s\} \subset \mathbb{Z}^n$. If $P = \operatorname{Conv}(A)$ is smooth of dimension n and A contains the nearest neighbors along the edges of each vertex of P, then* $\operatorname{Sec} X_A$ *has the expected dimension* $\min\{s, 2n + 1\}$ *unless A is $AGL_n(\mathbb{Z})$-equivalent to the set of all lattice points of one of*

$$2\Delta_n \quad (n \geq 2), \quad (2\Delta_n)_k \quad (0 \leq k \leq n - 3), \quad \Delta_\ell \times \Delta_{n-\ell} \quad (2 \leq \ell \leq n - 2).$$

Proof. For P in the last row of Table 1.4, $\dim \operatorname{Sec} X_A = \dim \operatorname{Sec} X_P = 2n + 1$ by Theorems 1.4 and 5.1. Now suppose that P is $AGL_n(\mathbb{Z})$-equivalent to a polytope in the first four rows. For these polytopes, all lattice points lie on edges and all edges have length ≤ 2. Hence A contains all lattice points of the polytope, so Corollary 1.5 applies. $\square$

We conclude with two examples, one in which the degrees of $\operatorname{Sec} X_P$ and $\operatorname{Sec} X_A$ are equal and one in which they are not.

Example 5.3. Let $P = 3\Delta_2$ and let A be the set of all lattice points in P minus the point $(1, 1)$. The variety $X_P \subset \mathbb{P}^9$ is the 3-uple Veronese embedding of $\mathbb{P}^2$ and $X_A \subset \mathbb{P}^8$ is a projection of X_P from a point. Our results imply that $\dim \mathrm{Sec}\, X_P = \dim \mathrm{Sec}\, X_A = 5$, $\deg \mathrm{Sec}\, X_P = 15$, and $\deg \mathrm{Sec}\, X_A$ divides 15. A *Macaulay 2* computation shows that $\deg \mathrm{Sec}\, X_A = 15$ in this case. Note also that a general point of $\mathrm{Sec}\, X_P$ or $\mathrm{Sec}\, X_A$ lies on a unique secant line of X_P or X_A, respectively.

Example 5.4. Let $A = \{(0, 0), (1, 0), (0, 1), (2, 1), (1, 2), (2, 2)\}$ and P be the convex hull of A. The set of lattice points in P is $A \cup \{(1, 1)\}$. The variety X_P is a smooth toric surface of degree 6 in $\mathbb{P}^6$ and X_A in $\mathbb{P}^5$ is the projection of X_P from a point. Note that X_P is the Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at three points, embedded by the complete anticanonical linear system. Our results imply that $\dim \mathrm{Sec}\, X_P = \dim \mathrm{Sec}\, X_A = 5$ and $\deg \mathrm{Sec}\, X_P = 3$. However, $\dim \mathrm{Sec}\, X_A = 5$ implies that $\mathrm{Sec}\, X_A$ fills all of $\mathbb{P}^5$ and hence $\deg \mathrm{Sec}\, X_A = 1$. Thus $\pi : \mathrm{Sec}\, X_P \to \mathrm{Sec}\, X_A$ has degree 3 and a general point of $\mathrm{Sec}\, X_A$ lies on three secant lines of X_A.

When X_A is a (possibly singular) monomial curve, the degree of $\mathrm{Sec}\, X_A$ has been computed explicitly in the recent paper [25].

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