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Journal of Symbolic Computation 41 (2006) 1091–1104
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Gröbner fan and universal characteristic sets of prime differential ideals☆

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Received 15 October 2004; accepted 18 May 2006
Available online 22 August 2006

Abstract

The concepts of Gröbner cone, Gröbner fan, and universal Gröbner basis are generalized to the case of characteristic sets of prime differential ideals. It is shown that for each cone there exists a set of polynomials which is characteristic for every ranking from this cone; this set is called a strong characteristic set, and an algorithm for its construction is given. Next, it is shown that the set of all differential Gröbner cones is finite for any differential ideal. A subset of the ideal is called its universal characteristic set, if it contains a characteristic set of the ideal w.r.t. any ranking. It is shown that every prime differential ideal has a finite universal characteristic set, and an algorithm for its construction is given. The question of minimality of this set is addressed in an example. The example also suggests that construction of a universal characteristic set can help in solving a system of nonlinear PDE's, as well as maybe providing a means for more efficient parallel computation of characteristic sets.

Keywords: Differential algebra; Prime differential ideal; Characteristic set

1. Introduction

Consider a differential ring of differential polynomials and a prime differential ideal in it. We study the dependence of the characteristic set of the ideal on the ranking of partial derivatives and give two invariants of the ideal—the differential Gröbner fan and the universal characteristic set.

☆ This is a full version of the article presented as a poster at ISSAC 2004.

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doi:10.1016/j.jsc.2006.05.003

In the algebraic case, a similar problem has been solved in Mora and Robbiano (1988) by introducing the concepts of Gröbner fan and universal Gröbner basis and giving an algorithm for their construction. This was followed by the development of several efficient algorithms for transformation of Gröbner bases from one admissible order to another, of which the most well-known are the FGLM (Faugère et al., 1993) and the Gröbner walk (Collart et al., 1993) algorithms.

These algorithms have been carried over to the case of characteristic sets of prime differential ideals (Boulier, 1999; Golubitsky, 2004). Now we carry over the theoretical basis for these algorithms to the differential case as well.

For a fixed ranking there are usually infinitely many characteristic sets, but their ranks coincide. One can pose the converse problem: given an autoreduced set of ranks (a rank is a power of a derivative), describe the sets of rankings with respect to which characteristic sets of the ideal have this set of ranks. We call this set of rankings a differential Gröbner cone. Then we show that

(1) For every differential Gröbner cone, there exists a set which is characteristic for each ranking from this cone. We call this set a *strong characteristic set* and describe an algorithm that, given any characteristic set of a prime differential ideal, constructs a strong one of the same rank.
(2) The set of all differential Gröbner cones corresponding to any differential ideal is finite; we call this set a *differential Gröbner fan*. The set of all rankings is equal to the disjoint union of the cones in the Gröbner fan.
(3) There exists a finite set of differential polynomials whose characteristic set, for every ranking, is a strong characteristic set of the prime ideal. We call this set a *universal characteristic set* and give an algorithm for its construction.

We illustrate the algorithm for construction of a universal characteristic set on an example from Boulier et al. (2001). For this particular example, we construct a minimal universal characteristic set; however, the problem of minimality in general is open. Moreover, we show how the system of nonlinear PDE's from this example can be easily solved with the help of the universal characteristic set (again, no claims are made about the usefulness of the universal characteristic sets for the solution of nonlinear PDE systems in general). Finally, the problem of efficient computation of the universal characteristic sets is discussed in relation to the problem of fast parallel computation of (ordinary) characteristic sets.

2. Basic concepts of differential algebra

Here we give a short summary of the basic concepts of differential algebra, referring the reader to Ritt (1950), Kolchin (1973) and Kondratieva et al. (1999) for a more complete exposition.

Let R be a commutative ring. A *derivation* over R is a mapping $\delta : R \to R$ which for every $a, b \in R$ satisfies

$$\delta(a + b) = \delta(a) + \delta(b), \qquad \delta(ab) = \delta(a)b + a\delta(b).$$

A *differential ring* is a commutative ring endowed with a finite set of derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ which commute pairwise. The commutative monoid generated by the derivations is denoted by Θ. Its elements are *derivation operators* $\theta = \delta_1^{i_1} \cdots \delta_m^{i_m}$, where $i_1, \dots, i_m$ are nonnegative integer numbers.

A *differential ideal* I of differential ring R is an ideal of R stable under derivations, i.e.

$$\forall A \in I,\ \delta \in \Delta \quad \delta A \in I.$$

For a subset $A \subset R$, $[A]$ denotes the smallest differential ideal containing A.

An ideal is called *prime*, if

$$\forall a, b \in R \;\; ab \in I \;\; \Rightarrow \;\; a \in I \;\text{ or }\; b \in I.$$

Let $U = \{u_1, \ldots, u_n\}$ be a finite set whose elements are called *differential indeterminates*. Derivation operators apply to differential indeterminates yielding *derivatives* θu. Denote by ΘU the set of all derivatives.

Let $\mathbb{K}$ be a differential field of characteristic zero. The differential ring of *differential polynomials* $\mathbb{K}\{U\}$ is the ring of polynomials of infinitely many variables $\mathbb{K}[\Theta U]$ endowed with the set of derivations Δ.

Let m be a nonnegative integer and n be a positive integer. Let

$$\mathbb{N} = \{0, 1, 2, \ldots\}, \quad \mathbb{N}_n = \{1, \ldots, n\}.$$

A *ranking* is a total order $\le$ of $\mathbb{N}^m \times \mathbb{N}_n$ such that for all $a, b, c \in \mathbb{N}^m$, $i, j \in \mathbb{N}_n$,

- $(a, i) \le (b, j) \iff (a + c, i) \le (b + c, j)$
- $(a, i) \ge (0, i)$.

Rankings on $\mathbb{N}^m \times \mathbb{N}_n$ correspond to rankings on the set of derivatives ΘU:

$$\delta_1^{i_1} \cdots \delta_m^{i_m} u_j \le \delta_1^{k_1} \cdots \delta_m^{k_m} u_l \iff (i_1, \ldots, i_m, j) \le (k_1, \ldots, k_m, l).$$

Theorem 1 (*Rust and Reid, 1997, Theorem 30*). *Let $\le$ be a ranking on $\mathbb{N}^m \times \mathbb{N}_n$. Let R be a finite subset of $(\mathbb{N}^m \times \mathbb{N}_n)^2$ such that for all pairs $((a, i), (b, j)) \in R$, $(a, i) \le (b, j)$. Then there exist non-singular $m \times m$ integer matrices $M_1, \ldots, M_n$, vectors $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^m$, and a permutation σ of $\mathbb{N}_n$, which uniquely specify a ranking $\le'$ on $\mathbb{N}^m \times \mathbb{N}_n$ such that for all $((a, i), (b, j)) \in R$, $(a, i) \le' (b, j)$.*

The specification of rankings described in the above theorem has been implemented in Maple by A.D. Wittkopf (Rust and Reid, 1997, Remark after Th. 30). This code was first included in the Rif package, but then replaced by a simpler and more efficient code that accounts for Riquier rankings only (see below).[1]

The problem of finding a ranking containing a given finite relation on partial derivatives has been solved for Riquier rankings and implemented in the Maple 9.01, package `diffalg`, procedure `differential_ring`. Below we briefly describe the corresponding algorithm. To the author's knowledge, for arbitrary rankings this problem remains open.[2]

A ranking is called a *Riquier ranking*, if for all $a, b \in \mathbb{N}^m$, $i, j \in \mathbb{N}_n$,

$$(a, i) \le (b, i) \iff (a, j) \le (b, j).$$

[1] From private communication with A.D. Wittkopf.

[2] The problem can be reduced to a linear programming problem using Theorem 1, namely by trying all permutations σ and numbers t_{ij} such that matrices M_i and M_j agree on the first t_{ij} rows [from private communication with C.J. Rust]. However, this solution is inefficient, and for this reason has not been implemented.

Let us embed $\mathbb{N}^m \times \mathbb{N}_n$ into $\mathbb{N}^{n+m}$ by using the following mapping:

$$\phi : (i_1, \ldots, i_m, j) \longmapsto (i_1, \ldots, i_m, 0, \ldots, \overset{(m+j)}{1}, \ldots, 0)^T.$$

Using this embedding, we can characterize Riquier rankings by matrices:

Theorem 2 (Rust and Reid, 1997, *Theorem 6*). *A Riquier ranking is a ranking $\leq$ for which there exists a positive integer s and an $s \times (m+n)$ real matrix M such that*

- *for $k = 1, \ldots, m$, kth column c_k of M satisfies*

$$c_k \geq_{\text{lex}} (0, \ldots, 0) \tag{1}$$

- *$(i_1, \ldots, i_m, j) \leq (k_1, \ldots, k_m, l)$ if and only if*

$$M\big(i_1, \ldots, i_m, 0, \ldots, \overset{(m+j)}{1}, \ldots, 0\big) \leq_{\text{lex}} M\big(k_1, \ldots, k_m, 0, \ldots, \overset{(m+l)}{1}, \ldots, 0\big).$$

Vice versa, any $s \times (m+n)$ real matrix M of rank $m+n$ satisfying Eq. (1) defines a Riquier ranking $\leq_M$.

Now, the question whether a finite set $R \subset (\mathbb{N}^m \times \mathbb{N}_n)^2$ is contained in a Riquier ranking reduces to the question whether the system

$$\{M(\phi(u) - \phi(v)) \leq_{\text{lex}} 0 \mid (u, v) \in R\},$$

in which matrix M is unknown and satisfies the requirements of the above theorem, has a solution. According to Sturmfels (1996, Proposition 1.11), this is the case if and only if there exists a vector $\mathbf{w} \in \mathbb{N}^{m+n}$ satisfying

$$\{\mathbf{w} \cdot (\phi(u) - \phi(v)) < 0 \mid (u, v) \in R\}.$$

The above linear system of inequalities is a linear programming problem, which can be solved, for example, by applying the simplex method (see Danzig (1998)).

3. Characteristic sets

Let $\leq$ be a ranking on the set of derivatives ΘU, and let $f \in \mathbb{K}\{U\}$, $f \notin \mathbb{K}$. The derivative θu_j of the highest rank present in f is called the *leader* of f (denoted $\mathrm{ld}_{\leq} f$ or $\mathbf{u}_f$ when the ranking is clear from the context). Let $d = \deg_{\mathbf{u}_f} f$. Then $f = \sum_{j=0}^{d} g_j \mathbf{u}_f^j$, where $g_0, \ldots, g_d$ are uniquely defined polynomials free of $\mathbf{u}_f$. Differential polynomial $\mathbf{i}_f = g_d$ is called the *initial* of f, and differential polynomial $\mathbf{s}_f = \sum_{j=1}^{d} j g_j \mathbf{u}^{j-1}$ is called the *separant* of f. The *rank* $\mathrm{rk}_{\leq} f$ is the monomial $(\mathbf{u}_f)^d$. For a set $A \subset \mathbb{K}\{U\}$, the set of its ranks is $\mathrm{rk}_{\leq} A = \{\mathrm{rk}_{\leq} f \mid f \in A\}$. Similarly, the set of initials and the set of terminals of A are denoted $\mathbf{i}_A$ and $\mathbf{s}_A$ respectively. Let $\mathbf{h}_A = \mathbf{i}_A \cup \mathbf{s}_A$.

Denote by $\mathbf{R} = \{t^d \mid t \in \Theta U,\ d > 0\}$ the set of all ranks. A ranking on the set of derivatives ΘU induces a linear order on the set of ranks $\mathbf{R}$, if we consider a rank $t^d \in \mathbf{R}$ as a pair (t, d) and compare two such pairs lexicographically. Let $f, p \in \mathbb{K}\{U\}$, $p \notin \mathbb{K}$. Differential polynomial f is *partially reduced* w.r.t. p, if f is free of all proper derivatives $\theta\mathbf{u}_p$ (i.e. $\theta \neq 1$) of the leader of p. If f is partially reduced w.r.t. p and $\deg_{\mathbf{u}_p} f < \deg_{\mathbf{u}_p} p$, then f is said to be *(fully) reduced*, or *irreducible*, w.r.t. p. A polynomial f is called *(partially) reducible* w.r.t. p, if it is not (partially) reduced w.r.t. p.

A differential polynomial f is called *(partially) reduced w.r.t. a set of differential polynomials* $A \subset \mathbb{K}\{U\}$, if it is (partially) reduced w.r.t. every polynomial $p \in A$.

A nonempty subset $A \subset \mathbb{K}\{U\}$ is called *(partially) autoreduced* if every $f \in A$ is (partially) reduced w.r.t. $A \setminus \{f\}$.

Every autoreduced set is finite (Kolchin, 1973, Chapter I, Section 9). If $A = \{p_1, \ldots, p_k\}$ is an autoreduced set, then any two leaders $\mathbf{u}_{p_i}, \mathbf{u}_{p_j}$ for $1 \le i \ne j \le r$ are distinct; we assume that elements of any autoreduced set are arranged in order of increasing rank of their leaders $\mathbf{u}_{p_1} < \mathbf{u}_{p_2} < \cdots < \mathbf{u}_{p_k}$.

Let $A = \{f_1, \ldots, f_k\}$, $B = \{g_1, \ldots, g_l\}$ be two autoreduced sets. We say that A has *lower rank than* B and write $\mathrm{rk}_\le A < \mathrm{rk}_\le B$, if either there exists $j \in \mathbb{N}$ such that $\mathrm{rk}_\le f_i = \mathrm{rk}_\le g_i$ $(1 \le i < j)$ and $\mathrm{rk}_\le f_j < \mathrm{rk}_\le g_j$, or $k > l$ and $\mathrm{rk}_\le f_i = \mathrm{rk}_\le g_i$ $(1 \le i \le l)$. If $k = l$ and $\mathrm{rk}_\le f_i = \mathrm{rk}_\le g_i$ $(1 \le i \le k)$, then we have $\mathrm{rk}_\le A = \mathrm{rk}_\le B$.

Any nonempty family of autoreduced sets contains an autoreduced set of the lowest rank (Kolchin, 1973, Chapter I, Section 10). For a subset $X \subset \mathbb{K}\{U\}$, an autoreduced subset of X of the lowest rank is called a *characteristic set* of X. It follows from the definition that all characteristic sets of X w.r.t. $\le$ have the same rank. An autoreduced set A is a characteristic set of X if and only if all nonzero elements of X are reducible w.r.t. A.

Lemma 1 (*Kolchin, 1973, Chapter IV, Section 9, proof of Lemma 2*). *If C is a characteristic set of a prime differential ideal I, then*

$$\mathbf{h}_C^\infty \cap I = \varnothing.$$

Lemma 2. *Let C be a characteristic set of a prime differential ideal I w.r.t. a ranking $\le$, and let $f \in I$, $f \ne 0$ be such that* $\mathrm{rk}_\le f$ *is irreducible w.r.t. C. Then $\mathbf{i}_f \in I$.*

Proof. Since $f \in I$, $f \ne 0$, and C is a characteristic set, f is reducible w.r.t. C. Since $\mathrm{rk}_\le f$ is irreducible w.r.t. C, two cases are possible:

(1) $\mathbf{i}_f$ is reducible w.r.t. a polynomial $p \in C$. Let q be the result of this reduction:

$$q = h \cdot \mathbf{i}_f - \tau\theta p, \quad h \in \mathbf{h}_p^\infty.$$

If $q = 0$, we obtain that $h \cdot \mathbf{i}_f \in I$.

If $q \ne 0$, consider the corresponding reduction of f:

$$f' = h \cdot f - \tau\, \mathrm{rk}_\le f \cdot \theta p.$$

Then $\mathrm{rk}_\le f' = \mathrm{rk}_\le f$ and $\mathbf{i}_{f'} = q$. Now, f' reduces to 0 in less steps than f, hence, by induction $\mathbf{i}_{f'} \in I$. This implies $h \cdot \mathbf{i}_f \in I$.

Since $h \notin I$ (by Lemma 1) and I is prime, $\mathbf{i}_f \in I$.

(2) $\mathbf{i}_f$ is irreducible w.r.t. C, and $f - \mathbf{i}_f\, \mathrm{rk}_\le f$ is reducible w.r.t. C. Let f' be the result of a reduction step of f w.r.t. C. Then $\mathrm{rk}_\le f' = \mathrm{rk}_\le f$ and $\mathbf{i}_{f'} = h \cdot \mathbf{i}_f$, where $h \in \mathbf{h}_C^\infty$, which, in particular, implies that $f' \ne 0$.

Now, since f' reduces to 0 in less steps than f, by induction we obtain that $\mathbf{i}_{f'} \in I$. Since also $h \notin I$ (by Lemma 1) and I is prime, we have $\mathbf{i}_f \in I$. □

4. Differential Gröbner cones

Definition 1. Let $X \subset \mathbb{K}\{U\}$, and let $\leq$ be a ranking. Define the **characteristic rank** of X w.r.t. $\leq$, $\operatorname{rkchar}_{\leq} X$, to be equal to the rank of any characteristic set of X w.r.t. $\leq$.

Definition 2. Let I be an ideal, and let $R = \operatorname{rkchar}_{\leq} I$ for some ranking $\leq$. Define the **differential Gröbner cone** corresponding to R as follows:

$$\Sigma_I(R) = \{\leq' \mid \operatorname{rkchar}_{\leq'} I = R\}.$$

Let also $\operatorname{cone}_I(\leq) = \Sigma_I(\operatorname{rkchar}_{\leq} I)$ denote the differential Gröbner cone containing ranking $\leq$.

The above definition is a generalization of the concept of Gröbner cone for algebraic ideals. Indeed, for a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and an admissible monomial ordering $\leq$, one can define the rank of a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ to be equal to its leading monomial w.r.t. $\leq$. For this definition of rank (which is rather unusual), a set A is autoreduced if and only if it is autoreduced w.r.t. $\leq$ in the usual algebraic sense. Hence, the autoreduced subset of I of the smallest rank is the reduced Gröbner basis of I (up to multiplication by constants from $\mathbb{K}$), and $\operatorname{rkchar}_{\leq} I$ is the set of its leading monomials. Furthermore, $\Sigma_I(R) = \{\leq' \mid \operatorname{rkchar}_{\leq'} I = R\}$ is the set of all admissible orderings, for which the set of leading monomials of the reduced Gröbner basis is equal to R. According to Mora and Robbiano (1988), this set corresponds to an open convex cone in $\mathbb{R}^n$, called the Gröbner cone. As is shown below, in the general (i.e., differential) case, the set $\Sigma_I(R)$ is convex in the following sense:

Definition 3. For two rankings $\leq_1, \leq_2$, the **segment** $[\leq_1, \leq_2]$ is defined as the following set of rankings:

$$[\leq_1, \leq_2] = \{\leq \mid \forall\, u, v \in \mathbb{N}^m \times \mathbb{N}_n \;\; (u \leq_1 v \wedge u \leq_2 v) \;\Rightarrow u \leq v\}.$$

A set of rankings R is called **convex**, if $\leq, \leq' \in R$ implies that $[\leq, \leq'] \subset R$.

Note that, unlike the algebraic case, in which the reduced Gröbner basis of I w.r.t. $\leq$ is unique, a characteristic set of a differential ideal I may not be unique. Moreover, if we take an arbitrary characteristic set w.r.t. a ranking $\leq \in \Sigma_I(R)$, it may not be characteristic for all rankings from $\leq \in \Sigma_I(R)$. For the purposes of construction of the universal characteristic set, we are interested in finding a set that is characteristic for all rankings from a differential Gröbner cone. The algorithm for constructing such a set, given an arbitrary characteristic set, is proposed in the following section.

5. Strong characteristic sets

Definition 4. Let C be a set of differential polynomials, and let $\leq$ be a ranking. Denote by $C^{\leq}$ the following set:

$$C^{\leq} = \{(f, \operatorname{rk}_{\leq} f) \mid f \in C\}$$

and call it a **marked set** of differential polynomials.

Definition 5. Let $\leq$ be a ranking, and let C be a characteristic set of ideal I w.r.t. $\leq$. Define the **subcone** corresponding to the marked characteristic set $C^{\leq}$ as follows:

$$\operatorname{subcone}(C^{\leq}) = \{\leq' \mid C^{\leq'} = C^{\leq}\}.$$

It follows from the above definition that for any $\leq' \in \text{subcone}(C^{\leq})$, set C is a characteristic set of I w.r.t. $\leq'$ and has the same rank as w.r.t. $\leq$.

Lemma 3. *The set* $\text{subcone}(C^{\leq})$ *is convex.*

Proof. Let $\leq_1, \leq_2 \in \text{subcone}(C^{\leq})$. Then $C^{\leq_1} = C^{\leq_2} = C^{\leq}$. Hence, by Definition 3, for any ranking $\preceq \in [\leq_1, \leq_2]$ we have $C^{\preceq} = C^{\leq_1} = C^{\leq_2}$. Thus, $\preceq \in \text{subcone}(C^{\leq})$, and $\text{subcone}(C^{\leq})$ is convex. □

Definition 6. A characteristic set C of ideal I w.r.t. a ranking $\leq$ is called **strong**, if

$$\text{subcone}(C^{\leq}) = \text{cone}_I(\leq).$$

In other words, a characteristic set C w.r.t. ranking $\leq$ is strong, if for any ranking $\leq' \in \text{cone}_I(\leq)$, $\text{rk}_{\leq'} C = \text{rk}_{\leq} C$. Note also that the above equality of two sets can be replaced by inclusion $\text{subcone}(C^{\leq}) \supseteq \text{cone}_I(\leq)$, since the inverse inclusion always holds.

For example, consider prime differential ideal $I = [u_x]$. Then, for any ranking $\leq$, sets $\{u_x\}$ and $\{uu_x\}$ are characteristic sets of I of rank $\{u_x\}$ w.r.t. $\leq$. Hence, the corresponding subcones, $\text{subcone}(\{u_x\}^{\leq})$ and $\text{subcone}(\{uu_x\}^{\leq})$, are equal to the set of all rankings $\mathbf{R}$. Also, since $\text{rkchar}_{\leq} I = \{u_x\}$ for any ranking $\leq$, we have $\text{cone}_I(\leq) = \mathbf{R}$. Therefore, sets $\{u_x\}$ and $\{uu_x\}$ are strong characteristic sets of I w.r.t. any ranking $\leq$.

Now, consider sets

$$\{u_y u_x\}, \{u_{yy} u_x\}, \ldots, \{v u_x\}, \ldots.$$

Each of these sets is a characteristic set of I w.r.t. a proper subset of the set of all rankings $\mathbf{R}$ (e.g., $\{u_y u_x\}$ is a characteristic set of I for all rankings $\leq$ satisfying $u_y < u_x$), hence the corresponding subcones are proper subsets of $\mathbf{R}$, and none of these sets is a strong characteristic set of I.

If there exists a strong characteristic set of I w.r.t. $\leq$, then we obtain that the differential Gröbner cone $\text{cone}_I(\leq)$ is convex. Below we prove that a strong characteristic set exists for any prime differential ideal w.r.t. any ranking.

For a polynomial f, denote by $\text{allrk}(f)$ the set of its ranks w.r.t. all possible rankings; for every $t \in \text{allrk}(f)$, $\mathbf{i}_t f$ denotes the initial of f w.r.t. a ranking $\leq$ such that $t = \text{rk}_{\leq} f$.

Theorem 3. *Let C be a characteristic set of a prime differential ideal I w.r.t. $\leq$ satisfying the following condition:*

$$\forall f \in C \quad \forall t \in \text{allrk}(f) \quad \mathbf{i}_t f \notin I.$$

Then C is strong.

Proof. Assume that C is not strong. Then there exists a ranking

$$\leq' \in \text{cone}_I(\leq) \setminus \text{subcone}(C^{\leq}).$$

Since $\leq' \notin \text{subcone}(C^{\leq})$, there exists polynomial $f \in C$ such that $\text{ld}_{\leq'} f \neq \text{ld}_{\leq} f$.

Let C' be a characteristic set of I w.r.t. $\leq'$. Then $\text{rk}_{\leq'} f$ is irreducible w.r.t. C' and $\leq'$. Indeed, suppose that $\text{rk}_{\leq'} f$ is reducible by $g \in C'$. Then $\text{rk}_{\leq'} f$ is also reducible by $\text{rk}_{\leq'} g$. Since $\leq' \in \Sigma_I(\text{rk}_{\leq} C)$, there exists a polynomial $f_1 \in C$ such that $\text{rk}_{\leq} f_1 = \text{rk}_{\leq'} g$, which implies that f is reducible by f_1 w.r.t. $\leq$. It remains to notice that $f_1 \neq f$, since $\text{rk}_{\leq'} f < \text{rk}_{\leq} f$ and therefore $\text{rk}_{\leq'} f$ cannot be reducible by $\text{rk}_{\leq} f$. Contradiction.

Hence, Lemma 2 applies and we obtain that $\mathbf{i}_{\leq'} f \in I$. This contradicts the condition

$$\forall f \in C \quad \forall t \in \text{allrk}(f) \quad \mathbf{i}_t f \notin I. \quad \square$$

The above theorem provides an algorithm for construction of a strong characteristic set, given any characteristic set of a prime differential ideal.

Algorithm StrongCharSet($C, \leq$)
  repeat
    flag = false
    for $f \in C$ **do**
      for $t \in \text{allrk}(f)$ **do**
        if $\mathbf{i}_t f \in I$ **then**
          $C = C \setminus \{f\} \cup \{\mathbf{i}_t f\}$
          flag = true
        end if
      end for
    end for
  until flag = false
  return C
end

The algorithm terminates, because each replacement of f by $\mathbf{i}_t f$ reduces the total number of derivatives present in C. Also, since initially C is a characteristic set w.r.t. $\leq$, and for any polynomial f and derivative t, $\text{rk}_\leq \mathbf{i}_t f \leq \text{rk}_\leq f$, we obtain that $\text{rk}_\leq C$ is an invariant of the above algorithm. Hence, the result of the algorithm is a characteristic set of I w.r.t. $\leq$. According to the above Theorem 3, it is a strong characteristic set of I.

We illustrate this algorithm on our example. Let a characteristic set of ideal $I = [u_x]$ be given, e.g. $\{u_y u_x\}$. For $f = u_y u_x$, we have $\text{allrk}(f) = \{u_x, u_y\}$. Next, for $t = u_y$, we have $\mathbf{i}_t f = u_x \in I$, hence the algorithm replaces f by u_x and thereby obtains a strong characteristic set of I.

6. Gröbner fan and universal characteristic sets

It follows from Definition 2 that distinct differential Gröbner cones do not intersect each other and that every ranking belongs to a cone, i.e., the set of all rankings can be represented as a disjoined union of differential Gröbner cones. The set of all differential Gröbner cones is called the *differential Gröbner fan*. As follows from Golubitsky (2004, Theorem 3), the fan is finite for any differential ideal. To make this paper self-contained, we also include the proof of this result here.

Let

$$\text{Ld}(I) = \{\text{ld}_\leq(A) \mid \leq \text{ is a ranking and } A \text{ is a char. set of } I \text{ w.r.t. } \leq\}.$$

In other words, $\text{Ld}(I)$ is the family of sets of leaders of characteristic sets of I w.r.t. all possible rankings. We will first show that the set $\text{Ld}(I)$ is finite for any ideal I.

Let $t_1, t_2 \in \Theta U$ be two derivatives. We say that t_2 *is a derivative of* t_1, if there exists $\theta \in \Theta$ such that $t_2 = \theta t_1$.

Lemma 4 (*Kolchin, 1973, Chapter 0, Lemma 15*). *Let $t_1, t_2, \ldots \subset \Theta U$ be an infinite sequence of derivatives. Then there exist indices $i < j$ such that t_j is a derivative of t_i.*

Lemma 5. *For any differential ideal $I \subset \mathbb{K}\{U\}$, family* $\text{Ld}(I)$ *is finite.*

Proof. Suppose that $\mathrm{Ld}(I)$ is infinite. For each $L \in \mathrm{Ld}(I)$, denote by $\le_L$ the corresponding ranking. Then the set $\Sigma = \{\le_L \mid L \in \mathrm{Ld}(I)\}$ is infinite.

Let $f_1 \in I$ be a differential polynomial, and let $A_1 = \{f_1\}$. Since f_1 contains only a finite number of derivatives, according to the pigeonhole principle, there exists an infinite subset $\Sigma_1 \subset \Sigma$ such that for all $\le, \le' \in \Sigma_1$, $\mathrm{ld}_\le f_1 = \mathrm{ld}_{\le'} f_1$.

Suppose A_1 is a characteristic set of I w.r.t. a ranking $\le_1 \in \Sigma_1$. Then A_1 is also a characteristic set of I w.r.t. any ranking $\le \in \Sigma_1$, since the reduction relations w.r.t. A_1 and any $\le \in \Sigma_1$ coincide. However, this contradicts the definition of the set of rankings Σ, because characteristic sets corresponding to different rankings in Σ have different sets of leaders. Therefore, A_1 is autoreduced but not a characteristic set of I. Hence, there exists a polynomial $f_2 \in I$ reduced w.r.t. A_1 and any $\le \in \Sigma_1$.

According to the pigeonhole principle, there exists an infinite subset $\Sigma_2 \subset \Sigma_1$ such that for all $\le \in \Sigma_2$, the characteristic set of $A_1 \cup \{f_2\}$ is the same and the polynomials in it have the same leaders; call this characteristic set A_2.

The set A_2 cannot be a characteristic set of I for some $\le \in \Sigma_2$ (according to the definition of Σ), hence there exists a polynomial $f_3 \in I$ reduced w.r.t. A_2 and any ranking $\le \in \Sigma_2$.

According to the pigeonhole principle, there exists an infinite subset $\Sigma_3 \subset \Sigma_2$ such that for all $\le \in \Sigma_3$, the characteristic set of $A_2 \cup \{f_3\}$ is the same and the polynomials in it have the same leaders; call this characteristic set A_3.

Proceeding in the same way, we construct an infinite sequence of polynomials $f_1, f_2, \ldots$, an infinite sequence of autoreduced sets $A_1, A_2, \ldots$, and an infinite sequence of sets of rankings $\Sigma_1 \supset \Sigma_2 \supset \ldots$. For each polynomial f_i, one of the following two options is possible:

(1) For all $j > i$, $\mathrm{ld}_\le f_j > \mathrm{ld}_\le f_i$ $(\le \in \Sigma_j)$. In this case $f_i \in A_j$ for all $j \ge i$, and we say that f_i *remains in the sequence*.

(2) There exists $j > i$ such that $\mathrm{ld}_\le f_j < \mathrm{ld}_\le f_i$ $(\le \in \Sigma_j)$. In this case we say that f_i *is followed by a smaller derivative*, and denote the smallest such j by $\nu(i)$.

Denote by $\nu^k(i)$ the expression $\nu(\nu(\ldots\nu(i)\ldots))$, where ν is applied k times.

Now we will construct a subsequence of $f_1, f_2, \ldots$ contradicting Lemma 4.

If f_1 remains in the sequence, let $i_1 = 1$. Otherwise, if $f_{\nu(1)}$ remains in the sequence, let $i_1 = \nu(1)$. Otherwise, if $f_{\nu^2(1)}$ remains in the sequence, let $i_1 = \nu^2(1)$, and so on. We will either find an index i_1 such that f_{i_1} remains in the sequence, or will construct an infinite sequence

$$f_1, f_{\nu(1)}, f_{\nu^2(1)}, \ldots.$$

But the latter is not possible. Indeed, it follows from the definition of $\nu(i)$ that for all $i < j$, $\mathrm{ld}_\le f_{\nu^j(1)} < \mathrm{ld}_\le f_{\nu^i(1)}$ $(\le \in \Sigma_{\nu^j(1)})$. Hence, $\mathrm{ld}_\le f_{\nu^j(1)}$ is not a derivative of $\mathrm{ld}_\le f_{\nu^i(1)}$, which contradicts Lemma 4.

If f_{i_1+1} remains in the sequence, let $i_2 = i_1+1$. Otherwise, if $f_{\nu(i_1+1)}$ remains in the sequence, let $i_2 = \nu(i_1 + 1)$, and so on. Applying the above argument, we show that the process will eventually stop and we will find an index i_2 such that f_{i_2} remains in the sequence.

Continuing in the same way, we obtain an infinite sequence of indices $i_1 < i_2 < \ldots$ such that for all j, f_{i_j} remains in the sequence. But the fact that both f_{i_j} and f_{i_k} $(i_j < i_k)$ remain in the sequence means that they both belong to the autoreduced set A_{i_k}, therefore $\mathrm{ld}_\le f_{i_k}$ is not a derivative of $\mathrm{ld}_\le f_{i_j}$ $(\le \in \Sigma_{i_k})$. Thus we have constructed an infinite sequence of derivatives $\{\mathrm{ld}_{\le_j} f_{i_j} \mid \le_j \in \Sigma_{i_j}\}$, none of which is a derivative of another one. This contradicts Lemma 4. □

Theorem 4. *The set of ranks of characteristic sets of any differential ideal I w.r.t. all possible rankings is finite.*

Proof. According to Lemma 5, the set

$$\mathrm{Ld}(I) = \{\mathrm{ld}_{\leq}(C) \mid \leq \text{ is a ranking and } C \text{ is a char. set of } I \text{ w.r.t. } \leq\}$$

is finite.

Suppose that the set of ranks of characteristic sets

$$\mathrm{Rk}(I) = \{\mathrm{rk}_{\leq}(C) \mid \leq \text{ is a ranking and } C \text{ is a char. set of } I \text{ w.r.t. } \leq\}$$

is infinite. Then there exists an infinite subset $R \subset \mathrm{Rk}(I)$ such that for all sets of ranks $r \in R$, the set of derivatives present in r is the same; denote this set of derivatives $\{l_1, \ldots, l_k\}$. It follows from Dickson's Lemma (Cox et al., 1996) that there exist two distinct ranks $r_1 = \{l_1^{i_1}, \ldots, l_k^{i_k}\}, r_2 = \{l_1^{j_1}, \ldots, l_k^{j_k}\} \in R$ such that for all $m \in \{1, \ldots, k\}$, $i_m \leq j_m$. Take an index m such that $i_m < j_m$. Take also the corresponding characteristic sets C_1, C_2 and rankings $\leq_1$, $\leq_2$, i.e., $\mathrm{rk}_{\leq_i} C_i = r_i$ $(i = 1, 2)$. Let $f \in C_1$ be the polynomial with $\mathrm{rk}_{\leq_1} f = l_m^{i_m}$. Then f is irreducible w.r.t. C_2 and $\leq_2$, which contradicts the fact that C_2 is a characteristic set of I w.r.t. $\leq_2$ and $f \in I$. □

The finiteness of the differential Gröbner fan implies the existence of a finite universal characteristic set in the following sense:

Definition 7. A subset C of ideal I is called a **universal characteristic set** of I, if for any ranking $\leq$, the characteristic set of C w.r.t. $\leq$ is a characteristic set of I w.r.t. $\leq$.

In fact, the union of strong characteristic sets corresponding to the cones in the differential Gröbner fan is a universal characteristic set. Below we give an algorithm which, given a characteristic set of a prime differential ideal, constructs a universal characteristic set.

Algorithm UniversalCharSet(C, $\leq$)
  U =StrongCharSet(C, $\leq$)
  $R = \{\mathrm{rk}_{\leq} U\}$
  while $\exists$ $\leq'$ such that $\mathrm{rkchar}_{\leq'} U \notin R$ **do**
    C' =convert(U, $\leq$, $\leq'$)
    C' =StrongCharSet(C',$\leq'$)
    $U = U \cup C'$
    $R = R \cup \{\mathrm{rk}_{\leq'} C'\}$
  end while
  return U
end

The **while**-loop in the above algorithm has the following invariant: for each set of ranks stored in R, U contains the corresponding characteristic set. This invariant implies correctness of the algorithm. Termination is guaranteed by the above Theorem 4. In fact, since family $\{\mathrm{rkchar}_{\leq} I \mid \leq \text{ is a ranking}\}$ is finite, set R in the above algorithm cannot grow infinitely. On the other hand, set R grows with each iteration of the **while**-loop. Note also that the conversion step C' =convert(C, $\leq$, $\leq'$) can be performed using any of the following algorithms: PARDI (Boulier et al., 2001), DFGLM (Boulier, 1999), or differential Gröbner walk (Golubitsky, 2004). In the latter case, from the efficiency point of view, it is worthwhile to add intermediate characteristic sets to U immediately once they are constructed.

The above algorithm works for general rankings, except for the following part, which is restricted to Riquier rankings only. The diffalg package of Maple 9.01 contains a subroutine (which is embedded in the procedure for specification of differential rings) that, given a finite marked set of polynomials, determines whether there exists a Riqiuier ranking w.r.t. which the marked derivatives are leaders, and if so, constructs such a ranking (see also end of Section 2). In particular, this subroutine allows us to generate all possible Riquier rankings on a finite set of derivatives. Thus, given a finite set U, we can generate all Riquier rankings on the derivatives present in it, compute $\mathrm{rkchar}_{\leq} U$ for each of them (the computation of the characteristic set U is trivial, since U is finite), and check whether at least one of the resulting characteristic ranks is not in R.

We illustrate the performance of the algorithm on an example from Boulier et al. (2001). Consider the prime differential ideal

$$I = [u_x^2 - 4u,\ u_{xy}v_y - u + 1,\ v_{xx} - u_x].$$

Before we started our experiments, we already knew some characteristic sets of this ideal w.r.t. several different rankings. These characteristic sets contained 7 derivatives: $u, u_x, u_y, v_y, v_{xx}, v_{xy}, v_{yy}$. In an attempt to simplify our implementation, we decided, instead of implementing the generation of all possible rankings on the derivatives present in U at each iteration of the **while**-loop, to generate first all possible rankings on the above 7 derivatives and compute the union of characteristic sets w.r.t. them. The latter was done using the Rosenfeld–Gröbner algorithm implemented in the diffalg package of Maple 9.01. It turned out that there are 37 Riquier rankings on the above derivatives (call the set of these rankings R_0), and the union of the corresponding characteristic sets consists of the following 19 polynomials:

$$\begin{gathered} -32v_y^2 + 16 - 8v_{xx}^2 + v_{xx}^4,\ -2v_y^2 + 1 - 2u + u^2, \\ -2v_y^2 + 1 - 2v_{yy}^2 + v_{yy}^4,\ v_{xx} - u_x,\ -v_{xx} + uv_{xx} - 2v_y u_y, \\ u_x u - u_x - 2v_y u_y,\ 4u - v_{xx}^2,\ 4v_{xx} + 8v_y v_{xy} - v_{xx}^3, \\ 4v_y u + u_x u_y - u_x u_y u,\ u_y^2 - 2u,\ -v_{yy}^2 + u,\ v_y v_{xy} + v_{yy} - v_{yy}^3, \\ v_{xx} - uv_{xx} + 2v_y v_{xy},\ -v_{yy}u + v_y v_{xy} + v_{yy},\ -v_{xx} + 2v_{yy}, \\ u_y u - u_y - v_y u_x,\ v_{xxx} - 2,\ u_x^2 - 4u,\ u_y u - u_y - v_y v_{xx}. \end{gathered}$$

We have obtained a set of polynomials that contains a characteristic set of I for each ranking from R_0. Now notice that all polynomials in this set, except for the polynomial $v_{xxx} - 2$, depend only on the above 7 derivatives, and the rank of $v_{xxx} - 2$ is equal to v_{xxx} for any ranking. Thus, we have obtained a universal characteristic set of I.

A rather strange phenomenon has been observed during this computation. Our algorithm, implemented in Maple 9.01 on a Celeron 900 MHz under Red Hat Linux 9, computes the above universal characteristic set in 9 seconds. As mentioned above, the algorithm proceeds by calling the Rosenfeld_Gröbner function from the diffalg package for 37 different rankings; each time the Rosenfeld–Gröbner algorithm is called, all previously constructed polynomials are used as its input. However, when the Rosenfeld–Gröbner algorithm was applied directly to the initial set of generators of the ideal I for one of these rankings,[3] the computation took more than 5 h and 1 GB of memory, after which it was interrupted. Thus, the computation of the universal

[3] The following ranking was used: the derivatives of u and v are ordered by weights $u = 6,\ v = 0,\ x = 4,\ y = 1$; $\theta_1\mathbf{u}_1 > \theta_2\mathbf{u}_2$ when weight($\theta_1\mathbf{u}_1$)>weight($\theta_2\mathbf{u}_2$), or weights are equal and $\mathbf{u}_1 = v,\ \mathbf{u}_2 = u$, or weights and indeterminates are equal and $\theta_1 > \theta_2$ for the lexicographic order $x > y$.

characteristic set turned out to be faster than the direct computation of a characteristic set for a particular ranking!

The above universal characteristic set is not minimal, since some polynomials can be removed from it. Applying a greedy minimization algorithm, we reduce the set to 11 polynomials:

$$v_{xx}^4 - 32v_y^2 - 8v_{xx}^2 + 16,\ -2v_y^2 + u^2 - 2u + 1,$$
$$v_{yy}^4 - 2v_y^2 - 2v_{yy}^2 + 1,\ -u_x + v_{xx},\ -2u_y v_y + v_{xx}u - v_{xx},$$
$$-v_{xx}^2 + 4u,\ u_y^2 - 2u,\ -v_{yy}^2 + u,\ v_y v_{xy} - v_{yy}u + v_{yy},\ v_{xxx} - 2,\ u_x^2 - 4u.$$

This universal characteristic set turns out to be minimal. To prove it, we write down the ranks of all characteristic sets w.r.t. different rankings; in total, there are 9 different sets of ranks, each of which corresponds to a cone in the differential Gröbner fan:

$$\begin{array}{ccc} \{v_{xx}, v_{xy}, u^2, v_{yy}^2\} & \{v_{xy}, v_{yy}, u^2, v_{xx}^2\} & \{u_x, v_{xx}, v_y^2, u_y^2\} \\ \{u_y, v_{xx}, v_y^2, u_x^2\} & \{v_{xy}, v_{yy}, u, v_{xx}^4\} & \{v_y, v_{xx}, u_x^2, u_y^2\} \\ \{v_{xx}, v_{xy}, u, v_{yy}^4\} & \{u_x, u_y, v_y^2, v_{xx}^2\} & \{u, v_y^2, v_{xxx}\}. \end{array}$$

Now we show that at least 11 polynomials are necessary to produce the above sets of ranks. Indeed, 4 polynomials are necessary to produce the characteristic set w.r.t. a ranking $\le$ with ranks $\{u_x, u_y, v_y^2, v_{xx}^2\}$. Since these 4 polynomials must form an autoreduced set w.r.t. $\le$, they cannot contain any of the following ranks:

$$v_{yy},\ v_{yy}^2,\ v_{yy}^4,\ v_{xx}^4,\ v_{xxx},\ u_x^2,\ u_y^2.$$

The first three of the above ranks require at least 3 different polynomials to produce them. Again, if one of these 3 polynomials contained at least one of the remaining 4 ranks, then the characteristic set containing this polynomial would not be autoreduced (this can be verified by considering all possibilities for these 3 polynomials to participate in the 9 characteristic sets). Applying the same considerations to the remaining 4 polynomials, we obtain that each of the presented 11 ranks requires a separate polynomial to produce it.

Note also that, in this particular example, the universal characteristic set allows us to obtain the general solution of the original PDE system[4]:

$$\begin{array}{l} v_{xxx} - 2 = 0 \ \Rightarrow\ v = \frac{1}{3}x^3 + x^2 f_1(y) + x f_2(y) + f_3(y) \\ v_{xx} - 2v_{yy} = 0 \ \Rightarrow \\ \qquad 2x + 2f_1(y) - 2x^2 f_1''(y) - 2x f_2''(y) - 2f_3''(y) = 0 \\ \qquad f_1''(y) = 0 \ \Rightarrow\ f_1(y) = c_1 y + c_2 \\ \qquad f_2''(y) = 1 \ \Rightarrow\ f_2(y) = \frac{1}{2}y^2 + c_3 y + c_4 \\ \qquad f_3''(y) = f_1(y) \ \Rightarrow\ f_3(y) = \frac{c_1}{6}y^3 + \frac{c_2}{2}y^2 + c_5 y + c_6 \\ v_{xx}^2 - 4u = 0 \ \Rightarrow\ u = (x + c_1 y + c_2)^2. \end{array}$$

Substitute this solution in the original system

$$u_x^2 - 4u,\ u_{xy}v_y - u + 1,\ v_{xx} - u_x.$$

Observe that the first and last equations are fulfilled automatically, and the second one implies

$$c_1 = \pm\frac{1}{\sqrt{2}},\quad c_2 = c_1 c_3,\quad c_5 = \frac{c_2^2 - 1}{c_1}$$

(the solution depends on three parameters: c_3, c_4, c_6).

[4] This observation was pointed out to the author by Vladimir Gerdt.

7. Conclusion

The concepts of differential Gröbner fan and universal characteristic set introduced in this paper raise several further questions about the algorithmic properties of differential ideals:

(1) How efficiently can a universal characteristic set be constructed?

It seems obvious that since the universal characteristic set includes characteristic sets for all rankings, the complexity of its construction cannot be less than the complexity of construction of a particular characteristic set. Oddly enough, for the above example we have observed just the opposite! Our program computes the universal characteristic in several seconds, while there exists a particular ranking, for which the direct application of the Rosenfeld–Gröbner algorithm takes much longer.

This suggests that not only the complexity of computation of the universal characteristic set is much less than the sum of complexities of computations of all particular characteristic sets (because many polynomials are shared by several characteristic sets), but there is another mechanism involved. It may well be the case that if a polynomial participates in several characteristic sets corresponding to several different rankings, the cost of its computation depends on the ranking. Thus, an algorithm which computes several characteristic sets simultaneously (possibly in parallel) and uses intermediate polynomials obtained for one ranking to perform reductions w.r.t. another ranking, may perform better than the direct computation of a single characteristic set.

The same idea can be applied in the algebraic case to compute characteristic sets or Gröbner bases of algebraic ideals more efficiently, especially when no monomial ordering is given a priori.

(2) Can universal characteristic sets be used to solve PDE systems?

It has been the case in our example that the universal characteristic set happened to contain a linear polynomial $v_{xxx} - 2$, which has played a key role for the solution of the system. Since a priori it is not clear whether there exists a ranking which would yield a linear polynomial in the characteristic set and, if so, which ranking it is, it seems reasonable to compute the universal characteristic set, in order to look for the polynomials that facilitate the solution of the system.

(3) How to find the minimal universal characteristic set?

It has been just a matter of chance that we succeeded at obtaining a minimal characteristic set for our example, as well as at proving the minimality. However, the following general approach seems feasible: one can consider the sum of algebraic ideals corresponding to particular characteristic sets, compute the universal Gröbner basis of this ideal, and try to obtain a universal characteristic set from this basis.

(4) Do arbitrary (radical) differential ideals always have strong characteristic sets?

If this turns out to be true, the differential Gröbner fan is finite for these ideals as well. However, the role of characteristic sets for arbitrary (as opposed to prime) ideals is not as important, since, in general, the ideal that has a given characteristic set may not be unique.

(5) Can one efficiently construct a universal regular (Boulier et al., 1995)/characterizable (Hubert, 2000) decomposition of a radical differential ideal?

Acknowledgements

The author is very grateful to Carlo Traverso for the invitation to work in his research group at the University of Pisa and for financial support, to Vladimir Gerdt for productive discussions during the work on this paper, and to Alexey Ovchinnikov and Marina Kondratieva for reading the draft of the paper. The author is also thankful to the anonymous reviewers for their comments and suggestions.

The author was supported by the Progetto di Interesse Nazionale "Algebra Commutativa e Computazionale" of the Italian "Ministero dell'Istruzione dell'Università della Ricerca Scientifica Tecnologica", RFBR grant no. 02-01-01033, and NSERC grant PDF-301108-2004.

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