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On the Basis Property for the Root Vectors of Some Nonselfadjoint Operators*

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When does the root system of a nonselfadjoint operator form a Riesz basis of a Hilbert space? This question is discussed in the paper.

1. INTRODUCTION

Let A be a linear, densely defined operator on a Hilbert space H, of the form $A = L + T$, where L is a selfadjoint operator with discrete spectrum $\{\lambda_n\}$, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ $D(A) = D(L)$, $D(A) \equiv \operatorname{dom} A$. We assume that

$$\lambda_n = cn^p(1 + o(n^{-1}))\, c, \qquad c = \text{const} > 0, \qquad p > 0. \tag{1}$$

This assumption is satisfied by some elliptic differential and pseudo-differential operators (PDO). An operator T is said to be subordinate to L if

$$|Tf| \leqslant M\, |L^a f|, \qquad a < 1, \quad \forall f \in D(L^a); \tag{2}$$

M here and in the sequel denotes various constants, and $|T|$ the norm of operator T on H.

Under assumptions (1), (2) the operator $A = L + T$ has a discrete spectrum, that is, every point of its spectrum is an eigenvalue of finite algebraic multiplicity. If λ is an eigenvalue of A, then the linear hull of the corresponding eigenvectors is called the eigenspace corresponding to λ. Let h_j be an eigenvector, $Ah_j = \lambda h_j$. If the equation $Ah_j^{(1)} = \lambda h_j^{(1)} + h_j$ is solvable then the chain $\{h_j, h_j^{(1)}, \ldots, h_j^{(s_j)}\}$, $Ah_j^{(s_j)} = \lambda h_j^{(s_j)} + h_j^{(s_j-1)}$ is called the Jordan chain corresponding to the pair (λ, h_j). The number $s_j + 1$ is called the length of this chain if the equation $Ah - \lambda h = h_j^{(s_j)}$ has no solutions. If λ has a finite algebraic multiplicity then $s_j < \infty$. The vectors $h_j^{(m)}$ are called root vectors (or associated vectors). The union of eigen and root vectors is called the root system of A. A system $\{g_j\}_{j=1}^{\infty}$ of vectors is called linearly independent if any

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finite set of these vectors is linearly independent. Consider a system $\{g_j\}$ of linearly independent vectors in H. If for all j the vector g_j does not belong to the closure of the linear hull of vectors $g_1,..., g_{j-1}, g_{j+1},...$ then the system $\{g_j\}$ is called minimal. A minimal system $\{g_j\}$ forms a basis of H if any $g \in H$ can be uniquely represented as $g = \sum_{j=1}^{\infty} c_j g_j$. We shall write $A \in B(A)$ (or $A \in B$) if its root system forms a basis for H.

A minimal system $\{g_j\}$ forms a Riesz basis of H if there exists a homomorphism B (linear bijection of H onto H) which sends an orthonormal basis $\{f_j\}$ onto $\{g_j\}$, i.e., $Bf_j = g_j, \forall j$. A minimal system $\{g_j\}$ forms a Riesz basis with brackets of H if there exists a homomorphism B which sends $\{F_j\}$ onto $\{G_j\}$, i.e., $BF_j = G_j$. Here $\{F_j\}$ is the collection of subspaces constructed as follows. Let $m_1 < m_2 < \cdots$ be an infinite increasing sequence of integers; then F_1 is the hull of vectors $f_1,..., f_{m_1}$, F_j is the hull of vectors $f_{m_{j-1}+1}, f_{m_{j-1}+2},..., f_{m_j}$, and G_j is defined similarly. Now we can give the basic definition in which a new word "basisness" is used.

DEFINITION. A linear operator A with discrete spectrum possesses the basisness property if its root system forms a Riesz basis with brackets for H. In this case we write $A \in R_b(H)$ (or $A \in R_b$). If the root system of A forms a Riesz basis we write $A \in R(H)$ (or $A \in R$).

The purpose of this paper is to give some conditions for $A \in R$ to be true. These conditions will be essentially conditions (1), (2). In the literature there are some results related to the question of basisness. In Kato [1, Section V.4] a theorem on basisness for an operator $L + T$ is proved under the following assumptions: The eigenvalues of L are simple and $\lambda_j - \lambda_{j-1} \to +\infty$ as $j \to \infty$, and T is bounded. In [2] some conditions for completeness of root system of some nonselfadjoint operators are given. In [3–7] some conditions for $A \in R_b$ are given and in [6, 7] applications to diffraction and scattering theory are presented. One of the main results [4] can be formulated as follows: $A \in R_b$ if $p(1-a) \geqslant 1$. The assumption about the selfadjointness of L can often be replaced by the assumption of the normality of L, provided that it is known a priori that the eigenvalues of L are concentrated near some rays in the complex plane.

In this paper we give a simple method to prove that $A \in R$ under the assumption $p(1-a) \geqslant 2$. The method is based on some estimates of the resolvent of A [10].

The main result is the following:

THEOREM. *Let* (1) *and* (2) *hold and* $p(1-a) \geqslant 2$. *Then* $A \in R$.

2. Proof

Let

$$P_j = -\frac{1}{2\pi i}\int_{C_j} (A - \lambda I)^{-1}\, d\lambda \tag{3}$$

denote the projector on the root space L_j of the operator A, corresponding to the eigenvalue $\lambda_j(A)$, where C_j is a circle with the center $\lambda_j(A)$ so small that there are no other eigenvalues inside the circle. In order to prove that $A \in B$ it is sufficient to prove that

$$\sum_{j=1}^{N} P_j f \to f \qquad \text{as} \quad N \to \infty, \quad \forall f \in H, \tag{4}$$

where the arrow denotes convergence in H. In order to prove additionally that $A \in R$ it is necessary and sufficient to prove that [2, p. 310, 334]

$$\sup_J \left\| \sum_{j \in J} P_j \right\| < \infty, \tag{5}$$

where J is an arbitrary finite subset of the set (1, 2, 3,...) of all integers.

We start with the identity

$$(2\pi i\lambda)^{-1} f = -(2\pi i)^{-1} R_\lambda f + (2\pi i\lambda)^{-1} R_\lambda A f,$$
$$f \in D(A), \quad R_\lambda = (A - \lambda I)^{-1} \tag{6}$$

and integrate this identity over the contour $\Gamma_m : |\lambda| = r_m = (\lambda_m + \lambda_{m+1})/2$. Note that the distance d_m between $\{\lambda_j\}$ and the circle $|\lambda| = r_m$ satisfies the inequality

$$d_m \geqslant (\lambda_{m+1} - \lambda_m)/2. \tag{7}$$

After integration we get

$$f = \sum_{j=1}^{N_m} P_j f + a_m + b_m, \tag{8}$$

where

$$a_m = (2\pi i)^{-1} \int_{\Gamma_m} \lambda^{-1} R_\lambda L f\, d\lambda,$$
$$b_m = (2\pi i)^{-1} \int_{\Gamma_m} \lambda^{-1} R_\lambda T f\, d\lambda. \tag{9}$$

It is easy to prove Lemma 1.

LEMMA 1. *Under assumptions* (1), (2) *operator* $A = L + T$ *is closed, its spectrum is discrete and the eigenvalues of* A *lie in the set*:

$$K = \bigcup_{j=1}^{\infty} \{\lambda : |\lambda - \lambda_j| < |\lambda_j|^a Mq\}, \qquad q > 1, \tag{10}$$

where M *and* a *are the constants from* (2).

While this statement can be found in the literature [1, 4, 6] we give its proof for the convenience of the reader after the proof of the theorem.

To prove that $A \in B$ it is sufficient to prove that

$$a_m \to 0, \qquad b_m \to 0 \quad \text{as} \quad m \to \infty. \tag{11}$$

Both terms can be considered similarly. Let us consider the first term. If $R^\circ_\lambda = (L - \lambda)^{-1}$, then

$$R_\lambda = \{(L-\lambda)(I + R^\circ_\lambda L^a L^{-a} T)\}^{-1} = (I + R^\circ_\lambda L^a T_1)^{-1} R^\circ_\lambda,$$
$$T_1 = L^{-a} T, \quad \|T_1\| \leqslant M, \tag{12}$$

$$|R^\circ_\lambda L^a| = \sup_j \frac{|\lambda_j|^a}{|\lambda - \lambda_j|} \leqslant \sup_j \frac{|\lambda_j|^a}{|r_m - \lambda_j|}$$
$$\leqslant M \frac{|\lambda_m|^a}{d_m} \leqslant \frac{M}{m^{p(1-a)-1}}. \tag{13}$$

Here M denotes various constants, m is assumed to be large, so that from (1) and (7) it follows that $\lambda_m \sim cm^p$, $d_m \geqslant Mm^{p-1}$. It is clear now that $p(1-a) > 1$ implies the following estimate provided that $|\lambda|$ is sufficiently large and runs through the set $\{r_m\}$:

$$|R^\circ_\lambda L^a| \leqslant M |\lambda|^{-\gamma}, \qquad \gamma = p^{-1}\{p(1-a) - 1\} = 1 - a - p^{-1} > 0. \tag{14}$$

Further we get

$$|R^\circ_\lambda| \leqslant \max_j \frac{1}{|\lambda - \lambda_j|} \leqslant \frac{M}{d_m} \leqslant \frac{M}{|\lambda|^{1-p^{-1}}} \tag{15}$$

since for large m from $\lambda_m \sim cm^p$ it follows that $m \sim c_1 \lambda_m^{1/p}$.

From (12), (14), (15) it follows that

$$|R_\lambda| \leqslant \frac{M}{|\lambda|^{1-p^{-1}}} \tag{16}$$

provided that $\gamma > 0$, i.e., $1 - p^{-1} > a$. All estimates (13)–(16) are given under the assumptions that $|\lambda| = r_m$, and m is sufficiently large.

It is well known that the eigensystem of the selfadjoint operator L with discrete spectrum forms an orthogonal basis for H. For $A = L$ an identity of the type (8) is

$$f = \sum_{j=0}^{N_m} P_j^\circ f + a_m^\circ, \qquad a_m^\circ = (2\pi i)^{-1} \int_{\Gamma_m} \lambda^{-1} R_\lambda^\circ Lf \, d\lambda, \tag{17}$$

where

$$P_j^\circ = -(2\pi i)^{-1} \int_{C_j^\circ} R_\lambda^\circ \, d\lambda, \tag{18}$$

and C_j° is a small circle with the center λ_j.

For the selfadjoint operator

$$f = \lim_{m\to\infty} \sum_{j=1}^{N_m} P_j f \quad \text{and} \quad a_m^\circ \to 0 \quad \text{as} \quad m \to \infty. \tag{19}$$

Thus in order to prove that $a_m \to 0$ as $m \to \infty$ it is sufficient to prove that

$$a_m - a_m^\circ \to 0 \quad \text{as} \quad m \to \infty. \tag{20}$$

To this end consider

$$\begin{aligned} |(R_\lambda - R_\lambda^\circ) Lf| &= |R_\lambda T R_\lambda^\circ Lf| \\ &\leqslant M |R_\lambda| \, |L^a R_\lambda^\circ| \, |Lf| \\ &\leqslant M |\lambda|^{-2(1-p^{-1})+a} |Lf|, \\ |R_\lambda Tf| &\leqslant |(R_\lambda - R_\lambda^\circ) Tf| + M |R_\lambda^\circ L^a| \, |f| \\ &\leqslant |R_\lambda^\circ T R_\lambda Tf| + M |\lambda|^{-\gamma} |f| \\ &\leqslant M |R_\lambda^\circ L^a| \, |R_\lambda Tf| + M |\lambda|^{-\gamma} |f| \\ &\leqslant M |\lambda|^{-\gamma} |R_\lambda Tf| + M |\lambda|^{-\gamma} |f|. \end{aligned} \tag{21}$$

If $\gamma > 0$ and $|\lambda|$ is sufficiently large we get

$$|R_\lambda Tf| \leqslant M |\lambda|^{-\gamma} |f|. \tag{22}$$

If $\gamma > 0$ and $\gamma + 1 - p^{-1} > 0$, i.e., $p(1-a) > 1$ and $p(2-a) > 2$, then from (21), (22) and (9) equalities (11) follow for $f \in D(L)$. The idea of the following argument is to prove (11) for any $f \in H$ and therefore prove that $A \in B$. To this end let us first give the proof for a simple case when $A = L$.

In this case the proof that $a_m^\circ \to 0$ as $m \to \infty$ for any $f \in H$ can be given as follows:

$$a_m^\circ = f - \sum_{j=1}^{N_m} P_j^\circ f$$

is a linear operator which is a bounded operator since P_j° are orthogonal projectors. Thus if $a_m^\circ = a_m^\circ(f) \to 0$ on a dense set in H this is true on all H. To apply this idea to a_m we must prove that $|\sum_{j=1}^{N_m} P_j| \leqslant M$, where M does not depend on m. To prove this it is sufficient to prove that

$$I_m \equiv \left| \sum_{j=1}^{N_m} (P_j - P_j^\circ) \right| \leqslant M. \tag{23}$$

We have

$$I_m \leqslant \frac{1}{2\pi} \left| \int_{\Gamma_m} (R_\lambda - R_\lambda^\circ) f \, d\lambda \right| \leqslant \frac{1}{2\pi} \left| \int_{\Gamma_m} R_\lambda T R_\lambda^\circ f \, d\lambda \right|$$
$$\leqslant M \frac{|\lambda| \, |f|}{|\lambda|^{1-p^{-1}+\gamma}} = \frac{M |f|}{|\lambda|^{1-a-2p^{-1}}}. \tag{24}$$

Therefore if

$$p \geqslant \frac{2}{1-a}, \qquad a < 1 \tag{25}$$

the above argument shows that $a_m(f) \to 0$ for all $f \in H$, so that $A \in B$. But actually inequality (24) shows more: if (25) holds then $A \in R$ (i.e., the root system of A forms a Riesz basis without brackets of H). Indeed

$$\left| \sum_J P_j \right| \leqslant \left| \sum_J P_j^\circ \right| + \left| \sum_J (P_j - P_j^\circ) \right| \leqslant M_1 + M_2 \leqslant M \tag{26}$$

for any subset J of integers. This completes the proof of the theorem.

Remark 1. From (25) both inequalities $p(1-a) > 1$ and $p(2-a) > 2$ follow.

Proof of Lemma 1. From (12) it follows that $\lambda \notin \sigma(A)$ if $|R_\lambda^\circ L^a| M < 1$. From (13) and (10) it follows that if $\lambda \notin K$, then

$$M |R_\lambda^\circ L^a| \leqslant M \sup_j \frac{|\lambda_j|^a}{|\lambda - \lambda_j|} < \sup_j \frac{M |\lambda_j|^a}{Mq |\lambda_j|^a} \leqslant q^{-1} < 1,$$

so that $\lambda \notin \sigma(A)$. Thus $\sigma(A) \subset K$, where K is defined in (10). Discreteness of $\sigma(A)$ and the closedness of A can be proved under weaker assumptions [8, 10].

3. Generalizations

Assumption (1) can be substituted by the following assumption:

$$\lambda_m^{a+1}(\lambda_{m+1} - \lambda_m)^{-2} \to 0 \qquad \text{as} \quad m \to \infty, \tag{1'}$$

where a is defined by formula (2).

PROPOSITION 1. *From* (1′) *and* (2) *it follows that* $A \in R$.

Proof. Let $|\lambda| = (\lambda_{m+1} + \lambda_m)/2$, $d_m = \lambda_{m+1} - \lambda_m$, M be various positive constants which do not depend on m. We need to prove that: (i) $|\lambda|\,|R_\lambda - R_\lambda^\circ| \to 0$ as $|\lambda| \to \infty$, (ii) $|(R_\lambda - R_\lambda^\circ)L| \to 0$ as $|\lambda| \to \infty$, (iii) $|R_\lambda T| \to 0$ as $|\lambda| \to \infty$. We have: $R_\lambda - R_\lambda^\circ = -R_\lambda T R_\lambda^\circ$, $|R_\lambda^\circ| \leqslant M d_m^{-1}$, $|R_\lambda^\circ L^a| \leqslant M |\lambda_m|^a d_m^{-1}$, $|R_\lambda| \leqslant |R_\lambda^\circ|\,|(I + R_\lambda^\circ T)^{-1}| \leqslant M d_m^{-1}$, $|TR_\lambda^\circ| + |R_\lambda^\circ T| \leqslant M |\lambda_m|^a d_m^{-1}$. Without loss of generality we can assume that L^{-1} exists (otherwise we can substitute L by $L + \varepsilon I$ where ε is a small number and $(L + \varepsilon I)^{-1}$ exists; in this case T should be substituted by $T - \varepsilon I$ and condition (2) holds for $T - \varepsilon I$ and $L + \varepsilon I$). From (1′) it follows that $\lambda_m^a d_m^{-1} \to 0$ as $m \to \infty$, because $\lambda_m \to +\infty$ and $a < 1$. We have: (i) $|\lambda|\,|R_\lambda - R_\lambda^\circ| \leqslant |\lambda|\,|R_\lambda T R^0| \leqslant M\lambda_m^{1+a} d_m^{-2} \to 0$, $m \to \infty$ (ii) $|(R_\lambda - R_\lambda^\circ)L| = |R_\lambda T R_\lambda^\circ L| \leqslant M\lambda_m^{1+a} d_m^{-2} \to 0$, $m \to \infty$ (iii) $|R_\lambda T| \leqslant |(R_\lambda - R_\lambda^\circ)T| + |R_\lambda^\circ T| \leqslant M\lambda_m^{2a} d_m^{-2} + M\lambda_m^a d_m^{-1} \to 0$, $m \to \infty$.

Remark 2. If $\lambda_m \sim cm^p$ and $d_m \geqslant Mm^{p-1}$ then (1′) implies that $p(1-a) > 2$. To get the condition $p(1-a) \geqslant 2$ as a sufficient condition for $A \in R$ we add the argument given in the paragraph above Eq. (23).

Remark 3. If a in (2) can be taken arbitrarily large negative and there exists some $b \in (-\infty, \infty)$ such that

$$d_m \geqslant M\lambda_m^b, \tag{1''}$$

then (1′) holds.

Instead of (1) for a wide class of PDO the following estimate is known:

$$\lambda_n = cn^p(1 + O(n^{-\delta})), \qquad c > 0, \quad p > 0, \quad \delta > 0. \tag{27}$$

In this case our arguments lead to

PROPOSITION 2. *Let* $p(1-a) > 2$, $0 < \delta_1 < \delta$, *where* δ *is defined in* (27) *and* $c_1 > 0$ *be a constant. Then there exists a sequence of integers* $m_n \sim c_1 n^{1/\delta_1}$ *such that the system of the subspaces* $\{P^{(n)}H\}_{n=1}^{\infty}$ *forms a Riesz basis of* H, *where* $P^{(n)} = \sum_{j=m_n}^{m_{n+1}} P_j$ *and* P_j *is defined by formula* (3). *It means that* $A \in R_b$ *and the sequence* m_n *defines the bracketing.*

The sequence $\{P^{(n)}H\}$ plays the role of the sequence $\{G_n\}$ of the subspaces defined in the Introduction. We need a few lemmas to prove this proposition.

LEMMA 1. *If* $\lambda_n = cn^p(1 + O(n^{-\delta}))$, $\delta > 0$ *then* $N(\lambda) = \sum_{\lambda_n < \lambda} 1 = (\lambda c^{-1})^{1/p}(1 + O(\lambda^{-\delta/p}))$.

Proof. This statement follows from the fact that $\lambda = \lambda(n)$ and $N(\lambda)$ are reciprocal functions.

In what follows we assume that the assumption of Lemma 1 holds.

LEMMA 2. *For sufficiently large n and m,* $n < m$, $0 < q_1 \leqslant nm^{-1} \leqslant q_2 < 1$ *there exist eigenvalues* $\lambda^{(1)}$ *and* $\lambda^{(2)}$, $\lambda_n \leqslant \lambda^{(1)} < \lambda^{(2)} \leqslant \lambda_m$ *such that* $\lambda^{(2)} - \lambda^{(1)} \geqslant c_1 m^{p-1}$ *and the interval* $(\lambda^{(2)}, \lambda^{(1)})$ *is free from the eigenvalues.*

Proof. There are $m - n$ eigenvalues (counting multiplicity) on the segment $(\lambda_n, \lambda_m]$. Thus there exists at least a couple of eigenvalues $\lambda_n < \lambda^{(1)} < \lambda^{(2)} \leqslant \lambda_m$ such that there are no eigenvalues in the interval $(\lambda^{(1)}, \lambda^{(2)})$ and $\lambda^{(2)} - \lambda^{(1)} \geqslant (\lambda_m - \lambda_n)/(m - n) \geqslant c(m^p - n^p)/(m - n) - O(m^{p-\delta}/(m - n)) \geqslant c_1 m^{p-1}$, where $c_1 = c_1(q_1, q_2)$. By c_1 we denote various positive constants.

LEMMA 3. *Suppose that* $m = m(n)$, $1 - d(n) \leqslant m^{-1}(n)\, n \leqslant 1 - b(n)$, $b(n)/d(n) \geqslant c_1$, $b(n)\, n^{\delta} \to \infty$ *as* $n \to \infty$, $d(n) \to 0$, $n \to \infty$. *Then the conclusion of Lemma* 2 *holds.*

Proof. It is similar to the proof of Lemma 2. The last step is slightly different:

$$
\begin{aligned}
c\,\frac{m^p - n^p - O(m^{p-\delta})}{m - n} &= cm^{p-1}\,\frac{1 - (n/m)^p - O(m^{-\delta})}{1 - (n/m)} \\
&\geqslant cm^{p-1}\,\frac{1 - (1 - b(n))^p - O(m^{-\delta})}{d(n)} \\
&\geqslant cm^{p-1}\,\frac{0.5pb(n)(1 - O(n^{-\delta}b^{-1}(n))}{d(n)} \\
&\geqslant c_1 m^{p-1}.
\end{aligned}
$$

Here we used the inequality $1 - (1 - x)^p \geqslant 0.5px$ which holds for small x.

Proof of Proposition 2. We can take $b(n) = n^{-\delta_1}$, $0 < \delta_1 < \delta$, $d(n) = b(n)$. In this case $(m_{n+1}/m_n) = 1 + b/m_n^{\delta_1}$ and $m_n \sim (\delta_1 b)^{1/\delta_1} n^{1/\delta_1}$. From this and Lemmas 3, 2 and the argument given in the proof in Section 2 Proposition 2 follows.

EXAMPLE 1. Let $Qf = \int_\Gamma r_{st}^{-1} \exp(ikr_{st}) f(t)\, dt$, where Γ is a smooth closed surface in R^3, $k > 0$, $r_{st} = |s - t|$. Then $Q = Q_0 + Q_1$, where $Q_0 = \operatorname{Re} Q$, $Q_1 = i \operatorname{Im} Q$,

$$Q_0 f = \int_\Gamma r_{st}^{-1} \cos(kr_{st}) f(t)\, dt,$$

$$Q_1 f = i \int_\Gamma r_{st}^{-1} \sin(kr_{st}) f(t)\, dt.$$

Operators Q_0, Q_1 are pseudo-differential of orders -1 and $-\infty$, respectively [5, 6], $\lambda_n(Q_0) \sim c_1 n^{-1/2}$, $c_1 = \text{const}$.

Let us assume that $L = Q_0^{-1}$ exists (without loss of generality, see [6]). Then $\lambda_n(L) \sim cn^{1/2}$, $c = \text{const}$, so that $p = 0.5$, where p is defined in (1). Since in the theorem the unperturbed operator is unbounded we denote $A = (Q_0 + Q_1)^{-1} = (I + LQ_1)^{-1} L = L + T$, $T \equiv -(I + LQ_1)^{-1} LQ_1 L$, we assumed that $(Q_0 + Q_1)^{-1}$ exists again without loss of generality; where $k > 0$ and k^2 is not an eigenvalue of the Laplace operator for the interior Dirichlet problem in the domain D with the boundary Γ it is easy to prove that $(Q_0 + Q_1)^{-1}$ exists [6]. Since $\operatorname{ord} LQ_1 L = -\infty$ we can take the number a in (2) negative and large, so that $p(1 - a) > 2$. Thus $Q \in R$, if (1″) holds, and $Q \in R_b$.

For complex k the order of $\operatorname{Im} Q = -3$, $a = -1$ so that $p(1 - a) = 1$ and $Q \in R_b$ but we cannot assert that $Q \in R$ [10].

EXAMPLE 2. Let $Qf = \int \exp(ikr_{xy})\, r_{xy}^{-1} q(y) f(y)\, dy$, $k > 0$, $\int \equiv \int_{R^3}$. Operator Q plays the principal role in the potential scattering theory. Let us assume that $q \in C_0^\infty(R^3)$, $q(x) \geqslant 0$. Then the operator$Q_1 f = \int \cos(kr_{xy})\, r_{xy}^{-1} q(y)\, dy$ is selfadjoint pseudo-differential operator of order -2 in $H = L^2(R^3; q(x))$; the operator $Q_2 f = i \int \sin(kr_{xy})\, r_{xy}^{-1} q(y)\, dy$ has order $-\infty$ because its kernel is infinitely smooth and $q(y)$ is compactly supported; $\lambda_n(Q_1) \sim cn^{-2/3}$. Thus in this case $p = 2/3$, a can be taken negative and as large as we want, inequality $p(1 - a) \geqslant 2$ holds and the root system of Q forms a Riesz basis of H if (1″) holds, and $Q \in R_b$. If q is not compactly supported additional consideration is needed. It is easy to prove the $Qf \equiv 0$ implies $f = 0$, so that Q^{-1} exists.

In both examples it is an open question whether $Q \in R$ or not.

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REFERENCES

1. T. KATO, "Perturbation Theory for Linear Operators," Springer–Verlag, Berlin/New York, 1966.
2. I. GOHBERG AND M. KREIN, Introduction to the theory of linear nonselfadjoint operators, *Amer. Math. Soc. Transl.* **18**, (1969).
3. A. MARCUS, The root vector expansion of a weakly perturbed selfadjoint operator, *Soviet Math. Dokl.* **3** (1962), 104–108.
4. V. KACNELSON, Conditions under which systems of eigenvectors of some losses of operators form a basis, *Functional Anal. Appl.* **1** (1967), 122–132.
5. M. AGRANOVICH, Summability of series in root vectors of nonselfadjoint elliptic operators, *Functional Anal. Appl.* **10** (1976), 65–74.
6. A. G. RAMM, Nonselfadjoint operators in diffraction and scattering, *Math. Methods Appl. Sci.* **2** (1980), 327–346.
7. A. G. RAMM, Eigenfunction expansion of a discrete spectrum in diffraction problems, *Radio Engrg. Electron. Phys.* **18** (1973), 364–369.
8. A. G. RAMM, Perturbations preserving asymptotics of spectrum, *J. Math. Anal. Appl.* **76** (1980), 10–17.
9. L. HÖRMANDER, The spectral function of an elliptic operator, *Acta Math.* **121** (1968), 193–218.
10. A. G. RAMM, "Theory and Applications of Some New Classes of Integral Equations," Springer-Verlag, New York, 1980.