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ScienceDirect
Journal of Functional Analysis 253 (2007) 287–302
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Non-commutative Arens algebras and their derivations

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Received 6 March 2007; accepted 19 April 2007

Available online 8 June 2007

Communicated by Dan Voiculescu

Abstract

Given a von Neumann algebra M with a faithful normal semi-finite trace τ, we consider the non-commutative Arens algebra $L^{\omega}(M,\tau)=\bigcap_{p\geqslant 1}L^{p}(M,\tau)$ and the related algebras $L_2^{\omega}(M,\tau)=\bigcap_{p\geqslant 2}L^{p}(M,\tau)$ and $M+L_2^{\omega}(M,\tau)$ which are proved to be complete metrizable locally convex $*$-algebras. The main purpose of the present paper is to prove that any derivation of the algebra $M+L_2^{\omega}(M,\tau)$ is inner and all derivations of the algebras $L^{\omega}(M,\tau)$ and $L_2^{\omega}(M,\tau)$ are spatial and implemented by elements of $M+L_2^{\omega}(M,\tau)$. In particular we obtain that if the trace τ is finite then any derivation on the non-commutative Arens algebra $L^{\omega}(M,\tau)$ is inner.
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Keywords: Von Neumann algebras; Non-commutative integration; Arens algebras; Derivations; Spatial derivations; Inner derivations; Operator algebras; Quantum statistical mechanics

1. Introduction

The present paper is devoted to the study of derivations on certain classes of unbounded operator algebras.

Given a (complex) algebra $\mathcal{A}$, a linear operator $d:\mathcal{A}\to\mathcal{A}$ is called a *derivation* if $d(xy)=d(x)y+xd(y)$ for all $x,y\in\mathcal{A}$. Each element $a\in\mathcal{A}$ generates a derivation $d_a:\mathcal{A}\to\mathcal{A}$ defined as $d_a(x)=ax-xa$, $x\in\mathcal{A}$. Such derivations are called *inner* derivations.

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doi:10.1016/j.jfa.2007.04.010

If the element a implementing the derivation d_a belongs to a larger algebra $\mathcal{B}$ containing $\mathcal{A}$ then d_a is called a *spatial* derivation.

It is a general algebraic problem to find algebras which admit only inner derivations. Such examples are:

- finite-dimensional simple central algebras;
- simple unital C^*-algebras;
- algebras $B(X)$ of all bounded linear operators on a Banach space X (cf. [8,16]).

A more general problem is the following one: given an algebra $\mathcal{A}$, does there exist an algebra $\mathcal{B}$ such that:

(i) $\mathcal{A}$ is an ideal in $\mathcal{B}$, so that any element $a \in \mathcal{B}$ defines a derivation on $\mathcal{A}$ by $d_a(x) = ax - xa$, $x \in \mathcal{A}$;
(ii) any derivation of $\mathcal{B}$ is inner;
(iii) any derivation of the algebra $\mathcal{A}$ is spatial and implemented by an element from $\mathcal{B}$?

Examples of algebras for which the answer is positive are:

- simple (non-unital) C^*-algebras;
- the algebra $\mathcal{F}(X)$ of finite rank operators on an infinite-dimensional Banach space X;
- more general standard operator algebras on X, i.e. subalgebras of $B(X)$ which contain $\mathcal{F}(X)$ (cf. [8,16]).

The theory of derivations in operator algebras is an important and well-investigated part of the general theory of operator algebras, with applications in mathematical physics (see, e.g., [7,16,17]). It is well known that every derivation of a C^*-algebra is norm-continuous and that every derivation of a von Neumann algebra is inner. For a detailed exposition of the theory of bounded derivations we refer to the monographs of Sakai [16,17]. A comprehensive study of derivations in general Banach algebras is given in the monograph of Dales [9] devoted to automatic continuity of derivations on various classes of Banach algebras.

Investigations of general unbounded derivations (and derivations on unbounded operator algebras) began much later and were motivated mainly by needs of mathematical physics, in particular by the problem of constructing the dynamics in quantum statistical mechanics.

The development of a non-commutative integration theory was initiated by I. Segal [19], who considered new classes of (not necessarily Banach) algebras of unbounded operators, in particular the algebra $L(M)$ of all measurable operators affiliated with a von Neumann algebra M. Algebraic, order and topological properties of the algebra $L(M)$ are somewhat similar to those of von Neumann algebras, therefore in [4,5] we initiated the study of derivations on the algebra $L(M)$. In the particular commutative case where $M = L^\infty(0;1)$ is the algebra of all essentially bounded measurable complex functions on $(0;1)$, the algebra $L(M)$ is isomorphic to the algebra $L^0(0;1)$ of all measurable functions on $(0;1)$. Recent results of [6] (see also [13]) show that $L^0(0;1)$ admits non-zero (and hence discontinuous) derivations. Therefore the properties of derivations on the unbounded operator algebra $L(M)$ are very far from being similar to those on C^*- or von Neumann algebras.

There are many other classes of unbounded operator algebras, which are important in analysis and mathematical physics like O^*-algebras, GB^*-algebras, EW^*-algebras (see, e.g., [2,18]).

These algebras also can be equipped by appropriate topologies and it is natural to study properties (such as automatic continuity, innerness, spatiality etc.) of their derivations. It is known that each derivation of the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D})$ is inner, and if a subalgebra $\mathcal{A}$ of $\mathcal{L}^+(\mathcal{D})$ contains all finite rank operators on $\mathcal{D}$, then any derivation of $\mathcal{A}$ is spatial and implemented by an element of $\mathcal{L}^+(\mathcal{D})$, where $\mathcal{D}$ is a dense linear subspace of a Hilbert space (see for details [18, Proposition 6.3.2, Corollary 6.3.3]).

Interesting examples of the mentioned algebras are given by the Arens algebra $L^\omega(0,1) = \bigcap_{p\geqslant 1} L^p(0,1)$, introduced in [3], and by its non-commutative generalizations $L^\omega(M,\tau) = \bigcap_{p\geqslant 1} L^p(M,\tau)$, where M is a von Neumann algebra with a faithful normal semi-finite trace τ, and $L^p(M,\tau) = \{x \in L(M)\colon\ \tau(|x|^p) < \infty\}$. Non-commutative Arens algebras were introduced by Inoue [11] and their properties were investigated in [1,22].

The main purpose of the present work is to give a complete description of derivations on the non-commutative Arens algebras $L^\omega(M,\tau)$ and related algebras. In particular for these algebras we obtain the complete solution of the problems mentioned above.

In Section 2 given a von Neumann algebra M with a faithful normal semi-finite trace τ, along with the non-commutative Arens algebra $L^\omega(M,\tau)$ we consider some basic properties of the related algebras $L_2^\omega(M,\tau) = \bigcap_{p\geqslant 2} L^p(M,\tau)$ and $M + L_2^\omega(M,\tau)$ and prove that they are also complete metrizable locally convex *-algebras. Applying the theory of Banach pairs from [12] we describe the predual space of the algebra $M + L_2^\omega(M,\tau)$. This result enables us to apply *-weak compactness of closed bounded sets in the algebra $M + L_2^\omega(M,\tau)$ for the proof of the main results. Namely, in Section 3 we prove that all derivations on the algebra $M + L_2^\omega(M,\tau)$ are inner and any derivation of the Arens algebra $L^\omega(M,\tau)$ is automatically continuous.

Since the algebras $L^\omega(M,\tau)$ and $L_2^\omega(M,\tau)$ are (two-sided) ideals in $M + L_2^\omega(M,\tau)$ any element $a \in M + L_2^\omega(M,\tau)$ defines a derivation on $L^\omega(M,\tau)$ (respectively on $L_2^\omega(M,\tau)$) by $d_a(x) = ax - xa$, $x \in L^\omega(M,\tau)$ (respectively $x \in L_2^\omega(M,\tau)$). The main results (Theorems 3.7 and 3.8) assert that any derivation of the Arens algebra $L^\omega(M,\tau)$ (respectively of the algebra $L_2^\omega(M,\tau)$) is spatial and implemented by some element $a \in M + L_2^\omega(M,\tau)$. In particular if the trace τ is finite then all the above algebras coincide and therefore all derivations on the Arens algebra $L^\omega(M,\tau)$ are inner. As a corollary we obtain that commutative Arens algebras (in particular the algebra $L^\omega(0;1)$) admit only zero derivations.

2. Non-commutative Arens algebras

Let M be a von Neumann algebra with a faithful normal semi-finite trace τ, and denote by $L(M)$ the algebra of all measurable operators affiliated with M.

Given $p \geqslant 1$ put $L^p(M,\tau) = \{x \in L(M)\colon\ \tau(|x|^p) < \infty\}$. It is known that $L^p(M,\tau)$ is a Banach space with respect to the norm

$$\|x\|_p = \bigl(\tau\bigl(|x|^p\bigr)\bigr)^{1/p}, \quad x \in L^p(M,\tau).$$

Consider the intersection

$$L^\omega(M,\tau) = \bigcap_{p\geqslant 1} L^p(M,\tau).$$

It is proved in [1] that $L^{\omega}(M,\tau)$ is a locally convex complete metrizable *-algebra with respect to the topology t generated by the family of norms $\{\|\cdot\|_p\}_{p\geqslant 1}$. Moreover the topology can be defined also by the countable system (sequence) of norms

$$\|x\|'_n=\max\bigl\{\|x\|_1,\|x\|_n\bigr\},\quad n\in\mathbb{N}.$$

The algebra $L^{\omega}(M,\tau)$ is called a (non-commutative) *Arens algebra*.

Non-commutative Arens algebras are special cases of O^*-algebras in the sense of K. Schmüdgen (see, e.g., [2,18]). If the trace τ is finite then $L^{\omega}(M,\tau)$ is also an EW^*-algebra [22]. The dual space for $(L^{\omega}(M,\tau),t)$ was completely described in [1], where it was also proved that $(L^{\omega}(M,\tau),t)$ is a reflexive space if and only if the trace τ is finite.

Recall that a subset K in a linear topological space E is said to be bounded if given any zero neighborhood V in E there exists $\alpha>0$ such that $K\subset\beta V$ for all $\beta>\alpha$.

Since $(L^{\omega}(M,\tau),t)$ is a countably normed space, a subset K in $L^{\omega}(M,\tau)$ is bounded if and only if there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of positive numbers such that $\|x\|_n\leqslant r_n$ for all $x\in K$ and $n\in\mathbb{N}$ (see [21, p. 368]).

A linear topological space E is said to be locally bounded if it admits a bounded neighborhood of zero.

Automatic continuity of derivations on Banach algebras and more general locally bounded F-algebras was investigated in the monograph [9] of H.G. Dales. Arens algebras are not locally bounded in general; moreover it is not difficult to see that $L^{\omega}(M,\tau)$ is locally bounded if and only if the von Neumann algebra M is finite-dimensional. Therefore most of results from [9] cannot be applied to the case of Arens algebras.

Now let us recall the notion of a Banach pair (see [12]). Let (E,t_E) be a Hausdorff linear topological space over the field of complex numbers, and let $(A,\|\cdot\|_A)$ and $(B,\|\cdot\|_B)$ be Banach spaces which are linear subspaces of (E,t_E) such that the topology t_E induces on A and B topologies which are weaker than the topologies defined by the norms $\|\cdot\|_A$ and $\|\cdot\|_B$, respectively. This means exactly that A and B are topologically imbedded into (E,t_E). In this case we say that A and B define a *Banach pair*. Each Banach pair defines a couple of Banach spaces $A\cap B$ and $A+B$ with the norms

$$\|x\|_{A\cap B}=\max\bigl\{\|x\|_A,\|x\|_B\bigr\},\quad x\in A\cap B,$$
$$\|x\|_{A+B}=\inf\bigl\{\|a\|_A+\|b\|_B\colon x=a+b,\ a\in A, b\in B\bigr\},$$

respectively.

A Banach space $(Z,\|\cdot\|_Z)$ is said to be *intermediate* for the Banach pair A and B, if the continuous embeddings

$$A\cap B\subset Z\subset A+B$$

are valid.

Further we shall need the following result from [12, p. 27].

Lemma 2.1. *If the intersection $A\cap B$ is dense in each member of the Banach pair A and B, then the dual spaces A' and B' also form a Banach pair. Moreover $(A\cap B)'$ is isometrically isomorphic to the space $A'+B'$, and $(A+B)'$ is isometrically isomorphic to the space $A'\cap B'$.*

It is known that [20] for the Banach pair $L^{p_1}(M,\tau)$ and $L^{p_2}(M,\tau)$, $p_1 < p_2$, any space $L^p(M,\tau)$, $p \in [p_1; p_2]$, is intermediate. Therefore

$$\|x\|_p \leqslant c(p_1, p_2, p)\max\bigl\{\|x\|_{p_1}, \|x\|_{p_2}\bigr\}$$

for all $x \in L^{p_1}(M,\tau) \cap L^{p_2}(M,\tau)$, where $c(p_1, p_2, p)$ is a fixed positive number depending on p_1, p_2, p.

On the spaces

$$\bigcap_{n=2}^{m} L^n(M,\tau) \quad \text{and} \quad \sum_{n=2}^{m} L^{q_n}(M,\tau),$$

consider respectively the norms

$$\|x\|_m^0 = \max\bigl\{\|x\|_n\colon n = \overline{2,m}\bigr\}, \quad x \in \bigcap_{n=2}^{m} L^n(M,\tau),$$

$$\|y\|_m' = \inf\left\{\sum_{n=2}^{m} \|y_n\|_{q_n}\colon y_n \in L^{q_n}(M,\tau),\ y = \sum_{n=2}^{m} y_n\right\},$$

where $\frac{1}{n} + \frac{1}{q_n} = 1$, $n = \overline{2,m}$, $m \in \mathbb{N} \setminus \{1\}$.

Consider the Banach pair $L^{p_1}(M,\tau)$ and $L^{p_2}(M,\tau)$, $2 \leqslant p_1 \leqslant p_2$. Since the space

$$M \cap L^1(M,\tau) = \bigl\{x \in M\colon \tau\bigl(|x|\bigr) < \infty\bigr\}$$

is dense in all spaces $L^p(M,\tau)$, $p \geqslant 1$, the intersection

$$L^{p_1}(M,\tau) \cap L^{p_2}(M,\tau)$$

is dense in $L^{p_i}(M,\tau)$, $i = 1, 2$. Therefore from Lemma 2.1 by induction on m we obtain the following result.

Proposition 2.2. *The dual space for* $(\bigcap_{n=2}^{m} L^n(M,\tau), \|\cdot\|_m^0)$ *is isometrically isomorphic to* $(\sum_{n=2}^{m} L^{q_n}(M,\tau), \|\cdot\|_m')$ *and the dual space for* $(\sum_{n=2}^{m} L^{q_n}(M,\tau), \|\cdot\|_m')$ *is isometrically isomorphic to* $(\bigcap_{n=2}^{m} L^n(M,\tau), \|\cdot\|_m^0)$. *The duality is given by the bilinear form*

$$\langle x, y\rangle = \tau(xy), \quad x \in \bigcap_{n=2}^{m} L^n(M,\tau), \quad y \in \sum_{n=2}^{m} L^{q_n}(M,\tau).$$

Since $L^1(M,\tau)$ and $\sum_{n=2}^{m} L^{q_n}(M,\tau)$ also form a Banach pair and $L^1(M,\tau) \cap M$ is dense in both of $L^1(M,\tau)$ and $\sum_{n=2}^{m} L^{q_n}(M,\tau)$, Lemma 2.1 and Proposition 2.2 imply the following.

Proposition 2.3. *The dual space for the Banach space $L^1(M,\tau)\cap(\sum_{n=2}^m L^{q_n}(M,\tau))$ is isometrically isomorphic to $M+\bigcap_{n=2}^m L^n(M,\tau)$. Moreover, given any $f\in(L^1(M,\tau)\cap(\sum_{n=2}^m L^{q_n}(M,\tau)))'$ there exists a unique element $a\in M+\bigcap_{n=2}^m L^n(M,\tau)$ such that*

$$f(x)=\tau(xa),\quad x\in L^1(M,\tau)\cap\left(\sum_{n=2}^m L^{q_n}(M,\tau)\right).$$

Now consider the following space:

$$L_2^\omega(M,\tau)=\bigcap_{p\geqslant 2} L^p(M,\tau)$$

with the topology t_2 generated by the family of norms $\{\|\cdot\|_p\}_{p\geqslant 2}$.

Proposition 2.4. *$(L_2^\omega(M,\tau),t_2)$ is a complete metrizable locally convex *-algebra.*

Proof. From the inequality $\|xy\|_p\leqslant\|x\|_{2p}\|y\|_{2p}$ it easily follows that $L_2^\omega(M,\tau)$ is closed under the multiplication. It is also clear that $L_2^\omega(M,\tau)$ is closed under the involution, i.e. it forms a *-algebra.

Let us show that

$$\bigcap_{n=2}^\infty L^n(M,\tau)=L_2^\omega(M,\tau).$$

Clearly $L_2^\omega(M,\tau)\subset\bigcap_{n=2}^\infty L^n(M,\tau)$. For any $p\geqslant 2$ take natural numbers n_1 and n_2 such that $n_1\leqslant p\leqslant n_2$. Since the space $L^p(M,\tau)$ is intermediate for the Banach pair $L^{n_1}(M_1,\tau)$ and $L^{n_2}(M_2,\tau)$, we have

$$L^{n_1}(M,\tau)\cap L^{n_2}(M,\tau)\subset L^p(M,\tau)$$

and

$$\|x\|_p\leqslant c(n_1,n_2,p)\max\bigl\{\|x\|_{n_1},\|x\|_{n_2}\bigr\}\leqslant c(n_1,n_2,p)\|x\|_{n_2}^0.$$

This means that $\bigcap_{n=2}^\infty L^n(M,\tau)=L_2^\omega(M,\tau)$, and the topology t_2 is generated by the system of norms $\{\|\cdot\|_n^0\}_{n\geqslant 2}$.

Let us show that $(L_2^\omega(M,\tau),t_2)$ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(L_2^\omega(M,\tau),t_2)$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L^p(M,\tau),\|\cdot\|_p)$ for all $p\geqslant 2$ and hence there exists $a_p\in L^p(M,\tau)$ such that $\|x_n-a_p\|_p\to 0$ as $n\to\infty$. We have $a_p=a_q$ for all $p,q\geqslant 2$. Indeed, let $e\in M$ be a projection and $\tau(e)<\infty$. Then

$$(a_p-a_q)e\in L^2(M,\tau),$$

moreover

$$\bigl\|(a_p-a_q)e\bigr\|_2\leqslant\bigl\|(a_p-x_n)e\bigr\|_2+\bigl\|(x_n-a_q)e\bigr\|_2\leqslant\|e\|_{p_1}\bigl\|(a_p-x_n)\bigr\|_p+\|e\|_{q_1}\bigl\|(x_n-a_q)\bigr\|_q,$$

where $p_1, q_1 \in (2; \infty]$, $\frac{1}{p_1} + \frac{1}{p} = \frac{1}{2}$, $\frac{1}{q_1} + \frac{1}{q} = \frac{1}{2}$. Therefore $\|(a_p - a_q)e\|_2 = 0$ and $a_p e = a_q e$ for each projection $e \in M$ with $\tau(e) < \infty$. Since τ is a semi-finite trace this means that $a_p = a_q$.

Therefore, $L_2^\omega(M, \tau)$ is a locally convex complete metrizable *-algebra, with respect to the topology t_2, generated by the family of norms $\{\|\cdot\|_n\}_{n \geqslant 2}$. The proof is complete. □

Note that if $\tau(\mathbf{1}) < \infty$ then $L_2^\omega(M, \tau) = L^\omega(M, \tau)$, and the topology t_2 coincides with the topology t.

On the space $M + L_2^\omega(M, \tau)$ consider the family of norms $\{\|\cdot\|_n''\}_{n \geqslant 2}$ defined by

$$\|x\|_n'' = \inf\bigl\{\|x_1\|_\infty + \|x_2\|_n^0\colon\ x = x_1 + x_2,\ x_1 \in M,\ x_2 \in L_2^\omega(M, \tau)\bigr\}.$$

Let t_0 be the topology on $M + L_2^\omega(M, \tau)$ generated by the family of norms $\{\|\cdot\|_n''\}_{n \geqslant 2}$.

Lemma 2.5. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $M + L_2^\omega(M, \tau)$ such that $\|x_n\|_n'' \to 0$ as $n \to \infty$. Then $x_n \xrightarrow{t_0} 0$ as $n \to \infty$.*

Proof. Let $k \in \mathbb{N} \setminus \{1\}$. Since $\|x_n\|_k'' \leqslant \|x_n\|_n''$ for $n \geqslant k$, then $\|x_n\|_k'' \to 0$ as $n \to \infty$ for all $k \geqslant 2$. Therefore $x_n \xrightarrow{t_0} 0$ as $n \to \infty$. The proof is complete. □

Proposition 2.6. *The algebra $(M + L_2^\omega(M, \tau), t_0)$ is a complete metrizable locally convex *-algebra. Moreover $L^\omega(M, \tau)$ is an ideal in $M + L_2^\omega(M, \tau)$.*

Proof. For $x \in M$ and $y \in L_2^\omega(M, \tau)$ it is clear that $xy, yx \in L_2^\omega(M, \tau)$. Since $L_2^\omega(M, \tau)$ is closed under the multiplication, it follows that $M + L_2^\omega(M, \tau)$ forms a *-algebra.

Take $x, y \in M + L_2^\omega(M, \tau)$. We have

$$\|xy\|_n'' \leqslant \|x\|_{2n}'' \|y\|_{2n}''.$$

Indeed, let $\varepsilon > 0$. Take $x_1, y_1 \in M$, $x_2, y_2 \in L_2^\omega(M, \tau)$ such that $x = x_1 + x_2$, $y = y_1 + y_2$, $\|x\|_n'' \geqslant \|x_1\|_\infty + \|x_2\|_n^0 - \varepsilon$, $\|x\|_n'' \geqslant \|y_1\|_\infty + \|y_2\|_n^0 - \varepsilon$.

Then $\|xy\|_n'' = \|x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2\|_n'' \leqslant \|x_1y_1\|_\infty + \|x_1y_2\|_n^0 + \|x_2y_1\|_n^0 + \|x_2y_2\|_n^0 = \|x_1y_1\|_\infty + \max_{2 \leqslant i \leqslant n}\{\|x_1y_2\|_i\} + \max_{2 \leqslant i \leqslant n}\{\|x_2y_1\|_i\} + \max_{2 \leqslant i \leqslant n}\{\|x_2y_2\|_i\} \leqslant \|x_1y_1\|_\infty + \|x_1\|_\infty \max_{2 \leqslant i \leqslant n}\{\|y_2\|_i\} + \|y_1\|_\infty \max_{2 \leqslant i \leqslant n}\{\|x_2\|_i\} + \max_{2 \leqslant i \leqslant n}\{\|x_2\|_{2i}\} \max_{2 \leqslant i \leqslant n}\{\|y_2\|_{2i}\} \leqslant \|x_1\|_\infty \|y_1\|_\infty + \|x_1\|_\infty \|y_2\|_{2n}^0 + \|y_1\|_\infty \|x_2\|_{2n}^0 + \|x_2\|_{2n}^0 \|y_2\|_{2n}^0 \leqslant (\|x_1\|_\infty + \|x_2\|_{2n}^0)(\|y_1\|_\infty + \|y_2\|_{2n}^0) \leqslant (\|x\|_{2n}'' + \varepsilon)(\|y\|_{2n}'' + \varepsilon)$, i.e. $\|xy\|_n'' \leqslant (\|x\|_{2n}'' + \varepsilon)(\|y\|_{2n}'' + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary this implies that $\|xy\|_n'' \leqslant \|x\|_{2n}'' \|y\|_{2n}''$.

Clearly $\|x^*\|_n'' = \|x\|_n''$. This means that the multiplication and the involution on $M + L_2^\omega(M, \tau)$ are continuous.

Let us show that $M + L_2^\omega(M, \tau)$ is complete. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $M + L_2^\omega(M, \tau)$. Then $\|x_n - x_m\|_k'' \to 0$ as $n, m \to \infty$ for all $k \geqslant 2$. Therefore, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\|_k'' < 2^{-k-1}, \quad k \geqslant 2.$$

Set $y_k = x_{n_{k+1}} - x_{n_k}$, $k \geqslant 2$. Take $b_k \in M$, $b_k' \in L_2^\omega(M, \tau)$ such that $b_k + b_k' = y_k$ and

$$\|b_k\|_\infty + \bigl\|b_k'\bigr\|_k^0 < \|y_k\|_k'' + 2^{-k-1} < 2^{-k}, \quad k \geqslant 2.$$

Then the series $\sum_{k=2}^{\infty} b_k$ and $\sum_{k=2}^{\infty} b'_k$ converge in M and $L_2^{\omega}(M,\tau)$, respectively. Put $b = \sum_{k=2}^{\infty} b_k$ and $b' = \sum_{k=2}^{\infty} b'_k$ and consider the sums $s_n = \sum_{k=2}^{n} b_k$ and $s'_n = \sum_{k=2}^{n} b'_k$. Then $\|s_n + s'_n - (b+b')\|''_n \to 0$ as $n \to \infty$, and by Lemma 2.5 we have $s_n + s'_n - (b+b') \xrightarrow{t_0} 0$. On the other hand, $s_k + s'_k = x_{n_{k+1}} - x_{n_2}$, i.e. $x_{n_k} \xrightarrow{t_0} x_{n_2} + b + b'$. Since (x_n) is a Cauchy sequence in $M + L_2^{\omega}(M,\tau)$ this implies that $x_n \xrightarrow{t_0} x_{n_2} + b + b'$.

Now we shall show that $L^{\omega}(M,\tau)$ is an ideal in $M + L_2^{\omega}(M,\tau)$. Take $x \in L^{\omega}(M,\tau)$ and $a \in L_2^{\omega}(M,\tau)$. Then $x \in L^{2p}(M,\tau)$ and $a \in L^{2p}(M,\tau)$ for all $p \geqslant 1$. Thus $ax, xa \in L^p(M,\tau)$ for all $p \geqslant 1$, and therefore $ax, xa \in L^{\omega}(M,\tau)$, i.e. the algebra $L^{\omega}(M,\tau)$ is an ideal in $L_2^{\omega}(M,\tau)$. Since for $x \in L^{\omega}(M,\tau)$ and $a \in M$ we have that $ax, xa \in L^{\omega}(M,\tau)$, this implies that $L^{\omega}(M,\tau)$ is an ideal in $M + L_2^{\omega}(M,\tau)$. The proof is complete. $\square$

Remark 1. In a similar way it follows that the algebra $L_2^{\omega}(M,\tau)$ is also an ideal in $M + L_2^{\omega}(M,\tau)$.

Remark 2. Note that if $\tau(\mathbf{1}) < \infty$ then $M + L_2^{\omega}(M,\tau) = L^{\omega}(M,\tau)$, and the topology t_0 coincides with the topology t.

Lemma 2.7. $M + L_2^{\omega}(M,\tau) = \bigcap_{m=2}^{\infty}(M + \bigcap_{n=2}^{m} L^n(M,\tau))$.

Proof. It is sufficient to show that $\bigcap_{m=2}^{\infty}(M + \bigcap_{n=2}^{m} L^n(M,\tau)) \subseteq M + L_2^{\omega}(M,\tau)$. Take $a \in \bigcap_{m=2}^{\infty}(M + \bigcap_{n=2}^{m} L^n(M,\tau))$. Then $a \in M + \bigcap_{n=2}^{m} L^n(M,\tau)$ for all $m \geqslant 2$. Therefore there exist $b_m \in M, c_m \in \bigcap_{n=2}^{m} L^n(M,\tau)$ such that $a = b_m + c_m$. Since $L^1(M,\tau) \cap M$ is dense in $\bigcap_{n=2}^{m} L^n(M,\tau)$, there exists $d_m \in L^1(M,\tau) \cap M$ such that $\|c_m - d_m\|_m^0 < 1/m$ for all $m \geqslant 2$. Then $\|a - b_m - d_m\|''_m = \|c_m - d_m\|''_m \leqslant \|c_m - d_m\|_m^0 \to 0$. By Lemma 2.5 we have $b_m + d_m \xrightarrow{t_0} a$. Since $M + L_2^{\omega}(M,\tau)$ is t_2-complete and $b_m + d_m \in M + L_2^{\omega}(M,\tau)$, one has $a \in M + L_2^{\omega}(M,\tau)$. The proof is complete. $\square$

Now let us prove the following equality:

$$\bigcup_{m=2}^{\infty} \sum_{n=2}^{m} L^{q_n}(M,\tau) = \mathrm{Lin}\Bigl(\bigcup_{1<q\leqslant 2} L^q(M,\tau)\Bigr),$$

where $\frac{1}{n} + \frac{1}{q_n} = 1$ and $\mathrm{Lin}(E)$ denotes the linear span of E.

Let $q \in (1,2]$. Take q_n and q_m such that $q_n \leqslant q \leqslant q_m$. Since $L^q(M,\tau)$ is intermediate for the Banach pair $L^{q_n}(M,\tau)$, $L^{q_n}(M,\tau)$, we have

$$L^q(M,\tau) \subset L^{q_n}(M,\tau) + L^{q_n}(M,\tau).$$

Therefore,

$$\bigcup_{m=2}^{\infty} \sum_{n=2}^{m} L^{q_n}(M,\tau) = \mathrm{Lin}\Bigl(\bigcup_{1<q\leqslant 2} L^q(M,\tau)\Bigr).$$

On the space $\bigcup_{m=2}^{\infty}\sum_{n=2}^{m} L^{q_n}(M,\tau)$ consider the norm

$$\|y\|_1' = \inf\left\{\sum_{n=2}^{m}\|y_n\|_{q_n}\colon\ y_n \in L^{q_n}(M,\tau),\, y = \sum_{n=2}^{m} y_n\right\},$$

where $\frac{1}{n}+\frac{1}{q_n}=1$, $n=\overline{2,m}$, $m\in\mathbb{N}\setminus\{1\}$, and on the space $L^1(M,\tau)\cap(\bigcup_{m=2}^{\infty}\sum_{n=2}^{m}L^{q_n}(M,\tau))$ define the norm as

$$\|x\|_1^0 = \max\bigl\{\|x\|_1, \|x\|_1'\bigr\},\quad x\in L^1(M,\tau)\cap\left(\bigcup_{m=2}^{\infty}\sum_{n=2}^{m}L^{q_n}(M,\tau)\right).$$

The main result of this section is the following theorem which describes the predual space of the algebra $M+L_2^{\omega}(M,\tau)$.

Theorem 2.8. *The dual space for $L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau)))$ is isomorphic to $M+L_2^{\omega}(M,\tau)$.*

Proof. Let $a=b+c\in M+L_2^{\omega}(M,\tau)$. Then putting

$$f_a(x)=\tau(xb)+\tau(xc) \tag{1}$$

we define a continuous linear functional f_a on $L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau)))$.

Conversely, let us show that any continuous linear functional on

$$L^1(M,\tau)\cap\left(\mathrm{Lin}\left(\bigcup_{1<q\leqslant 2}L^q(M,\tau)\right)\right)$$

has the form (1).

Let $f\in(L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau))))'$. Since the restriction of the linear functional f on $L^1(M,\tau)\cap(\sum_{n=2}^{m}L^{q_n}(M,\tau))$ is continuous, by Proposition 2.3 there exists $a_m=b_m+c_m\in M+\bigcap_{n=2}^{m}L^n(M,\tau)$ such that

$$f(x)=\tau(xb_m)+\tau(xc_m),\quad x\in L^1(M,\tau)\cap M.$$

Thus $\tau(xb_i)+\tau(xc_i)=\tau(xb_j)+\tau(xc_j)$ for all $i,j\geqslant 2$. Therefore,

$$\tau\bigl(x(b_i-b_j)\bigr)=\tau\bigl(x(c_j-c_i)\bigr),\quad x\in L^1(M,\tau)\cap M. \tag{2}$$

Since $L^1(M,\tau)\cap M$ is dense in $L^1(M,\tau)\cap(\sum_{n=2}^{m}L^{q_n}(M,\tau))$ for all $m\in\mathbb{N}\setminus\{1\}$, by (2) we obtain that $b_i-b_j=c_j-c_i$, i.e. $a_i=b_i+c_i=b_j+c_j=a_j$ for all $i,j\geqslant 2$. Thus $a_i=a_j$ for all $i,j\geqslant 2$. By Lemma 2.7 it follows that $a_2\in M+L_2^{\omega}(M,\tau)$. Since $L^1(M,\tau)\cap M$ is dense in $L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau)))$, we have $f(x)=\tau(xb_1)+\tau(xc_1)$ for all $x\in L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau)))$. The uniqueness of the element a_2 follows from the density of the set $L^1(M,\tau)\cap M$ in $L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau)))$.

Now in a standard way one proves that the map $a\mapsto f_a$ is a linear isomorphism between $M+L_2^{\omega}(M,\tau)$ and $(L^1(M,\tau)\cap(\mathrm{Lin}(\bigcup_{1<q\leqslant 2}L^q(M,\tau))))'$. The proof is complete. $\square$

Theorem 2.8 implies the following

Corollary 2.9. *If $\tau(\mathbf{1}) < \infty$ then the dual space for* $\mathrm{Lin}(\bigcup_{1<q\leqslant 2} L^q(M,\tau))$ *is isomorphic to* $L^\omega(M,\tau)$.

For general Arens algebras the predual space does not exist but the dual space was described in [1] in the following way.

Proposition 2.10. *The dual space for $L^\omega(M,\tau)$ is isomorphic to* $\mathrm{Lin}(\bigcup_{1<q\leqslant\infty} L^q(M,\tau))$.

Since any closed bounded subset in the dual space of a locally convex space is *-weakly compact, Theorem 2.8 implies the following.

Corollary 2.11. *Each closed bounded subset in $M + L_2^\omega(M,\tau)$ is *-weakly compact.*

3. Derivations of Arens algebras

Let $\mathcal{A}$ be a complex algebra and let E be a complex linear space. Recall that E is called a left $\mathcal{A}$-module (respectively right $\mathcal{A}$-module) if a bilinear map $(a,x) \mapsto a\cdot x$ (respectively $(a,x)\mapsto x\cdot a$) from $\mathcal{A}\times E$ into E is defined such that

$$a\cdot(b\cdot x) = ab\cdot x \quad \text{(respectively } (x\cdot a)\cdot b = x\cdot ab\text{)},$$

for all $a,b\in\mathcal{A}$, $x\in E$.

E is said to be $\mathcal{A}$-bimodule if E is simultaneously a left and right $\mathcal{A}$-module such that

$$a\cdot(x\cdot b) = (a\cdot x)\cdot b,$$

for all $a,b\in\mathcal{A}$, $x\in E$.

Let $\mathcal{A}$ be a Banach algebra and let E be a Banach space. If E is a $\mathcal{A}$-bimodule and the maps $(a,x)\mapsto a\cdot x$ and $(a,x)\mapsto x\cdot a$ are continuous, then E is called a Banach $\mathcal{A}$-bimodule.

For example, the Banach space $L^p(M,\tau)$, $p\geqslant 1$, is a Banach M-bimodule. Indeed, since for all $a\in M$ and $x\in L^p(M,\tau)$ one has $ax\in L^p(M,\tau)$, $xa\in L^p(M,\tau)$ and $\|ax\|_p\leqslant\|a\|_\infty\|x\|_p$, where $\|\cdot\|_\infty$ is the C^*-norm on M, the space $L^p(M,\tau)$ is a Banach M-bimodule. Therefore, the space $M+L^p(M,\tau)$ $(p\geqslant 1)$ is also a Banach M-bimodule.

Further we need the following result due Ringrose (see [15, Theorem 2], also [9, p. 638]).

Theorem 3.1. *Let $\mathcal{A}$ be a C^*-algebra and E be a Banach $\mathcal{A}$-bimodule. Then each derivation $D:\mathcal{A}\to E$ is continuous.*

One of the main results of the present work is the following

Theorem 3.2. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ. Then each derivation of the algebra $M+L_2^\omega(M,\tau)$ is inner.*

Proof. Since $M+\bigcap_{n=2}^m L^n(M,\tau)$ $(m\geqslant 2)$ is a Banach M-bimodule, by Theorem 3.1 the derivation d is a continuous map from M into $M+\bigcap_{n=2}^m L^n(M,\tau)$ for all $m\geqslant 2$. Therefore,

$\|x_k\|_\infty \to 0$ $(k \to \infty)$ implies $\|d(x_k)\|_n'' \to 0$ $(k \to \infty)$ for all $n \geqslant 2$. Since $M + L_2^\omega(M,\tau)$ is a countably normed space with the family of norms $\{\|\cdot\|_n''\}_{n\in\mathbb{N}\setminus\{1\}}$, the operator $d : M \to M + L_2^\omega(M,\tau)$ is continuous.

Let U be the group of all unitary elements in M. For $u \in U$ put

$$T_u(x) = uxu^* + d(u)u^*, \quad x \in M + L_2^\omega(M,\tau).$$

Then for $u, v \in U$, one has

$$\begin{aligned} T_u\big(T_v(x)\big) &= T_u\big(vxv^* + d(v)v^*\big) = u\big(vxv^* + d(v)v^*\big)u^* + d(u)u^* \\ &= uvxv^*u^* + ud(v)v^*u^* + d(u)u^* = \big(uvx + d(u)v + ud(v)\big)(uv)^* \\ &= uvx(uv)^* + d(uv)(uv)^* = T_{uv}(x), \end{aligned}$$

i.e.

$$T_uT_v = T_{uv}, \quad u, v \in U. \tag{3}$$

Since the operator d is continuous there exist $C_n > 0$, $n \in \mathbb{N} \setminus \{1\}$, such that

$$\big\|d(x)\big\|_n'' \leqslant C_n\|x\|_\infty, \quad x \in M.$$

From $\|u\|_\infty = 1$ $(u \in U)$ it follows

$$\big\|T_u(0)\big\|_n'' = \big\|d(u)u^*\big\|_n'' \leqslant \big\|d(u)\big\|_n''\|u^*\|_\infty = \big\|d(u)\big\|_n'' \leqslant C_n,$$

i.e. $\|T_u(0)\|_n'' \leqslant C_n$ for all $u \in U$. Therefore, the set $K_d = \{T_v(0)\colon v \in U\}$ is bounded. Moreover, the set $K = \mathrm{cl}(\mathrm{co}(K_d))$—the closure of the convex hull of K_d, is a closed convex bounded subset in $M + L_2^\omega(M,\tau)$. By Corollary 2.11 K is a non-empty convex *-weakly compact set. By (3) we have $T_u(K_d) \subset K_d$ for all $u \in U$. Since T_u is an affine homeomorphism, $T_u(\mathrm{cl}(\mathrm{co}(K_d))) = \mathrm{cl}(\mathrm{co}(T_u(K_d))) \subset \mathrm{cl}(\mathrm{co}(K_d))$ $(u \in U)$, i.e. $T_u(K) \subseteq K$.

For $x, y \in M + L_2^\omega(M,\tau)$ we have

$$\big\|\big(T_u(x) - T_u(y)\big)\big\|_n'' = \big\|u(x-y)u^*\big\|_n'' = \|x-y\|_n''.$$

Therefore by Ryll-Nardzewski's fixed point theorem [14] there exists $a \in K$ such that $T_u(a) = a$ for all $u \in U$. Therefore $uau^* + d(u)u^* = a$, i.e. $d(u) = au - ua$ for all $u \in U$. Since any element of M is a finite linear combination of unitary elements in M, we have $d(x) = ax - xa$ for all $x \in M$.

Now let us show that $d(x) = ax - xa$ for all $x \in M + L_2^\omega(M,\tau)$. First suppose that $x \in M + L_2^\omega(M,\tau)$, $x \geqslant 0$. Then the element $\mathbf{1} + x$ is invertible and $(\mathbf{1} + x)^{-1} \in M$.

Let $b \geqslant 0$ be an invertible elements of $M + L_2^\omega(M,\tau)$. Since $\mathbf{1} = \mathbf{1}^2$, then $d(\mathbf{1}) = d(\mathbf{1}^2) = d(\mathbf{1})\mathbf{1} + \mathbf{1}d(\mathbf{1}) = 2d(\mathbf{1})$, i.e. $d(\mathbf{1}) = 0$. Therefore $0 = d(\mathbf{1}) = d(bb^{-1}) = d(b)b^{-1} + bd(b^{-1})$, i.e. $d(b) = -bd(b^{-1})b$.

Using this equality we obtain

$$d(x) = d(\mathbf{1} + x) = -(\mathbf{1} + x)d\big((\mathbf{1} + x)^{-1}\big)(\mathbf{1} + x).$$

On the other hand, since $(\mathbf{1}+x)^{-1} \in M$ one has

$$d\big((\mathbf{1}+x)^{-1}\big) = a(\mathbf{1}+x)^{-1} - (\mathbf{1}+x)^{-1}a.$$

Therefore, $-(\mathbf{1}+x)d((\mathbf{1}+x)^{-1})(\mathbf{1}+x) = -(\mathbf{1}+x)[a(\mathbf{1}+x)^{-1} - (\mathbf{1}+x)^{-1}a](\mathbf{1}+x) = -(\mathbf{1}+x)a + a(\mathbf{1}+x) = ax - xa$, i.e.

$$d(x) = -(\mathbf{1}+x)d\big((\mathbf{1}+x)^{-1}\big)(\mathbf{1}+x) = ax - xa.$$

Since any element of $M + L_2^{\omega}(M,\tau)$ is a finite linear combination of positive elements in $M + L_2^{\omega}(M,\tau)$, we have $d(x) = ax - xa$ for all $M + L_2^{\omega}(M,\tau)$. The proof is complete. □

Corollary 3.3. *Consider the Arens algebra $L^{\omega}(M,\tau)$, where τ is a finite trace. Then any derivation d on $L^{\omega}(M,\tau)$ is inner. In particular, d is t-continuous and *-weakly continuous. Moreover, the element $a \in L^{\omega}(M,\tau)$ implementing d can be taken such that $\|a\|_n'' \leqslant \|d\|_n$, $n \geqslant 2$.*

The continuity of derivations can be proved also in the general case. Namely, we have the following

Proposition 3.4. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ. Then each derivation of the algebra $L^{\omega}(M,\tau)$ (respectively $L_2^{\omega}(M,\tau)$) is t-continuous (respectively t_2-continuous).*

Proof. Let us prove the assertion for the algebra $L^{\omega}(M,\tau)$, the case of $L_2^{\omega}(M,\tau)$ is similar.

Let d be a derivation on $L^{\omega}(M,\tau)$. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ in $L^{\omega}(M,\tau)$, such that $a_n \xrightarrow{t} 0$ and $d(a_n) \xrightarrow{t} y$ for some $y \in L^{\omega}(M,\tau)$. Since $L^{\omega}(M,\tau)$ is a complete metrizable space, by the closed graph theorem it is sufficient to show that $y = 0$.

Let $e \in M$ be a projection with finite trace and consider the Arens algebra $L^{\omega}(eMe,\tau_e)$ associated with the von Neumann algebra eMe and the faithful normal finite trace τ_e, where τ_e is the restriction of τ on eMe.

Put

$$d_e(x) = ed(exe)e, \quad x \in L^{\omega}(eMe,\tau_e). \tag{4}$$

For $x, y \in eMe$, since $x = exe$, $y = eye$, one has $d_e(xy) = ed(exye)e = ed(exeeye)e = ed(exe)eye + exed(eye)e = d_e(x)y + xd_e(y)$, i.e. d_e is a derivation on $L^{\omega}(eMe,\tau_e)$.

By Corollary 3.3 the derivation d_e is continuous. Thus from $ea_ne \xrightarrow{t} 0$ it follows that $d_e(ea_ne) \xrightarrow{t} 0$ as $n \to \infty$. On other hand $d_e(ea_ne) = ed(ea_ne)e = ed(e)a_ne + ed(a_n)e + ea_nd(e)e \xrightarrow{t} eye$ as $n \to \infty$. Therefore,

$$eye = 0 \tag{5}$$

for all projections $e \in M$ with finite trace. Since $a_ny^* \xrightarrow{t} 0$ and $d(a_ny^*) = d(a_n)y^* + a_nd(y^*) \xrightarrow{t} yy^*$, so in (5) the element y can be a replaced by yy^*. Thus $eyy^*e = 0$, i.e. $(ey)(ey^*) = 0$, and therefore, $ey = 0$, i.e.

$$y^*ey = 0 \tag{6}$$

for each projection $e \in M$ with finite trace. Since the map $x \mapsto y^*xy$ is positive and monotone continuous, taking $e \uparrow \mathbf{1}$ in (6), we obtain that $yy^* = 0$. Therefore $y = 0$. The proof is complete. $\square$

It is easy to see that in commutative Arens algebras any derivation is equal to zero on projections. Since the linear span of projection is t-dense in any Arens algebra Proposition 3.4 implies the following

Corollary 3.5. *If M is an abelian von Neumann algebra with a faithful normal semi-finite trace τ then all derivations on $L^\omega(M,\tau)$ are identically zero.*

Remark 3. As it was noted above the commutative algebra $L^0(0,1)$ of all complex measurable functions on $(0,1)$ admits non-zero derivations (see [6,13]). On the other hand the Corollary 3.5 shows that the Arens algebra $L^\omega(0,1)$ admits only zero derivations (similar to the algebra $L^\infty(0,1)$), though it contains unbounded elements.

The following proposition gives one more type of continuity for derivations of Arens algebras.

Proposition 3.6. *Let $d : L^\omega(M,\tau) \to L^\omega(M,\tau)$ be a derivation. Then d maps any weakly converging net from $L^\omega(M,\tau)$ into a net converging in the *-weak topology in $M + L_2^\omega(M,\tau)$.*

Proof. Let a net $(x_\alpha)_{\alpha\in A} \subset L^\omega(M,\tau)$ weakly converge to zero, i.e.

$$\tau(x_\alpha a) \to 0$$

for all $a \in \mathrm{Lin}(\bigcup_{1<q\leqslant\infty} L^q(M,\tau)) \cong L^\omega(M,\tau)'$, where $\cong$ denote the isomorphism (see Proposition 2.10). By Proposition 3.4 d is t-continuous, and hence by [10, Proposition 8.6.5] d is weakly continuous. Thus $(d(x_\alpha))_{\alpha\in A}$ weakly converges to zero, i.e.

$$\tau\big(d(x_\alpha)a\big) \to 0 \tag{7}$$

for all $a \in \mathrm{Lin}(\bigcup_{1<q\leqslant\infty} L^q(M,\tau))$, and, in particular, for all

$$a \in L^1(M,\tau) \cap \left(\mathrm{Lin}\left(\bigcup_{1<q\leqslant 2} L^q(M,\tau)\right)\right).$$

Since

$$\left(L^1(M,\tau) \cap \left(\mathrm{Lin}\left(\bigcup_{1<q\leqslant 2} L^q(M,\tau)\right)\right)\right)' \cong M + L_2^\omega(M,\tau),$$

from (7) it follows that $(d(x_\alpha))_{\alpha\in A}$ is *-weakly converging to zero in $M + L_2^\omega(M,\tau)$. The proof is complete. $\square$

As it was mentioned in Proposition 2.6 the Arens algebra $L^\omega(M,\tau)$ is an ideal in $M+L_2^\omega(M,\tau)$, and therefore any element $a\in M+L_2^\omega(M,\tau)$ generates a spatial derivation on $L^\omega(M,\tau)$ defined as

$$d(x)=ax-xa,\quad x\in L^\omega(M,\tau).$$

In this connection a natural problem arises whether the converse assertion is also true, i.e. can any derivation on the Arens algebra be represented in this form?

The main result of the present work is the following theorem, which answers this question in affirmative and gives a complete description of derivations on the Arens algebra $L^\omega(M,\tau)$.

Theorem 3.7. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ. Then any derivation d on $L^\omega(M,\tau)$ is spatial, moreover it is implemented by an element of $M+L_2^\omega(M,\tau)$, i.e.*

$$d(x)=ax-xa,\quad x\in L^\omega(M,\tau)$$

for some $a\in M+L_2^\omega(M,\tau)$.

Proof. Since the trace τ is semi-finite there exists a net of projections (e_α) with $\tau(e_\alpha)<\infty$ for all α, such that $e_\alpha\uparrow\mathbf{1}$. Consider the derivations

$$d_{e_\alpha}:L^\omega(e_\alpha Me_\alpha,\tau_{e_\alpha})\to L^\omega(e_\alpha Me_\alpha,\tau_{e_\alpha})$$

defined as in (4). By Corollary 3.3 there exist $a_\alpha\in L^\omega(e_\alpha Me_\alpha,\tau_{e_\alpha})$ such that

$$d_{e_\alpha}(x)=a_\alpha x-xa_\alpha,\quad x\in L^\omega(e_\alpha Me_\alpha,\tau_{e_\alpha}),$$

and moreover the net (a_α) is bounded in $M+L_2^\omega(M,\tau)$.

By Corollary 2.11 the net (a_α) contains a subnet which *-weakly converges in $M+L_2^\omega(M,\tau)$. Without loss of generality we may assume that $a_\alpha\to a$ for some $a\in M+L_2^\omega(M,\tau)$.

Let $x\in M\cap L^1(M,\tau)$. If $\alpha\geqslant\beta$, then

$$e_\alpha d(e_\beta xe_\beta)e_\alpha=d_{e_\alpha}(e_\beta xe_\beta)=a_\alpha e_\beta xe_\beta-e_\beta xe_\beta a_\alpha.$$

Therefore

$$d(e_\beta xe_\beta)=ae_\beta xe_\beta-e_\beta xe_\beta a. \tag{8}$$

By Proposition 3.6 the derivation d maps any weakly convergent net into a *-weakly convergent one. Therefore from (8) it follows that $d(x)=ax-xa$ for all $x\in M\cap L^1(M,\tau)$.

Now since the set $M\cap L^1(M,\tau)$ is t-dense in $L^\omega(M,\tau)$ from the t-continuity of d it follows that $d(x)=ax-xa$ for all $x\in L^\omega(M,\tau)$. The proof is complete. □

The following theorem can be proved in a way similar to the proof of the Theorem 3.7.

Theorem 3.8. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ. Then any derivation d on $L_2^\omega(M,\tau)$ is spatial, moreover it is implemented by an element of $M + L_2^\omega(M,\tau)$, i.e.*

$$d(x) = ax - xa, \quad x \in L^\omega(M,\tau)$$

for some $a \in M + L_2^\omega(M,\tau)$.

Remark 4. For any integer $s \geqslant 3$ put $L_s^\omega(M,\tau) = \bigcap_{p\geqslant s}^{\infty} L^p(M,\tau)$. It is not difficult to show that the space $L_s^\omega(M,\tau)$ is also a complete metrizable locally convex *-algebra, with the topology generated by the family of norms $\{\|\cdot\|_p\}_{p\geqslant s}$, and $L_s^\omega(M,\tau)$ is an ideal in $M + L_2^\omega(M,\tau)$. Moreover, one can also prove similarly to Theorem 3.7 that any derivation of the algebra $L_s^\omega(M,\tau)$ is spatial and implemented by an element of the algebra $M + L_2^\omega(M,\tau)$.

Remark 5. In this work we have considered von Neumann algebra with faithful normal semi-finite traces, i.e. semi-finite von Neumann algebras. In order to consider type III von Neumann algebras, one has to construct an analogue of Arens algebras associated with a von Neumann algebra and faithful normal finite weight. To this and one has to find out an appropriate notion of corresponding L^p-spaces among different existing approaches.

Acknowledgments

The second and third named authors would like to acknowledge the hospitality of the Institut für Angewandte Mathematik, Universität Bonn (Germany). This work is supported in part by the DFG 436 USB 113/10/0-1 project (Germany) and the Fundamental Research Foundation of the Uzbekistan Academy of Sciences. The authors are indebted to the reviewer for useful comments.

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