Hereditary crossed product orders over discrete valuation rings

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Abstract

In this paper, we consider weak crossed product orders \( A_f = \sum Sx_\sigma \) with coefficients in the integral closure \( S \) of a discrete valuation ring \( R \) in a tamely ramified Galois extension of the field of fractions of \( R \). In the first section, we compute the Jacobson radical of \( A_f \) when \( S \) is local, and we give a characterization of the hereditarity of the order in terms of the cocycle values. In the second section, we prove (again in the local case) that every \( \sigma \) in the inertia group for \( S/R \) must belong to \( \{ \sigma \in G \mid f(\sigma, \sigma^{-1}) \text{ is a unit of } S \} \). In the final section, we compute the Jacobson radical in the general case (\( S \) is semilocal) and show how the hereditarity of \( A_f \) can be determined locally under an additional hypothesis.

1. Introduction

Let \( R \) be a discrete valuation ring with field of fractions \( F \), and let \( K/F \) be a finite Galois extension with automorphism group \( G \). Let \( f : G \times G \to K^\times \) be a 2-cocycle; that is, \( f \) satisfies \( f(\sigma_1, \sigma_2)f(\sigma_1\sigma_2, \sigma_3) = f(\sigma_2, \sigma_3)^{\sigma_1}f(\sigma_1, \sigma_2\sigma_3) \) for every \( \sigma_1, \sigma_2, \sigma_3 \in G \). We shall always assume that \( f \) is normalized so that \( f(1_G, \sigma) = f(\sigma, 1_G) = 1_K \) for every \( \sigma \in G \).

It is well known that every normalized 2-cocycle is cohomologous to one that takes values in the integral closure \( S \) of \( R \) in \( K \). Assuming, therefore, that \( f \) maps \( G \times G \) into \( S - \{0\} \), one may consider the subring \( A_f := \sum_{\sigma \in G} Sx_\sigma \) of the crossed product algebra \( \Sigma_f := \sum_{\sigma \in G} Kx_\sigma \). The subring \( A_f \) is finitely generated as an \( R \)-module and contains an \( F \)-basis for \( \Sigma_f \); such a subring is said to be an \( R \)-order in \( \Sigma_f \). We call \( A_f \) a weak crossed product order and \( f \) a weak cocycle, the adjective weak indicating that some of the values taken by \( f \) are permitted (though not required) to be nonunits.
in $S$. If $f$ takes only unit values in $S$, we say that $f$ is a classical cocycle. Finally, if every two-sided ideal of $A_f$ is a projective $A_f$-module, we say that $A_f$ is a hereditary order.

In Section 2, we study the relationship between the hereditaryity of $A_f$ when $S$ is local and $S/R$ is tamely ramified, the group $H := \{ \sigma \in G \mid f(\sigma, \sigma^{-1}) \}$ is a unit of $S = \{ \sigma \in G \mid x_{\sigma} \}$ is invertible in $A_f$, and the $\pi_S$-adic values $v_S(f(\sigma, \tau))$ of the cocycle values. It has been shown that $G/H$ is a partially ordered set (Haile [2], Haile, Larson, and Sweedler [3]), and it is interesting to study relationships between the properties of the order $A_f$ and the properties of the directed graph arising from the partial ordering on $G/H$. We will see that this partial ordering turns out to be a total ordering when $A_f$ is hereditary. In the final section, we consider the general (semilocal) case. We give a local criterion for $A_f$ to be hereditary under an additional hypothesis and derive some related structure results.

Many of the results of this paper bear a strong resemblance to those of Haile [2] concerning maximal orders in the case that $S/R$ is unramified. This similarity is explained, at least in part, by another chief result of this paper: If $S/R$ is tamely ramified, $S$ is local, and $A_f$ is hereditary, then the inertia group for the extension $S/R$ is a subgroup of $H$. In other words, if $\sigma$ belongs to the inertia group, then $x_{\sigma}$ must be invertible in $A_f$. Informally speaking, allowing $S/R$ to be tamely ramified does not contribute any “additional” cosets to $G/H$ nor any branching in the associated graph beyond what one sees in the unramified case.

2. The local case

In this section, we consider the following situation: $R$ is a DVR with field of fractions $F$ and residue field $\bar{R}$, $K/F$ is a finite dimensional Galois extension with Galois group $G$, and the integral closure $S$ of $R$ in $K$ is also a DVR with prime element $\pi_S$ and residue field $\bar{S} := S/\pi_S S$. Denote the $\pi_S$-adic valuation by $v_S$, so $v_S(s) := \max\{ z \in \mathbb{Z} : \pi_S^z \text{ divides } s \}$ for $s \in S$. Let $U := \ker(G \to \text{Gal}(\bar{S}/\bar{R}))$ be the inertia group. We assume that $S/R$ is tamely ramified (or unramified), so the characteristic of $R$ does not divide $|U|$. Finally, let $f : G \times G \to S - \{0\}$ be a weak 2-cocycle and $A_f := \sum_{\sigma \in G} Sx_{\sigma}$ a weak crossed product order in the crossed product algebra $\Sigma_f := \sum_{\sigma \in G} \text{Ker}(x_{\sigma})$.

Our first goal is to compute the Jacobson radical of $A_f$. We use the following result of Passman (Theorem 4.2 of Passman [9]) as a starting point.

**Lemma 2.1 (Passman).** Let $k$ be a ring on which a finite group $U_0$ acts. If $|U_0|$ is invertible in $k$, then the (classical) crossed product $B := \sum_{\sigma \in U_0} kx_{\sigma}$ has Jacobson radical $\text{Rad}(B) = \sum_{\sigma \in U_0} \text{Rad}(k)x_{\sigma}$. In particular, if $k$ is a field, then $\text{Rad}(B) = 0$.

We can extend this theorem to get a useful proposition, which we will later apply (with $k = \bar{S}$) to obtain a description of $\text{Rad}(A_f)$.

**Proposition 2.2.** Let $G_0$ be a group that acts on a field $k$ of characteristic $p$ and let $f : G_0 \times G_0 \to k^\times$ be a 2-cocycle. Construct the crossed product algebra $C_f := \sum_{\sigma \in G_0} kx_{\sigma}$ in the usual way. Then $C_f$ is semisimple if $p$ does not divide $|U_0|$, where $U_0 := \ker(G_0 \to \text{Aut}(k))$.

**Proof.** Let $B = \sum_{\sigma \in U_0} kx_{\sigma}$. Then $B$ is a subalgebra of $C_f$. Because $p$ does not divide $|U_0|$, $B$ is semisimple by Passman’s result (Lemma 2.1 above). We can write $C_f = \sum_{\tau \in G/\bar{U}_0} Bx_{\tau}$. We will show that any nonzero two-sided ideal of $C_f$ must intersect $B$ nontrivially.

Let $y = \sum_{i=1}^r b_ix_{\tau_i} \in I$ be a nonzero element of minimal length $r$. We may assume $x_{\tau_1} = 1$ (if not, we may replace $y$ by the product of $y$ and $x_{\tau_1}^{-1}$). We claim that $r = 1$. If $r > 1$, then (because $\tau_2 \notin U_0$) there is some $a \in k$ such that $\tau_2(a) \neq a$. Noting that $k \subseteq Z(B)$, we compute that $ay - ya = \sum_{i=2}^r (ab_i - b_i\tau_2(a))^x_{\tau_i}$. Because $\tau_2(a) \neq a$, $ay - ya$ is a nonzero element of $I$ that has length smaller than $r$, which is impossible. Therefore, $I$ must contain a nonzero element of the form $bx_{\tau}$. For any $\sigma$ such that $\tau \sigma \in U_0$, we have $bx_{\tau}x_{\sigma} \in I \cap B$.

If $\text{Rad}(C_f) \neq 0$, we can conclude that $\text{Rad}(C_f) \cap B$ has a nonzero element by taking $I = \text{Rad}(C_f)$ in the previous paragraph. But this contradicts the semisimplicity of $B$. It follows that $C_f$ is semisimple. $\square$
Define a subgroup $H \subseteq G$ by $H = \{\sigma \in G: f(\sigma, \sigma^{-1})$ is a unit of $S\}$; $H$ is called the inertial subgroup for $A_f$. Let $B_f = \sum_{\sigma \in H} Sx_\sigma$ and $N = \sum_{\sigma \in H} Sx_\sigma$, so $A_f = B_f \oplus N$ as an $R$-module. We will see that $\mathrm{Rad}(A_f)$ has a similar decomposition. Before we compute $\mathrm{Rad}(A_f)$, we state an easy fact about the inertial subgroup:

**Lemma 2.3.** If $\sigma_1, \sigma_2 \notin H$ but $\sigma_1\sigma_2 \in H$, then $f(\sigma_1, \sigma_2)$ is not a unit of $S$.

**Proof.** From the cocycle identity, we get

$$f(\sigma_1, \sigma_2) f(\sigma_1\sigma_2, (\sigma_1\sigma_2)^{-1}) = f(\sigma_2, (\sigma_1\sigma_2)^{-1}) f(\sigma_1, \sigma_1^{-1}).$$

Because $\sigma_1\sigma_2 \in H$ and $\sigma_1 \notin H$, we have that $f(\sigma_1\sigma_2, (\sigma_1\sigma_2)^{-1})$ is a unit of $S$ and $f(\sigma_1, \sigma_1^{-1})$ is not a unit of $S$, so $\nu_S(f(\sigma_1, \sigma_2)) \geq 1$. □

We now have the necessary tools to compute the radical of $A_f$.

**Theorem 2.4.** Let $A_f$ be a weak crossed product order over $R$ with $S$ a DVR. Put $B_f := \sum_{\sigma \notin H} Sx_\sigma$. If $S/R$ is tamely ramified or unramified, then the Jacobson radical of $A_f$ is given by $\mathrm{Rad}(A_f) = \pi_S B_f \oplus \sum_{\sigma \notin H} Sx_\sigma$.

**Proof.** It is easy to see that $\pi_S A_f$ is a two-sided ideal and is equal to $A_f \pi_S$: the relation $a_\sigma x_\sigma \cdot \pi_S = \pi_S \cdot a_\sigma x_\sigma \in \pi_S A_f$ shows that multiplying $\pi_S$ on either side by a single term $a_\sigma x_\sigma$ produces an element of $\pi_S A_f$. We also have $\pi_S S = \mathrm{Rad}(S) \subseteq \mathrm{Rad}(A_f)$ by Theorem 6.15 of Reiner [10] or Corollary 5.9 of Lam [8]. Let $\tilde{N}$ denote the image of $N := \sum_{\sigma \notin H} Sx_\sigma$ in $A_f/\pi_S A_f$. We claim that $\tilde{N}$ is a nilpotent ideal of $A_f/\pi_S A_f$.

We first verify that $\tilde{N}$ is an ideal. Let $\tau \in G$ and $\sigma \notin H$. If $\tau \sigma \notin H$, then for $a, b \in S$ we have $a_\tau \cdot b x_\sigma = a_\tau (b) f(\tau, \sigma)x_{\tau \sigma} \in N$. On the other hand, if $\tau \sigma \in H$, it must be that $\tau \notin H$, so that $f(\tau, \sigma) \in \pi_S S$ by Lemma 2.3 and $a_\sigma (b) f(\tau, \sigma)x_{\tau \sigma} \in \pi_S A_f$.

To prove that $\tilde{N}$ is nilpotent, we will show that $\tilde{N}$ has a basis of nilpotent elements. Let $a_\sigma \in N$, and let $r$ be the smallest integer such that $\sigma^r \in H$. Then $f(\sigma^{-r}, \sigma)$ is not a unit of $S$ by Lemma 2.3. Because $(a_\sigma)^r = c x_{\sigma^r}$ where $c \in S$ is divisible by $f(\sigma^{-r}, \sigma)$, we see that $(a_\sigma)^r \in \pi_S A_f$. Because $\tilde{N}$ has a basis of nilpotent elements, $\tilde{N}$ is a nilpotent ideal by a theorem of Wedderburn and therefore is part of $\mathrm{Rad}(A_f/\pi_S A_f)$ (cf. Lam [8], 4.11, 4.13). Therefore, $\pi_S B_f \oplus N = \pi_S A_f + N \subseteq \mathrm{Rad}(A_f)$.

In order to see that $\pi_S B_f \oplus N$ is the entire radical of $A_f$, we will show that the classical crossed product algebra $C_f := A_f/(\pi_S B_f \oplus N) = \sum_{\sigma \in H} \tilde{S}x_\sigma$ is semisimple. (We note that $\pi_S B_f \oplus N$ is an ideal of $A_f$ for the same reasons that $\tilde{N}$ is an ideal of $A_f/\pi_S A_f$.) The subgroup of $H$ that acts trivially on the field $\tilde{S}$ is $\tilde{H} \cap U$. The extension $S/R$ is tamely ramified or unramified, so the characteristic $p$ of $\tilde{R}$ does not divide $|\tilde{H} \cap U|$ because $p$ does not divide $|U|$. Thus, we can apply Proposition 2.2 with $U_0 = \tilde{H} \cap U$ and $k = \tilde{S}$ to conclude that $C_f$ is semisimple. Hence, $\pi_S B_f \oplus N = \mathrm{Rad}(A_f)$. □

Now that we have obtained a description of $\mathrm{Rad}(A_f)$, we are ready to determine the conditions on the cocycle $f$ that are equivalent to the hereditarity of $A_f$. We will start with some additional lemmas that will help us to identify the $\pi_S$-adic value of $f(\sigma_1, \sigma_2)$ in various cases.

**Lemma 2.5.** For every $h \in H$ and $g \in G$, the cocycle values $f(h, g)$ and $f(g, h)$ are units of $S$.

**Proof.** Using the cocycle relation, we have

$$f(h, g) f(h^{-1}, h g) = f(h^{-1}, h) f(h^{-1} h, g) = (a \text{ unit of } S) \cdot f(1, g) = (a \text{ unit of } S),$$
and

\[ f(g, h) f(gh, h^{-1}) = f(h, h^{-1})^g f(g, h^{-1}h) \]

\[ = (\text{a unit of } S) \cdot f(g, 1) \]

\[ = (\text{a unit of } S). \]

So \( f(h, g) \) and \( f(g, h) \) are units of \( S \). \( \square \)

Lemma 2.6. For every \( \sigma \in G \) we have \( v_S(f(\sigma, \sigma^{-1})) = v_S(f(\sigma^{-1}, \sigma)) \).

Proof. From the cocycle relation, we have

\[ f(\sigma, \sigma^{-1}) f(1, \sigma) = f(\sigma^{-1}, \sigma)^\sigma f(\sigma, 1). \]

Observe that \( f(1, \sigma) = f(\sigma, 1) = 1. \) \( \square \)

As in Haile, Larson, and Sweedler [3] and Haile [2], the cohomology class (over \( S \)) of the cocycle \( f \) gives rise to a partial ordering \( \leq \) on \( G/H \) defined by

\[ g_1 H \leq g_2 H \text{ if and only if } f(g_1, g_1^{-1}g_2) \text{ is a unit of } S. \]

This partial ordering enjoys a remarkable property known as lower subtractivity:

Given \( g_1, g_2, g_3 \in G \) with \( g_1 H \leq g_3 H \), we have

\[ g_1 H \leq g_2 H \leq g_3 H \text{ if and only if } g_1^{-1}g_2 H \leq g_1^{-1}g_3 H \]

(cf. Haile [1], Theorem 2.1, Haile, Larson, and Sweedler [3], Lemma 7.6, Theorem 7.13).

Lemma 2.7. If \( \tau H \leq \sigma H \), then \( v_S(f(\tau, \tau^{-1})) \leq v_S(f(\sigma, \sigma^{-1})). \)

Proof. From the cocycle identity, we have

\[ f(\tau, \tau^{-1}\sigma) f(\sigma, \sigma^{-1}) = f(\tau^{-1}\sigma, \sigma^{-1})^\tau f(\tau, (\tau^{-1}\sigma)\sigma^{-1}) \]

\[ = f(\tau^{-1}\sigma, \sigma^{-1})^\tau f(\tau, \tau^{-1}). \]

But \( f(\tau, \tau^{-1}\sigma) \) is a unit because \( \tau H \leq \sigma H \), so the expression on the left hand side of the equation has \( \pi_S \)-adic value \( v_S(f(\sigma, \sigma^{-1})). \) The factors on the right hand side of the equation have nonnegative \( \pi_S \)-adic values; therefore, \( v_S(f(\tau, \tau^{-1})) \) is at most equal to \( v_S(f(\sigma, \sigma^{-1})). \) \( \square \)

We will need the following useful theorem concerning the left order

\[ O_l(\text{Rad}(A_f)) := \{ x \in \Sigma_f \mid x\text{Rad}(A_f) \subseteq \text{Rad}(A_f) \} \]

of the Jacobson radical of \( A_f \).

Theorem 2.8. If \( A_f \) is hereditary, then \( O_l(\text{Rad}(A_f)) = A_f \).
The conclusion of Theorem 2.8 holds for any semihereditary order, including orders that are not crossed products, cf. Kauta [6]. The proof in the general case (i.e. R is not assumed to be complete) appears in Kauta [6], Proposition 1.4 and Theorem 1.5.

Proposition 2.9. If $O_l(\text{Rad}(A_f)) = A_f$ (in particular, if $A_f$ is hereditary by 2.8), then $v_S(f(\sigma, \sigma^{-1})) \leq 1$ for every $\sigma \in G$.

Proof. By Lemma 2.7, we only need to consider $\sigma$ for which $\sigma H$ is maximal with respect to the partial ordering on $G/H$. Suppose that some such $\sigma$ has $v_S(f(\sigma, \sigma^{-1})) \geq 2$. Choose any $a \in K$ with $v_S(a) = -1$. We will show that $ax_\sigma \in O_l(\text{Rad}(A_f))$. Recall that $\text{Rad}(A_f) = \sum_{\tau \in H} \pi_S x_\tau + \sum_{\tau \notin H} Sx_\tau$ (Theorem 2.4). We need to show that the products of $ax_\sigma$ with single terms of the form $x_\tau$ or $\pi_S x_\tau$ (depending on whether $\tau \in H$) are elements of $\text{Rad}(A_f)$.

Case 1: If $\tau \notin H$ but $\sigma \tau \in H$, then $\sigma^{-1} \tau = H$. In particular, $\sigma^{-1} H \leq \tau H$ so that $f(\sigma^{-1}, \sigma \tau)$ is a unit of $S$. We have

$$f(\sigma, \tau)^{\sigma^{-1}} f(\sigma^{-1}, \sigma \tau) = f(\sigma^{-1}, \sigma) f(1, \tau).$$

Because $f(\sigma^{-1}, \sigma \tau)$ and $f(1, \tau)$ are units of $S$, $v_S(f(\sigma, \tau)) = v_S(f(\sigma^{-1}, \sigma))$, which is the same as $v_S(f(\sigma, \sigma^{-1}))$ by Lemma 2.6. Hence, $v_S(f(\sigma, \tau)) \geq 2$. So

$$ax_\sigma \cdot x_\tau = a \cdot f(\sigma, \tau) x_{\sigma \tau} \in \text{Rad}(A_f)$$

because $v_S(a \cdot f(\sigma, \tau)) \geq 1$.

Case 2: If $\tau, \sigma \tau \notin H$, then $f(\sigma, \tau) = f(\sigma, \sigma^{-1}(\sigma \tau))$ is not a unit of $S$. (Otherwise we would have $\sigma H \leq \sigma \tau H \Rightarrow \sigma H = \sigma \tau H$ by maximality of $\sigma H$. This would imply that $\tau \in H$, but $\tau \notin H$.) Therefore,

$$ax_\sigma \cdot x_\tau = a \cdot f(\sigma, \tau) x_{\sigma \tau} \in \text{Rad}(A_f)$$

because $\sigma \tau \notin H$ and $v_S(a \cdot f(\sigma, \tau)) \geq 0$.

Case 3: If $\tau \in H$, then $ax_\sigma \cdot \pi_S x_\tau = a\sigma(\pi_S)f(\sigma, \tau) x_{\sigma \tau} \in \text{Rad}(A_f)$ because $\sigma \tau \notin H$ and $a\sigma(\pi_S)f(\sigma, \tau) \in S$.

It follows that $ax_\sigma \in O_l(\text{Rad}(A_f))$. But $O_l(\text{Rad}(A_f)) = A_f$ by hypothesis, so $ax_\sigma \in A_f$. This contradicts the fact that $v_S(a) = -1$. $\square$

Corollary 2.10. If $v_S(f(\sigma^{-1}, \sigma)) \in \{0, 1\}$ for every $\sigma \in G$, then $v_S(f(\sigma_1, \sigma_2)) \leq 1$ for all $\sigma_1, \sigma_2 \in G$. In particular, this conclusion holds whenever $S/R$ is a tamely ramified extension of DVRs and $A_f$ is hereditary (or more generally, when $O_l(\text{Rad}(A_f)) = A_f$) by the previous proposition.

Proof. Because $f(1, \sigma_2) = 1$, the right hand side of the equation

$$f(\sigma_1, \sigma_2)^{\sigma^{-1}} f(\sigma_1^{-1}, \sigma_1 \sigma_2) = f(\sigma_1^{-1}, \sigma_1) f(1, \sigma_2)$$

has $\pi_S$-adic value equal to either 0 or 1. $\square$

Corollary 2.11. If $A_f$ is hereditary (or more generally, if $O_l(\text{Rad}(A_f)) = A_f$), then $G/H$ is totally ordered.
Proof. Given \( g, \tau \notin H \), write \( g = \tau \sigma \). We may assume \( gH \neq \tau H \) so that \( \sigma \notin H \). Then \( f(\sigma, \sigma^{-1}) \) is not a unit of \( S \) and must have \( \pi_S \)-adic value equal to 1 by Lemma 2.9.

The cocycle condition says that
\[
f(\tau, \sigma) f(\tau \sigma, \sigma^{-1}) = f(\sigma, \sigma^{-1}) f(\tau, \sigma \sigma^{-1}).
\]

So exactly one of \( f(\tau, \sigma^{-1}(\tau \sigma)) = f(\tau, \sigma) \) or \( f(\tau \sigma, (\tau \sigma)^{-1}) \tau = f(\tau \sigma, \sigma^{-1}) \) has value 0 and is a unit of \( S \). Thus, either \( \tau H \leq \tau \sigma H = gH \) or \( gH = \tau \sigma H \leq \tau H \). □

Corollary 2.12. If \( A_f \) is hereditary (or if \( O_1(\text{Rad}(A_f)) = A_f \)), then \( H \) is a normal subgroup of \( G \).

Proof. Because \( f(1, 1^{-1}g) \) is a unit of \( S \), we have \( H = 1H \leq gH \) for every \( g \in G \). Thus, given \( g_1H \leq g_2H \) and any \( h \in H \), we have \( h^{-1}H = h \leq g_1H \leq g_2H \). Using lower subtractivity, we obtain \( h_1g_1H \leq h_2g_2H \). This shows that left multiplication by any element of \( H \) preserves the partial ordering on \( G/H \).

Thus, because \( G/H \) is totally ordered, we have \( h_1g_1H = gH \) for every \( g \in G \) and \( h \in H \). But then \( g^{-1}h_1g \in H \) so that \( g^{-1}hg \in H \). □

Lemma 2.13. Suppose that \( O_1(\text{Rad}(A_f)) = A_f \) (for example, if \( A_f \) is hereditary) and that \( H \neq G \). Let \( \tau \) be the unique minimal coset among \( \{gH: g \notin H \} \). Then \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{r-1}H \), where \( r \) is the least positive integer for which \( \tau^r \in H \).

Proof. Let \( 1 < i < r \), and assume that \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{i-1}H \). (This is clearly true for \( i = 2 \).) Because \( \tau H \leq \tau^iH \) (by the definitions of \( \tau \) and \( r \)) and because \( \tau^{-1}\tau^{i-1}H = \tau^{i-2}H \leq \tau^{i-1}H = \tau^{-1}\tau^iH \), we conclude by lower subtractivity that \( \tau H \leq \tau^{i-1}H \leq \tau^iH \). The lemma follows by induction. □

Theorem 2.14. Let \( S/R \) be a tamely ramified extension of DVRs (for which \( S \) is a DVR). If \( A_f \) is hereditary (or more generally, if \( O_1(\text{Rad}(A_f)) = A_f \)) and \( H \neq G \), then there exists \( \tau \in G \) such that \( G/H = \{\tau^iH: i = 1, \ldots, |G/H|\} \) and \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{[G/H]-1}H \).

Proof. Let \( \tau H \) be the unique minimal coset among \( \{gH: g \notin H \} \). Let \( g \notin H \) and let \( r \) be the least positive integer for which \( \tau^r \in H \). We need to show that \( gH \) is among the cosets forming the chain \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{r-1}H \) from Lemma 2.13.

Following an argument of Haile ([2], Theorem 2.3), we choose the greatest \( i \in \{1, 2, \ldots, r-1\} \) such that \( \tau^iH \leq gH \). If \( \tau^iH \neq gH \), then \( \tau^{-1}gH \neq \tau^{-1}H \), so \( \tau H \leq \tau^{-1}gH \) by the minimality of \( \tau H \). Because \( \tau^iH \leq gH \) and \( \tau^{-1}\tau^iH = \tau^{-1}gH \), we conclude by lower subtractivity that \( \tau^iH \leq \tau^{i+1}H \leq gH \), contradicting the maximality of \( i \). Thus, \( gH = \tau^iH \), so the chain \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{r-1}H \) includes every left coset of \( H \) in \( G \). □

We now prove a converse to Theorem 2.14 and summarize the relationships between the hereditariness of \( A_f \), the partial ordering on \( G/H \), conditions on the maximum \( \pi_S \)-adic value of \( f(\sigma_1, \sigma_2) \), and the condition \( O_1(\text{Rad}(A_f)) = A_f \).

Theorem 2.15. Let \( S/R \) be a tamely ramified or unramified extension of DVRs (\( S \) is assumed to be a DVR) and let \( A_f \) be a weak crossed product order. The following are equivalent:

1. \( v_S(f(\sigma, \sigma^{-1})) \leq 1 \) for all \( \sigma \in G \), or equivalently, \( v_S(f(\sigma_1, \sigma_2)) \leq 1 \) for all \( \sigma_1, \sigma_2 \in G \).
2. Either \( H = G \) or the partial ordering on \( G/H \) is given by \( H \leq \tau H \leq \tau^2H \leq \cdots \leq \tau^{[G/H]-1}H \) for some \( \tau \in G \) satisfying \( v_S(f(\tau, \tau^{-1})) = 1 \).
3. \( A_f \) satisfies \( O_1(\text{Rad}(A_f)) = A_f \).
4. \( A_f \) is hereditary.
If $H \neq G$, then these conditions imply that $G/H$ is a cyclic group with generator $\tau H$ and that $\text{Rad}(A_f) = x_\tau A_f$.

**Proof.** We first note that the two conditions given in (1) are equivalent by Corollary 2.10. We have (4) $\Rightarrow$ (3) $\Rightarrow$ (1) and (2) by 2.8, 2.9, and 2.14.

(1) $\Rightarrow$ (3): It is clear that $A_f \subseteq O_1(\text{Rad}(A_f))$ because $\text{Rad}(A_f)$ is an ideal of $A_f$. For the reverse inclusion, we will show that the elements of $\Sigma_f \setminus A_f$ cannot belong to $O_1(\text{Rad}(A_f))$. Thus, fix such an element $y := \sum_{\sigma \in G} a_\sigma x_\sigma \in \Sigma_f$ with $v_S(a_\rho) < 0$ for some $\rho$. We will find an element $z \in \text{Rad}(A_f)$ with $yz \notin \text{Rad}(A_f)$ so that $y \notin O_1(\text{Rad}(A_f))$. (Theorem 2.4 shows that $\text{Rad}(A_f) = \pi S B_f \oplus \bigoplus_{\sigma \notin H} \Sigma B_\sigma$, where $B_f$ denotes $\bigoplus_{\sigma \in H} \Sigma B_\sigma$.)

**Case 1:** If $\rho \in H$, then $z := \pi_S$ has the required property, for $z \in \text{Rad}(A_f)$ but $yz = \sum_{\sigma \in G} a_\sigma x_\sigma \pi_S = a_\rho (\pi_S) x_\rho + \sum_{\sigma \neq \rho} a_\sigma (\pi_S) x_\sigma \notin \text{Rad}(A_f)$ because $v_S(a_\rho(\pi_S) x_\rho) = 0$ and $\rho \in H$ (which implies that either $yz \in B_f \setminus \pi S B_f$ or $yz \notin A_f$.) Hence, $yz \notin \text{Rad}(A_f)$, so $y \notin O_1(\text{Rad}(A_f))$.

**Case 2:** If $\rho 
\notin H$, then $\rho^{-1} \notin H$, so $z := x_{\rho^{-1}}$ is an element of $\text{Rad}(A_f)$. We have $yz = \sum_{\sigma \in G} a_\sigma x_{\sigma^{-1}} = a_\rho f(\rho, \rho^{-1}) + \sum_{\sigma \neq \rho} a_\sigma f(\sigma, \rho^{-1}) x_{\sigma^{-1}}$. Thus, the constant term of $yz$ has $\pi_S$-adic value less than 1 because $v_S(a_\rho) < 0$ and $v_S(f(\rho, \rho^{-1})) = 1$. We see that $yz \notin \text{Rad}(A_f)$ so that $y \notin O_1(\text{Rad}(A_f))$.

We have shown that $O_1(\text{Rad}(A_f)) = A_f$.

(2) $\Rightarrow$ (1): This implication is clear if $G = H$ (in this case the cocycle values are units of $S$), so assume that $H$ is a proper subgroup of $G$. From the cocycle relation we get

$$f(\tau^{-1}, \sigma) f(\tau^{-1}, \tau^{[G/H]-1}) = f(\tau, (\tau^{[G/H]-1})^{-1}) f(\tau^{-1}, \tau^{[G/H]}).$$

By Lemmas 2.5 and 2.6, the left hand side of this equation has $\pi_S$-adic value equal to 1; therefore, the right hand side also has $\pi_S$-adic value 1. Because $\tau^{[G/H]} \in H$, $f(\tau^{-1}, \tau^{[G/H]})$ is a unit of $S$ (again by 2.5), so $v_S(f(\tau, (\tau^{[G/H]-1})^{-1})) = 1$.

Using the cocycle relation again, we have

$$f(\tau, \tau^{[G/H]-1}) f(\tau^{[G/H]}, (\tau^{[G/H]-1}^{-1})) = f(\tau^{[G/H]-1}, (\tau^{[G/H]-1})^{-1}) f(\tau, 1).$$

We have seen that the left side of this equation has $\pi_S$-adic value 1, so it is clear that $f(\tau^{[G/H]-1}, (\tau^{[G/H]-1}^{-1})$ also has value 1. For every $\sigma \in G$, we have $\sigma H \subseteq \tau^{[G/H]-1} H$, so by Lemma 2.7,

$$v_S(f(\sigma, \sigma^{-1})) \leq v_S(f(\tau^{[G/H]-1}, (\tau^{[G/H]-1}^{-1}))) \leq 1.$$ 

This proves (2) $\Rightarrow$ (1).

We will need the following lemma to prove the implication (3) $\Rightarrow$ (4).

**Lemma 2.16.** (See Harada [5], Lemma 3, p. 72.) Let $A$ be an order over $R$ in a central simple $K$-algebra $\Sigma$. Then $A$ is hereditary if and only if some power of $\text{Rad}(A)$ is principally generated by an element of $\Sigma$; that is, $(\text{Rad}(A))^t = x A_f$ for some $t > 0$ and $x \in \Sigma$.

(3) $\Rightarrow$ (4): We have already established the equivalence of (1), (2), and (3). If $H = G$, then $\text{Rad}(A_f) = \pi_S A_f$ by Theorem 2.4, and we are done by Lemma 2.16. If $H \neq G$, we claim that $\text{Rad}(A_f) = x \tau A_f$, where $\tau$ is given in (2).

We have $x A_f \subseteq \text{Rad}(A_f)$ by Theorem 2.4. By the same theorem, it suffices to check the reverse inclusion by showing that $x \tau A_f$ contains all monomial terms of the form $a x \sigma \sigma h$ where $h \in H$ and all monomial terms of the form $a x \sigma$ where $\sigma \notin H$. 


Thus, take $a \in S$ and $h \in H$. We show that $b \in S$ can be found so that $a\pi S x_h = x_f \cdot b x_{\tau^{-1}h}$. The right hand side of this equation is equal to $b^f f(\tau, \tau^{-1}h)x_h$, so $b = [a\pi S / f(\tau, \tau^{-1}h)]= a\pi S x_h \in x_f A_f$.

For the other type of monomial term, take $a \in S$ and $\sigma \notin H$. We need to find $b \in S$ so that $ax_\sigma = x_f \cdot bx_{\tau^{-1}\sigma}$. The right hand side of this equation is equal to $b^f f(\tau, \tau^{-1}\sigma)x_\sigma$, so $b = [a/f(\tau, \tau^{-1}\sigma)]^{-1}$ satisfies the requirement as long as $b \in S$. Because $\sigma \notin H$, we must have $\tau H \leq \sigma H$ by (2), which means that $f(\tau, \tau^{-1}\sigma)$ is a unit of $S$. Thus, $b \in S$ so that $ax_\sigma \in x_f A_f$.

Because $\text{Rad}(A_f) = x_f A_f$ is a principal ideal, we use Harada’s Lemma (2.16 above) to conclude that $A_f$ is hereditary. \hfill \Box

3. Tame ramification and the inertial subgroup

Keep the notation from the previous section. Throughout this section, we remain in the setting of a weak crossed product order $A_f$ arising from a tamely ramified extension $S/R$ of DVRs, and we retain the hypothesis that $S/R$ is local with maximal ideal $(\pi_S)$. Let $U = \ker(G \rightarrow \text{Gal}(\bar{S}/R))$ be the inertia group for the extension $K/F$, and let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) is a unit of $S$\}$ be the inertial group for $A_f$.

We can make use of the following theorem of Williamson ([11], Propositions 2.1 and 3.1) to find a relationship between $U$ and $H$.

**Theorem 3.1** (Williamson). If $S/R$ is tamely ramified and $S$ is a DVR, then the extension of $S$ over its inertia ring $S^U$ is cyclic, and $S$ contains a primitive $e$-th root of unity $\omega$, where $e = |U|$.

If $S/R$ is an extension of complete DVRs and $\rho$ is a generator of $U$, then the uniformizer $\pi_S$ of $S$ can be chosen in such a way that $\rho(\pi_S) = \omega \pi_S$, and $S^U$ contains a primitive $e$-th root of unity.

**Remark 3.2.** In fact, Williamson shows how to choose $\pi_S$ and how to find $c \in S^U$ so that $\pi_S$ satisfies the polynomial $x^e - e \pi_R \in S^U[x]$ in the complete case.

We will also need the following computation:

**Lemma 3.3.** For any $t > 1$ and any $\sigma \in G$, we have

$$(x_\sigma)^t = \left(\prod_{i=1}^{t-1} f(\sigma, \sigma^i)^{(\sigma^i-1)}\right) x_{\sigma^t}.$$  

**Proof.** We proceed by induction on $t$. The statement is clear for $t = 2$, and whenever it is true that $(x_\sigma)^{t-1} = (\prod_{i=1}^{t-2} f(\sigma, \sigma^i)^{(\sigma^i-1)}x_{\sigma^{t-1}}$, then we also have

$$(x_\sigma)^t = x_{\sigma^t} \cdot (x_\sigma)^{t-1} = x_{\sigma^t} \cdot \left(\prod_{i=1}^{t-2} f(\sigma, \sigma^i)^{(\sigma^i-1)}\right)x_{\sigma^{t-1}} = \left(\prod_{i=1}^{t-2} f(\sigma, \sigma^i)^{(\sigma^i-1)}\right) f(\sigma, \sigma^{t-1})x_{\sigma^t} = \left(\prod_{i=1}^{t-1} f(\sigma, \sigma^i)^{(\sigma^i-1)}\right)x_{\sigma^t}. \quad \Box$$
**Theorem 3.4.** Let $S/R$ be tamely ramified (with $S$ local). If $A_f$ is hereditary, then the inertia group $U$ in the sense of extensions of DVRs is a subgroup of the inertial group $H$ associated to the cocycle $f$.

**Proof.** Given such an order $A_f$, we immediately pass to the completion $A_f \otimes R \hat{R}$, where $\hat{R}$ denotes completion at the maximal ideal of $R$. The extension $(K \otimes \hat{F})/\hat{F}$ is Galois with group $G$ and inertia group $U$, and $i \circ f$ gives a cocycle for the $R$-order $A_f \otimes R \hat{R}$, where $\hat{S} = S \otimes R \hat{R}$. The cocycle $i \circ f$ takes values in $\hat{S}$ that have $\hat{S}$-adic value $\in \{0, 1\}$. Moreover, $v_S(f(\sigma, \tau)) = v_S((i \circ f)(\sigma, \tau))$ for every $\sigma, \tau \in G$, so $H_f = H_{i \circ f}$. By Theorem 2.15, $A_f \otimes R \hat{R}$ is a hereditary weak crossed product order with cocycle $i \circ f$. Any relationships between $U$ and $H$, etc., will be preserved under passage to the completion, so we shall assume that $R$ is complete for the remainder of this proof.

Let us write $U = \langle \rho \rangle$ and $\hat{H} := H \cap U$. If $U \subseteq H$, then there is some integer $r$, $1 < r < e$, with $\hat{H} = \langle \rho^r \rangle$, so $U/\hat{H} = \{ \rho^0 \hat{H}, \rho^2 \hat{H}, \ldots, \rho^r \hat{H} = \hat{H} \}$, By Williamson's result (Theorem 3.1 above), we may assume that $\pi_S$ satisfies $\rho(\pi_S) = \omega \pi_S$, where $\omega \in S^U$ is a primitive $e$-th root of unity.

Consider the weak crossed product order $C_f := \sum_{i=1}^e \sum_{\sigma \in G} Sx_{\rho^i}$ corresponding to the extension $S/U$. (Again abusing notation, we use the symbol $f$ to denote both the original cocycle defined on $G \times G$ and its restriction to $U \times U$.) Because $A_f$ is hereditary, we know that $v_S(f(\rho^i, \rho^{-1})) \in \{0, 1\}$ for all $i$, and therefore $C_f$ is also hereditary by Theorem 2.15. Because $C_f$ is hereditary, we may by Theorem 2.14 choose the generator $\rho$ for $U$ so that the associated partial ordering for $C_f$ is given by $\hat{H} \leq \rho \hat{H} \leq \rho^2 \hat{H} \leq \cdots \leq \rho^{r-1} \hat{H}$.

We now choose a more convenient basis for $C_f$ over $S^U$. Note that $\rho \hat{H} \leq \rho^i \hat{H}$ if and only if $r = |U/\hat{H}|$ does not divide $i$, so $f(\rho, \rho^{i-1}) = f(\rho, \rho^{-1}) \rho^i$ is a unit if and only if $r$ does not divide $i$. Moreover, whenever $f(\rho, \rho^{-1})$ is a nonunit, we have $v_S(f(\rho, \rho^{i-1})) = 1$. Now define $y_1 := x_1 = 1$, $y_{\rho} := x_{\rho}$ and for $i = 2, \ldots, e - 1$ let

$$y_{\rho^i} := \begin{cases} y_{\rho} \cdot y_{\rho^{i-1}} & \text{if } r(= |U/\hat{H}|) \text{ does not divide } i, \\ \frac{1}{\pi_S} y_{\rho} \cdot y_{\rho^{i-1}} & \text{if } i = mr \text{ for some } m. \end{cases}$$

Finally, we know that $y_{\rho} \cdot y_{\rho^{e-1}} = u \pi_S$ for some unit $u$ of $S$. (Because $\rho \hat{H} \leq \rho^i \hat{H}$, we have $v_S(f(\rho, \rho^{e-1})) = v_S(f(\rho, \rho^{-1} \rho^i)) = v_S(f(\rho, \rho^{-1})) = 1$.) We see that $Sy_{\rho^i} = Sx_{\rho^i}$ for each $i$ and that replacing each $x_{\rho^i}$ with the corresponding $y_{\rho^i}$ does not change whether or not $\alpha \hat{H} \leq \beta \hat{H}$ for any $\alpha$ or $\beta$ in $U$.

Thus, by making the appropriate substitutions we may assume that the cocycle $f$ satisfies

$$f(\rho, \rho^i) = \begin{cases} 1 & \text{if } i \neq mr - 1 \text{ for any } m, \\ \pi_S & \text{if } i = mr - 1 \text{ where } 1 \leq m < k, \\ u \pi_S & \text{if } i = e - 1, \end{cases}$$

where $k = e/r = |\hat{H}|$.

We now use Lemma 3.3 to compute that

$$(y_{\rho})^e = \pi_S^{\sigma_{e-r}} \cdot \pi_S^{\sigma_{e-2r}} \cdot \cdots \cdot u \pi_S^{\sigma_{e-kr}} y_{\rho^e} = \omega^e \pi_S^k u$$

for some integer $e$. (In fact, $(y_{\rho})^e = N_{K/K}(\pi_S)u$.) Thus we have

$$(y_{\rho})^{e+1} = (y_{\rho})^e \cdot y_{\rho} = \omega^e \pi_S^k u y_{\rho}$$
and

\[(y_\rho)^{e+1} = y_\rho \cdot (y_\rho)^e = y_\rho \omega^e \pi_S^k u = \omega^e \rho(\pi_S^k) \rho(u) y_\rho = \omega^{e+k} \pi_S^k \rho(u) y_\rho.\]

Setting these expressions equal to each other yields \(\rho(u) = \omega^{-k} u\). Write \(u = a + b\pi_S\), where \(a, b \in S^U\), \(a \notin \pi_S S\). Reducing the equation \(\rho(u) = \omega^{-k} u\) modulo \(\pi_S\) and remembering that any \(a \in S^U\) is fixed by \(\rho\), we have \(\omega^{-k} a = \rho(a) = a\) in \(S\). Thus, \(\omega^k = 1\) in \(S\).

This is not possible. \(S\) has primitive \(k\)-th roots of unity (indeed, primitive \(e\)-th roots of unity by Theorem 3.1), and the images of these \(k\)-th roots of unity satisfy the polynomial \(x^k - 1\) in \(S[x]\). This polynomial shares no roots with its derivation \(kx^{k-1}\), because \(k\) divides \(e\) and \(\gcd(e, \text{char}(S)) = 1\) by the hypothesis that \(S/R\) is (at most) tamely ramified. The roots in \(S\) of \(x^k - 1\), including \(\omega^k\) and 1, must therefore be distinct. Hence, \(\rho^j \in H\) for all \(i\), so \(U \subseteq H\). \(\square\)

**Corollary 3.5.** Let \(S/R\) be a totally tamely ramified extension of DVRs (\(S\) is local) with Galois group \(G = U\), and let \(A_f\) be a hereditary weak crossed product order. Then \(f\) takes unit values only, so \(A_f\) is a classical crossed product order.

**Corollary 3.6.** Let \(S/R\) be a tamely ramified or unramified extension of DVRs (\(S\) is local), and let \(A_f\) be a hereditary weak crossed product order. If \(H\) is trivial, then \(S/R\) is unramified and \(A_f\) is a maximal order.

**Proof.** By the theorem, \(U\) must be a subgroup of \(H\), so \(S/R\) is unramified. The order \(A_f\) is maximal by Theorem 2.3 of Haile [2]. \(\square\)

One may view this corollary and Theorem 2.15 as being complementary results to those of J.S. Kauta [7] on hereditary cyclic algebras.

These results, along with the repeated use of a certain form of the cocycle identity (shown in the following example), make it easy to write down explicit examples of weak crossed product orders where \(S/R\) is ramified.

**Example 3.7.** An example of a weak crossed product order in which \(S/R\) is a ramified extension of DVRs.

Let \(R := \mathbb{Q}(i)[[t]]\) be the ring of power series in the variable \(t\) with rational coefficients. Then \(R\) is a DVR and its field of fractions is \(F := \mathbb{Q}(i)(t)\). Let \(K := F[\pi]\), where \(\pi := \sqrt[3]{1/2}\). Then \(G = \text{Gal}(K/F)\) is cyclic of order 4, generated by the automorphism \(\sigma : \pi \mapsto i \pi\). The integral closure of \(R\) in \(K\) is \(S := R[\pi]\), the element \(\pi\) unifies \(S\), and the inertial group is \(U = \langle \sigma^2 \rangle\).

Let \(f\) be a cocycle for any hereditary weak crossed product order \(A_f = \sum_{j=0}^3 S\alpha_j\) whose cocycle assumes nonunit values in \(S\). We may assume that

\[f(1, \sigma^j) = f(\sigma^j, 1) = 1, \quad j = 0, 1, 2, 3.\]

Because \(U \subseteq H\) by Theorem 3.4, we must have \(H = U = \{1, \sigma^2\}\) in order to observe any nonunit values taken by the cocycle. By Lemma 2.5, \(f(\sigma, \sigma^2)\) is a unit of \(S\). Under the partial ordering on \(G/H\), we have \(\sigma^2 H = H \leq \sigma H = \sigma^3 H\); therefore, the cocycle values \(f(\sigma, \sigma) = f(\sigma, \sigma^{-1} \sigma^2)\) and \(f(\sigma, \sigma^3) = f(\sigma, \sigma^{-1})\) are nonunits having value \(v_S = 1\) (by Corollary 2.10). As in the proof of the theorem, we may choose the \(x_{\alpha_i}\)'s in such a way that

\[f(\sigma, \sigma) = \pi,\]
\[f(\sigma, \sigma^2) = 1,\]
\[f(\sigma, \sigma^3) = u \pi\]

for some unit \(u\) of \(S\).
Because $G$ is a cyclic group, the rest of the cocycle can be deduced by successive applications of the cocycle identity

$$f(\sigma^r, \sigma^c) = \frac{f((\sigma^{r-1})^\sigma f(\sigma, \sigma^{r-1+c}))}{f(\sigma, \sigma^{r-1})}.$$  

If one keeps track of the cocycle values in a $4 \times 4$ multiplication table (as in Table 1), then $r$ denotes the power on $\sigma$ in the value's row and $c$ denotes the power on $\sigma$ for the column. So any table entry with $r \geq 2$ is determined by entries with smaller $r$'s. In this way, we obtain the values in Table 1.

By using $g_1 = g_3 = \sigma$ and $g_2 = \sigma^3$ in the cocycle identity

$$f(g_1, g_2) f(g_1 g_2, g_3) = f(g_2, g_3)^{R_1} f(g_1, g_2 g_3),$$  

we obtain the relation $u^\sigma = -u$. No additional relations result from the choice of any other $(g_1, g_2, g_3)$. Hence, the hereditary weak-crossed product orders with $|H| = 2$ for the extension $S/R$ are given by the cocycles $f$ having the values listed above, where $u$ is any unit of $S$ for which $u^\sigma = -u$ (for example, $u$ = any element of $\sqrt{2} \cdot R^\times$).

4. The general case

Let $R$ be a DVR with maximal ideal $m$ and quotient field $F$, $K/F$ a tamely ramified Galois extension, and $S \subseteq K$ the integral closure of $R$ in $K$. Then $S$ is semilocal, and $mS = (M_1 \cdots M_k)^\ell$, where $M_i \subseteq S$ are the maximal ideals of $S$.

For each maximal ideal $M \subseteq S$, we define $D_M$ to be the decomposition group $D_M := \{ \sigma \in G \mid \sigma(M) = M \}$. Let $U_M$ denote the inertia subgroup for $M$, so $U_M := \ker(D_M \to \text{Gal}(S/M)/(R/m))$. We also define $H_M := \{ \sigma \in D_M \mid f(\sigma, \sigma^{-1}) \notin M \}$. Let $v_M$ denote the $M$-adic valuation on $S$, so $v_M(s) := \max\{ t \in \mathbb{Z} \mid s \in M^t \}$.

Let $\hat{R}$ denote the completion of $R$ at its maximal ideal $m$, and let $\hat{S} := S \otimes_R \hat{R}$. A well-known consequence of the Chinese Remainder Theorem and Hensel’s Lemma is that $\hat{S}$ has a decomposition

$$\hat{S} = S_1 \oplus \cdots \oplus S_k,$$

where each $S_i$ is the completion of the localization of $S$ at the maximal ideal $M_i$ of $S$. Let $e_i = 1_{S_i}$ be the primitive idempotent of $S_1 \oplus \cdots \oplus S_k$ corresponding to the identity element of $S_i$, so $e_i S = e_i S_i = S_i$, $e_i e_j = 0$ for $i \neq j$, and $\sum e_i = 1_{\hat{S}}$.

Let $A_f := \sum S_i x_i$ be a weak crossed product order inside the algebra $\Sigma_f := \sum K x_i$, and let $\hat{A}_f := A_f \otimes_R \hat{R}$. We remark that the elements of $e_i A_\hat{f} e_i$ all have the form $\sum_{\sigma \in D_{M_i}} e_i S_i x_i \sigma$, because for any $\sigma \notin D_{M_i}$, one has $e_i x_\sigma e_i = e_i e_i^\sigma x_\sigma = e_i e_i^\sigma e_i = 0$. We additionally note that, for elements of $e_i A_\hat{f} e_i$, the coefficient of each $x_i \sigma$ belongs to the discrete valuation ring $e_i \hat{S} = S_i$. Thus $e_i A_\hat{f} e_i$ can be viewed as a weak crossed product order $A_{fi} := \sum_{\sigma \in D_{M_i}} S_i x_i \sigma$ corresponding to the extension $S_i/\hat{R}$.

As in the local case, we begin by computing the radical of $A_f$. The following lemma of Williamson will allow us to make use of our local results.

**Lemma 4.1** (Williamson). (Cf. Lam [8], Proposition 5.6(1).) The Jacobson radicals of $A_f$ and $A_{fi}$ are related by $A_{fi} \cap \text{Rad}(A_f) \subseteq \text{Rad}(A_{fi})$. 


---

**Table 1**

<table>
<thead>
<tr>
<th>$f(\sigma^r, \sigma^c)$</th>
<th>$c = 0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r = 1$</td>
<td>1</td>
<td>$\pi$</td>
<td>1</td>
<td>$u\pi$</td>
</tr>
<tr>
<td>$r = 2$</td>
<td>1</td>
<td>$i$</td>
<td>$u$</td>
<td>$iu^\sigma$</td>
</tr>
<tr>
<td>$r = 3$</td>
<td>1</td>
<td>$iu\pi$</td>
<td>$u^2$</td>
<td>$iu^2\pi$</td>
</tr>
</tbody>
</table>
The proof of 4.1 is found in the second paragraph on page 108 of Williamson [11]. (Note that there are some “hats” missing from the \(\Delta_i\)'s in Williamson’s proof; \(\Delta_1\) and \(\Delta_i\) in [11] correspond to \(A_f\) and \(A_f\) in our notation).

For each \(\sigma \in G\), define

\[
I_{\sigma} := \prod_{\{M \subseteq S | f(\sigma, \sigma^{-1}) \in M\}} M,
\]

the product of the maximal ideals of \(S\) that are “missing” from a factorization of \(f(\sigma, \sigma^{-1})\). So

\[
I_{\sigma} = \{s \in S | s \cdot f(\sigma, \sigma^{-1}) \in M_1 \cdots M_g\}.
\]

Define also

\[
J := \sum I_{\sigma} x_{\sigma}.
\]

**Lemma 4.2.** (See Haile [2], Proposition 3.1b.) Let \(J\) be as above. Then \(J\) is a two-sided ideal of \(A_f\), and \(J\) is nilpotent modulo \((M_1 \cdots M_g)\).

Although this fact is stated in the unramified case in Haile [2], the same proof applies to our setting. (The only difference in the unramified case is that one may abbreviate \(M_1 \cdots M_g\) by \(m\).)

**Proof.** To see that \(J\) is a right ideal, it suffices to show that \(I_{\sigma} x_{\sigma} x_\tau \subseteq I_{\sigma \tau} x_{\sigma \tau}\). This amounts to showing that \(I_{\sigma} f(\sigma, \tau) \subseteq I_{\sigma \tau}\); that is, \(I_{\sigma} f(\sigma, \tau) f(\sigma \tau, (\sigma \tau)^{-1}) \subseteq M_1 \cdots M_g\). Using the cocycle relation,

\[
I_{\sigma} f(\sigma, \tau) f(\sigma \tau, (\sigma \tau)^{-1}) = I_{\sigma} f(\tau, (\sigma \tau)^{-1}) f(\sigma, (\sigma \tau)^{-1})
\]

\[
= I_{\sigma} f(\tau, (\sigma \tau)^{-1})^\sigma f(\sigma, \sigma^{-1})
\]

\[
\subseteq I_{\sigma} f(\sigma, \sigma^{-1}) \subseteq M_1 \cdots M_g.
\]

So \(J\) is a right ideal. A similar computation shows that \(J\) is a left ideal.

The ideal \(J\) is nilpotent modulo \(M_1 \cdots M_g\) if it has an \(\hat{R}\)-basis of nilpotent elements. It is enough to check that \(I_{\sigma} x_{\sigma}\) is nilpotent modulo \(M_1 \cdots M_g\). If \(r\) is the order of \(\sigma\) in \(G\), then

\[
(I_{\sigma} x_{\sigma})^r = I_{\sigma} (I_{\sigma})^{(r)} \cdots (I_{\sigma})^{(r)} f(\sigma, \sigma^{-1})^{(r-1)} f(\sigma, \sigma^{-2})^{(r-2)} \cdots f(\sigma, \sigma^{-r})
\]

\[
\subseteq I_{\sigma} f(\sigma, \sigma^{-1}) \subseteq M_1 \cdots M_g,
\]

so the image of \(I_{\sigma} x_{\sigma}\) in \(A_f/(M_1 \cdots M_g)\) is nilpotent as desired. \(\square\)

**Theorem 4.3.** If \(S\) is semilocal and tamely ramified (or unramified) over \(R\), then

\[
\text{Rad}(A_f) = \sum_{\sigma \in G} I_{\sigma} x_{\sigma},
\]

where
\[ I_\sigma := \prod_{\{M \leq S \mid f(\sigma, \sigma^{-1}) \notin M\}} M \]

is the product of the maximal ideals of \( S \) whose generators are missing from the factorization of \( f(\sigma, \sigma^{-1}) \).

**Proof.** Let \( J = \sum_{\sigma \in \mathbb{C}} I_\sigma \sigma x_\sigma \) as in the previous proposition. We will first show that \( J \subseteq \text{Rad}(A_f) \).

We know that \( mA_f \subseteq \text{Rad}(A_f) \). Because \( (M_1 \cdots M_g)A_f \) is nilpotent modulo \( mA_f \), we also have \( (M_1 \cdots M_g)A_f \subseteq \text{Rad}(A_f) \). But \( J \) is nilpotent modulo \( (M_1 \cdots M_g)A_f \) by the previous proposition, so \( J \) is part of \( \text{Rad}(A_f) \) as well.

**Claim:** \( J \) is the entire radical of \( A_f \).

If not, then we can find a maximal ideal \( M_1 \) (say) of \( S \) and an element \( \sum a_{\sigma_i} x_{\sigma_i} \in \text{Rad}(A_f) \) for which \( a_{\sigma_i} \notin M_1 \) and \( f(\sigma_1, \sigma_1^{-1}) \notin M_1 \). Multiplying \( \sum a_{\sigma_i} x_{\sigma_i} \) on the right by \( x_{\sigma_1^{-1}} \), we get \( \sum a_{\sigma_i} f(\sigma_1, \sigma_1^{-1}) x_{\sigma_i} x_{\sigma_1^{-1}} \in \text{Rad}(A_f) \). This element has constant term equal to \( a_{\sigma_1} f(\sigma_1, \sigma_1^{-1}) \notin M_1 \).

To summarize: Given an element of \( \text{Rad}(A_f) - J \), one can produce a maximal ideal (say, \( M_1 \)) of \( S \) and an element of \( \text{Rad}(A_f) - J \) whose constant term does not belong to \( M_1 \). Fix this \( M_1 \).

Let \( y = \sum b_i x_{\sigma_i} \) be an element of minimal length among those elements of \( \text{Rad}(A_f) - J \) that satisfy \( b_j \notin M_1 \), where \( x_{\sigma_1} = x_1 = 1 \).

**Sub-claim:** For each nonzero \( b_j \), we have \( \sigma_j(s) - s \in M_1 \) for all \( s \in S \).

Otherwise, \( \text{Rad}(A_f) \) would contain \( \sigma_j(s)y - ys = \sum_{i \neq j} (\sigma_j(s) - \sigma_i(s)) b_i x_{\sigma_i} \). This element is of shorter length than \( y \) and has constant term \( (\sigma_j(s) - s) b_1 \notin M_1 \), contradicting the minimality of the length of \( y \).

Thus for every nonzero term \( b_j x_{\sigma_j} \) of \( y \) and \( s \in S \), we have \( \sigma_j(s) - s \in M_1 \). The automorphism \( \bar{\sigma_j} \) of \( S/M_1 \) induced by \( \sigma_j \) is the identity on \( S/M_1 \), so \( \sigma_j \in U_{M_1} \) for every \( \sigma_j \) that appears in a term of \( y \). In particular, each \( \sigma_j \) is an element of \( D_{M_1} \).

Passing to the completion \( \hat{R} \) of \( R \), we have \( e_1 y \in A_{f_1} := e_1 A_f e_1 \) because each \( \sigma_j \) is in \( D_{M_1} \). By Lemma 4.1 we also have \( e_1 y \in \text{Rad}(A_{f_1}) \), and Theorem 2.4 gives us \( \text{Rad}(A_{f_1}) = \sum_{d \in H_{M_1}} \pi_{M_1} S_1 x_d + \sum_{d \in D_{M_1} - H_{M_1}} S_1 x_d \). But the constant term of \( e_1 y \) is a unit in \( S_1 \), a contradiction.

As in the local case, we will use our description of the radical to obtain a characterization of the hereditariness of \( A_f \) in terms of the values of the cocycle \( f \).

**Lemma 4.4.** If \( M \) is a maximal ideal of \( S \), then \( v_M(f(d, d^{-1})) = v_M(f(d^{-1}, d)) \) for every \( d \in D_M \).

**Proof.** The action of \( d \) on an element of \( S \) does not change its \( M \)-adic value because \( d \in D_M \). The lemma then follows from the cocycle equation:

\[ f(d, d^{-1}) f(1, d) = f(d^{-1}, d)^d f(d, 1). \]

**Lemma 4.5.** If \( M \) is a maximal ideal of \( S \), then

\[ v_M(f(h, d)) = v_M(f(d, h)) = 0 \]

for every \( d \in D_M \) and \( h \in H_M \).

**Proof.** Given \( h \in H_M \), we have \( f(h, h^{-1}) \notin M \). From the cocycle relation, we get

\[ f(h^{-1}, h) f(1, d) = f(h, d)^{h^{-1}} f(h^{-1}, hd). \]
Because \( h \in H_M \subseteq D_M \), we have

\[
v_M(f(h,d)) = v_M(f(h,d)^{h^{-1}}) = v_M(f(h^{-1}, h)).
\]

This is equal to \( v_M(f(h, h^{-1})) = 0 \) by Lemma 4.4.

The cocycle relation also gives us

\[
f(d, h)f(dh, h^{-1}) = f(h, h^{-1})^d f(d, 1).
\]

Because \( d \in D_M \) and \( h \in H_M \), we conclude that

\[
v_M(f(h, h^{-1})^d) = v_M(f(h, h^{-1})) = 0.
\]

The right hand side of this cocycle equation is not contained in \( M \), so the left side is not in \( M \) either. \( \square \)

We can define a relation \( \leq_M \) on \( D_M/H_M \): define \( g_1 H_M \leq_M g_2 H_M \) if and only if \( f(g_1, g_1^{-1}g_2) \notin M \).

The relation \( \leq_M \) is well defined: (1) Put \( g_1 = \sigma, g_2 = h, g_3 = h^{-1}\sigma^{-1}\tau \) into the cocycle equation

\[
f(g_1, g_2)f(g_1g_2, g_3) = f(g_2, g_3)^{g_1} f(g_1, g_2g_3)
\]

to get

\[
f(\sigma, h)f(\sigma h, h^{-1}\sigma^{-1}\tau) = f(h, h^{-1}\sigma^{-1}\tau)^{\sigma} f(\sigma, \sigma^{-1}\tau)
\]

and use the fact that \( f(h, g) \notin M \) and \( f(g, h) \notin M \) whenever \( h \in H_M \). This shows that the statement “\( \sigma H_M \leq_M \tau H_M \)” does not depend on choice of representative for the “lower” coset \( \sigma H_M \). (2) Put \( g_1 = \sigma, g_2 = \sigma^{-1}\tau, g_3 = h \) in the cocycle equation to see that the statement “\( \sigma H_M \leq_M \tau H_M \)” does not depend on the choice of representative \( \tau \) for the “upper” coset \( \tau H_M \).

It is noted in Haile [2] that \( \leq_M \) is not necessarily a partial ordering.

Let us state two useful facts concerning the relation \( \leq_M \).

**Lemma 4.6.**

1. For \( \tau, \sigma \in D_M \), we have \( v_M(f(\tau, \tau^{-1})) \leq v_M(f(\sigma, \sigma^{-1})) \) whenever \( \tau H_M \leq_M \sigma H_M \).
2. If \( v_M(f(\sigma, \sigma^{-1})) \in \{0, 1\} \) for every \( \sigma \in D_M \), then \( v_M(f(\sigma_1, \sigma_2)) \leq 1 \) for all \( \sigma_1 \) and \( \sigma_2 \in D_M \).

**Proof.** The proofs of these statements are nearly symbol-for-symbol the same as in the local case (2.7, 2.10); one need only replace each “\( v_S \)” with a “\( v_M \)” and replace each instance of the phrase “is a unit of \( S \)” with “is not an element of \( M \).” \( \square \)

Theorem 2.8 concerning the left order of \( \text{Rad}(A_f) \) holds in this case as well: If \( A_f \) is hereditary, then \( A_f \) satisfies \( O_l(\text{Rad}(A_f)) = A_f \). Thus, we are now in a position to see what hereditarity of \( A_f \) means in terms of the values taken by the cocycle \( f \), following our work in the local case.

**Proposition 4.7.** If \( O_l(\text{Rad}(A_f)) = A_f \) (as is the case when \( A_f \) is hereditary), then \( v_M(f(\sigma_1, \sigma_2)) \leq 1 \) for all \( \sigma_1 \) and \( \sigma_2 \in D_M \).
Proof. By part (2) of the previous lemma, it suffices to show $v_M(f(\sigma, \sigma^{-1})) \leq 1$ for every $\sigma \in D_M$. We know $\text{Rad}(A_f) = \sum I_{\sigma}x_{\sigma}$ where $I_{\sigma} = \prod_{(f(\sigma, \sigma^{-1}))_{M M}} M$. Suppose that there is a $\sigma \in D_M$ and maximal ideal $M$ of $S$ with $v_M(f(\sigma, \sigma^{-1})) \geq 2$. By Lemma 4.6, one may assume that $\sigma H_M$ is maximal with respect to the relation $\leq_M$ described above. Fix this ideal $M$ throughout this proof.

Choose $a \in K$ such that

1. $v_M(a) = -1$ for the ideal $M$ chosen above, and
2. $v_M(a) = 1$ for every other maximal ideal $M' \neq M$ of $S$.

For example, if $M$ is principally generated by $\pi_1$ and the other $M'$'s are principally generated by $\pi_2, \ldots, \pi_g$, then one could take $a = \pi_1^{-1}\pi_2 \cdots \pi_g$. We will show that $ax_{\sigma} \in O_f(\text{Rad}(A_f))$, which contradicts the conclusion of Theorem 2.8 because $a \notin A_f$. To do this, we consider various single terms of the form $x_{\tau}$ or $\pi_Mx_{\tau}$ (depending on whether $\tau \in H_M$, or equivalently, whether $\pi_M \in I_{\tau}$) from $\text{Rad}(A_f)$ and show that the product of $ax_{\sigma}$ with any of these terms gives an element of $\text{Rad}(A_f)$.

Case 1: Suppose that $\tau \notin H_M$ but $\sigma \in H_M$. So $f(\sigma, (\sigma \tau)^{-1}) \notin M$. We have $\tau H_M = \sigma^{-1}H_M$ (because $\sigma \tau H_M = H_M$) so $\sigma^{-1}H_M \leq_M \tau H_M$. Thus, $f(\sigma^{-1}, \sigma \tau) \notin M$. From the cocycle equation we get

$$f(\sigma, \tau)^{-1} f(\sigma, \sigma^{-1}) = f(\sigma^{-1}, \sigma) f(1, \tau)$$

so that $v_M(f(\sigma, \tau)) \geq 2$. (Here we have used the fact that $\sigma \in D_M$ to ensure that $v_M(f(\sigma, \tau)^{-1}) = v_M(f(\sigma, \tau))$.) Hence, $ax_{\sigma} \cdot x_{\tau} = af(\sigma, \tau)x_{\sigma \tau}$, and the coefficient $af(\sigma, \tau)$ has $M$-adic value at least 1 because $v_M(a) = -1$ and $v_M(f(\sigma, \tau)) \geq 2$. So $af(\sigma, \tau)$ is contained in the product ideal of all of the maximal ideals of $S$. Hence, $ax_{\sigma}x_{\tau} = af(\sigma, \tau)x_{\sigma \tau} \in \text{Rad}(A_f)$.

Case 2: Suppose $\tau \notin H_M$ and also $\sigma \tau \notin H_M$. Then $f(\sigma \tau, (\sigma \tau)^{-1}) \in M$. We have $f(\sigma, \tau) = f(\sigma, (\sigma \tau)^{-1})$, which is in $M$ (otherwise we get $\sigma H \leq_M \sigma \tau H$, which implies that $\sigma H = \sigma H$ by the maximality of $\sigma H$ with respect to $\leq_M$. But then $\tau \in H_M$, contradicting our assumptions for Case 2). Now we can compute that

$$ax_{\sigma} \cdot x_{\tau} = af(\sigma, \tau)x_{\sigma \tau} \in I_{\sigma \tau}x_{\sigma \tau} \subseteq \text{Rad}(A_f).$$

Case 3: If $\tau \in H_M$, then $f(\tau, \tau^{-1}) \notin M$ by the definition of $H_M$. Thus, every element of $I_{\tau}$ is divisible by $\pi_M$, where $\pi_M$ principally generates the ideal $M$. Noting that $\sigma \notin H_M$ (we chose $\sigma$ with $v_M(f(\sigma, \sigma^{-1})) \geq 2$) and that $\tau \in H_M$, we have $\sigma \tau \notin H_M$. Thus, $f(\sigma \tau, (\sigma \tau)^{-1}) \in M$, and so $M$ is not among the maximal ideals of $S$ whose product forms the ideal $I_{\sigma \tau}$. Finally, note that $f(\sigma, \tau) \notin M$ by Lemma 4.5. We now compute

$$ax_{\sigma} \cdot \pi_Mx_{\tau} = a\pi_M^{\sigma} \ f(\sigma, \tau)x_{\sigma \tau},$$

which is an element of $I_{\sigma \tau}x_{\sigma \tau} \subseteq \text{Rad}(A_f)$. (Here we again use the fact that $\sigma \in D_M$ to guarantee that $v_M(\pi_M^{\sigma}) = 1$.)

This proves the assertion that $ax_{\sigma} \in O_f(\text{Rad}(A_f))$. But this contradicts Theorem 2.8 as $v_M(a) = -1$ implies that $ax_{\sigma} \notin A_f$, so there cannot be any $\sigma \in D_M$ and maximal ideal $M$ of $S$ for which $v_M(f(\sigma, \sigma^{-1})) \geq 2$. \[\square\]
Lemma 4.9. but allowing nonunit cocycle values makes things a bit more complicated. (1) and (2) were proved in 4.5. (3) follows from the equation

\[ f(\sigma, \sigma^{-1}) f(1, d\sigma) = f(\sigma^{-1}, d\sigma) f(\sigma, \sigma^{-1} d\sigma). \]

Proof. We first note that the identity

\[ x_\sigma x_{\sigma^{-1}} = f(\sigma, \sigma^{-1}) \]

gives

\[ (x_\sigma)^{-1} = x_{\sigma^{-1}} f(\sigma, \sigma^{-1})^{-1}. \]

Although we do not know whether \( f(\sigma, \sigma^{-1})^{-1} \in S \), we do know from the hypothesis \( f(\sigma, \sigma^{-1}) \notin M_i \) that \( f(\sigma, \sigma^{-1})^{-1} e_i \) is a unit of \( S_i = e_i S \). Now we compute:

\[
(x_\sigma)^{-1} [S_i x_d] x_\sigma = (x_\sigma)^{-1} [e_i S x_d] x_\sigma
\]

\[
= (x_\sigma)^{-1} e_i S f(d, \sigma) x_d \sigma
\]

\[
= x_{\sigma^{-1}} f(\sigma, \sigma^{-1})^{-1} e_i S f(d, \sigma) x_d \sigma
\]

\[
= (f(\sigma, \sigma^{-1})^{-1} e_i)^{\sigma^{-1}} f(d, \sigma)^{\sigma^{-1}} f(\sigma^{-1}, d\sigma)(e_i S)^{\sigma^{-1}} x_{\sigma^{-1} d\sigma}
\]

\[
= (f(\sigma, \sigma^{-1})^{-1} e_i)^{\sigma^{-1}} f(d, \sigma)^{\sigma^{-1}} f(\sigma^{-1}, d\sigma)(S_i)^{\sigma^{-1}} x_{\sigma^{-1} d\sigma}.
\]

We claim that this is equal to \( S_i x_{\sigma^{-1} d\sigma} \). To see this, we must show that

1. \( f(\sigma, \sigma^{-1})^{-1} e_i)^{\sigma^{-1}} \notin M_i^{\sigma^{-1}} \),
2. \( f(d, \sigma)^{\sigma^{-1}} e_i^{-1} \notin M_i^{\sigma^{-1}} \), and
3. \( f(\sigma^{-1}, d\sigma) e_i^{\sigma^{-1}} \notin M_i^{\sigma^{-1}} \).
The first assertion is true because \( f(\sigma, \sigma^{-1})^{-1}e_i \) is a unit of \( S_i \), as noted above. The remaining assertions follow directly from parts (2) and (3) of the previous lemma.  

We immediately have the following important corollary:

**Proposition 4.10.** If \( f(\sigma, \sigma^{-1}) \notin M_i \), then 

\[
(x_\sigma)^{-1} A(D_i, S_i)x_\sigma = A(\sigma^{-1}D_i\sigma, S_i\sigma^{-1}) = A(D_j, S_j)
\]

for some \( j \). Moreover, for \( \sigma : S_j \to S_i \), the map 

\[
\phi_\sigma = \phi_{i, j} : \Sigma_f \otimes \hat{R} \to \Sigma_f \otimes \hat{R}
\]

given by 

\[
a x_g \mapsto x_{\sigma^{-1}}(f(\sigma, \sigma^{-1})^{-1}e_i)ax_gx_\sigma
\]
 restricts to a ring isomorphism 

\[
A(D_i, S_i) \xrightarrow{\cong} A(D_j, S_j) = A(\sigma^{-1}D_i\sigma, S_i\sigma^{-1})
\]

**Proof.** All that remains is to check that \( \phi_{i, j}|_{A(D_i, S_i)} \) is an isomorphism of rings. This restricted map is the inclusion of \( A(D_i, S_i) \) in \( \Sigma_f \) followed by the \( \Sigma_f \)-automorphism of conjugation by \( x_\sigma \) (whose inverse in \( \Sigma_f \) is given by \( (x_\sigma)^{-1} = x_{\sigma^{-1}}(f(\sigma, \sigma^{-1})^{-1}) \)). The image of the restriction \( \phi_{i, j}|_{A(D_i, S_i)} \) is equal to \( A(D_j, S_j) \) for some \( j \), so we are done.  

Consider now the following left \( R \)-module decomposition of \( \hat{R} \otimes_R A_f \):

\[
\hat{R} \otimes_R A_f = (\hat{R} \otimes_R S) \otimes S \sum R x_\alpha \\
= (\bigoplus_i S_i) \otimes S \sum R x_\alpha \\
= \bigoplus_i \left( \sum \alpha S_i x_\alpha \right).
\]

Assume that, for every maximal ideal \( M_i \) of \( S \), we have a decomposition of \( G = \bigcup_j D_i\sigma_{i,j} \) into disjoint right cosets of \( D_i \) in \( G \) such that \( f(\sigma_{i,j}, \sigma_{i,j}^{-1}) \notin M_i \) for every \( i \) and \( j \). For convenience, we choose our labeling scheme so that \( \sigma_{i,1} = 1_G \) for every \( i \). The decomposition of \( \hat{R} \otimes_R A_f \) then becomes

\[
\hat{R} \otimes_R A_f = A(D_1, S_1)x_{\sigma_{1,1}} \oplus A(D_1, S_1)x_{\sigma_{1,2}} \oplus \cdots \oplus A(D_1, S_1)x_{\sigma_{1,g}} \\
\oplus A(D_2, S_2)x_{\sigma_{2,1}} \oplus A(D_2, S_2)x_{\sigma_{2,2}} \oplus \cdots \oplus A(D_2, S_2)x_{\sigma_{2,g}} \\
\oplus \cdots \oplus A(D_i, S_i)x_{\sigma_{i,j}} \oplus \cdots \\
\oplus A(D_g, S_g)x_{\sigma_{g,1}} \oplus A(D_g, S_g)x_{\sigma_{g,2}} \oplus \cdots \oplus A(D_g, S_g)x_{\sigma_{g,g}}.
\]

Every term \( y \) of an element of \( A_f \) can be expressed in the form \( y = ax_{(d_i, \sigma_{i,j})} = af(d_i, \sigma_{i,j})^{-1}x_{d_i}x_{\sigma_{i,j}} \) for some \( d_i \in D_i \) and \( a \in S \).
By part (1) of Lemma 4.8, \(e_i a f(\sigma_{i,j}, d_i)^{-1} e_i \in S_i\) (though it could be the case that \(a f(\sigma_{i,j}, d_i)^{-1}\) is not in \(S\) as \(v_{M_k}(f(\sigma_{i,j}, d_i)^{-1})\) could be negative for some \(k \neq i\)). Therefore, we can define \((R\text{-module})\) projection maps

\[
p_{i,j} : \hat{R} \otimes_R A_f \to A(D_i, S_i)x_{\sigma_{i,j}}
\]

by

\[
p_{i,j}(b x_d x_{\sigma_{i,j}}) = (e_i b x_d e_i)x_{\sigma_{i,j}},
\]

\[
p_{i,j}(b x_d x_{\sigma_{i,j}}) = 0 \quad \text{if } i \neq i' \text{ or } j \neq j'
\]

for each \(b = a f(d_i, \sigma_{i,j})^{-1} \in S, d_i \in D_i,\) and \(\sigma_{i,j}\) as above.

Given an ideal \(I\) of \(A_f \otimes_R \hat{R}\), it is clear that \(I = \sum_{i,j} p_{i,j}(I)\). It turns out that we can learn quite a bit about \(I\) from the ideal \(p_{1,1}(I)\) of the crossed product \(A(D_1, S_1)\).

**Lemma 4.11.** Let \(I\) be an ideal of \(A_f \otimes_R \hat{R}\). Assume that \(f(\sigma_{i,j}, \sigma_{i,j}^{-1}) \notin M_i\) for each \(x_{\sigma_{i,j}}\) appearing above in the decomposition of \(A_f \otimes_R \hat{R}\). Then

1. For each \(i\), \(p_{i,1}(I)\) is an ideal of the ring \(A(D_i, S_i)\).
2. The two-sided ideal of \(A_f \otimes_R \hat{R}\) generated by \(p_{i,j}(I)\) and \(p_{i,1}(I)\) is the same ideal as the one generated by \(p_{i,1}(I)\) alone.
3. The two-sided ideal of \(A_f \otimes_R \hat{R}\) generated by \(p_{1,1}(I)\) is the same ideal as the one generated by \(p_{1,1}(I)\).

**Proof.**

(1) This is clear.

(2) The hypothesis that \(e_i a f(\sigma_{i,j}, \sigma_{i,j}^{-1})\) is a unit of \(S_i\) for any \(d \in D_i\) means that \(e_i a f(\sigma_{i,j}, \sigma_{i,j}^{-1})^{-1}\) is an element of \(S\). Let

\[
y = \sum_{d \in D_i} e_i b x_d x_{\sigma_{i,j}} \in p_{i,j}(I),
\]

say \(y = p_{i,j}(z)\) where \(z \in I\). Then \(z x_{(\sigma_{i,j})^{-1}} e_i f(\sigma_{i,j}, \sigma_{i,j}^{-1})^{-1}\) is also in \(I\) (because \(e_i f(\sigma_{i,j}, \sigma_{i,j}^{-1})^{-1} \in S\)), and

\[
p_{i,1}(z x_{(\sigma_{i,j})^{-1}} e_i f(\sigma_{i,j}, \sigma_{i,j}^{-1})^{-1}) = \sum_{d \in D_i} e_i b x_d \in p_{i,1}(I).
\]

Thus, any element \(y = \sum_{d \in D_i} e_i b x_d x_{\sigma_{i,j}}\) of \(p_{i,j}(I)\) can be obtained by multiplying an element \(\sum_{d \in D_i} e_i b x_d\) of \(p_{i,1}(I)\) by \(x_{\sigma_{i,j}}\).

(3) Let \(z \in I\) and \(y = p_{i,1}(z)\), and let \(\sigma = \sigma_{i,j}\) be the right coset representative of \(D_i\) for which \(f(\sigma, \sigma^{-1}) \notin M_i\) and \(\sigma : S_1 \to S_i\). Let \(\phi_{i,1} : A(D_i, S_i) \to A(D_1, S_1)\) be the map

\[
a x_d \mapsto x_{\sigma^{-1}} f(\sigma, \sigma^{-1}) a x_d x_{\sigma}.
\]

Then

\[
\phi_{i,1}(y) = x_{\sigma^{-1}} f(\sigma, \sigma^{-1}) e_i y x_{\sigma} = p_{i,1}(x_{\sigma^{-1}} f(\sigma, \sigma^{-1}) e_i y x_{\sigma}) = p_{i,1}(\phi_{i,1}(z)).
\]
Because \( z \in I \) implies \( \phi_{1,1}(z) = x_{\sigma^{-1}} f(\sigma, \sigma^{-1}) z x_{\sigma} \in I \), we see that \( \phi_{1,1} \) restricts to a map \( \phi_{1,1}|_{p_{i,1}(I)} : p_{i,1}(I) \to p_{i,1}(I) \). To complete the proof of (3), we must show that this restricted map is surjective.

For the remainder of the proof, we shall abuse notation and allow the symbol \( \phi_{i,1} \) to represent the restriction of \( \phi_{i,1} \) to \( p_{i,1}(I) \). Let \( z' \in I \) with \( p_{i,1}(z') = y' = \sum b_d e_d x_d \) (where \( b_d \in S \) and \( d \in D_1 \)). We have

\[
x_{\sigma}(e_1 f(\sigma^{-1}, \sigma)^{-1}) z'(e_1 f(\sigma^{-1}, \sigma)^{-1}) x_{\sigma^{-1}} \in I
\]

because \( I \) is a two-sided ideal. Here we have used the cocycle relation

\[
f(\sigma^{-1}, \sigma)f(1, \sigma^{-1}) = f(\sigma, \sigma^{-1}) f(\sigma^{-1}, 1)
\]

and the fact that \( f(\sigma, \sigma^{-1}) \notin M_1 \) to conclude that \( f(\sigma^{-1}, \sigma) \notin M_1 \sigma^{-1} = M_1 \); this allows us to multiply \( z' \) by \( e_1 f(\sigma^{-1}, \sigma)^{-1} \) in \( A_f \otimes \hat{R} \). Apply \( p_{i,1} \) to get

\[
p_{i,1}[x_{\sigma}(e_1 f(\sigma^{-1}, \sigma)^{-1}) z'(e_1 f(\sigma^{-1}, \sigma)^{-1}) x_{\sigma^{-1}}] = x_{\sigma}(e_1 f(\sigma^{-1}, \sigma)^{-1}) \sum b_d e_d x_d (e_1 f(\sigma^{-1}, \sigma)^{-1}) x_{\sigma^{-1}}.
\]

This is an element of \( p_{i,1}(I) \). Applying \( \phi_{i,1} \), we get

\[
\phi_{i,1}[x_{\sigma}(e_1 f(\sigma^{-1}, \sigma)^{-1}) \sum b_d e_d x_d (e_1 f(\sigma^{-1}, \sigma)^{-1}) x_{\sigma^{-1}}] = x_{\sigma^{-1}} x_{\sigma}(e_1 f(\sigma^{-1}, \sigma)^{-1}) \sum b_d e_d x_d (e_1 f(\sigma^{-1}, \sigma)^{-1}) x_{\sigma^{-1}} x_{\sigma} = \sum e_1 b_d x_d = y',
\]

so \( \phi_{i,1} \) is surjective as desired. Applying \( \phi_{i,1} \) to an element of \( p_{i,1}(I) \) amounts to multiplying that element by other elements of \( A_f \otimes \hat{R} \), so \( \phi_{i,1}(p_{i,1}(I)) \subseteq p_{i,1}(I) \). But \( \phi_{i,1}(p_{i,1}(I)) = p_{i,1}(I) \), so the two-sided ideal of \( A_f \otimes \hat{R} \) generated by \( p_{i,1}(I) \) contains all of the elements of the ideal generated by \( p_{i,1}(I) \). Containment in the other direction is proved in the same way. \( \square \)

We summarize these results in the following proposition:

**Proposition 4.12.** Suppose that, for every maximal ideal \( M_i \) of \( S \), there is a set of representatives \( D_i \sigma_1, \ldots, D_i \sigma_r \) for the cosets of \( D_i \) in \( G \) (so \( G \) is the disjoint union \( \bigcup_j D_i \sigma_j \)) satisfying \( f(\sigma_j, \sigma_j^{-1}) \notin M_i \) for every \( j \). Then every ideal of \( A_f \otimes \hat{R} \) is generated by an ideal of \( A(D_1, S_1) \).

This proposition leads to a local characterization of hereditariness for the crossed product order \( A_f \). We will also need the following lemma.

**Lemma 4.13.** The Jacobson radicals of \( A_f \otimes \hat{R} \) and \( A(D_1, S_1) \) are related by

\[
p_{i,1}(\text{Rad}(A_f \otimes \hat{R})) = \text{Rad}(A(D_1, S_1)).
\]
Proof. Put \( \hat{A}_f := A_f \otimes \hat{R} \). Then Theorem 3.1 says that

\[
A(D_1, S_1) \cap \text{Rad}(\hat{A}_f) \subseteq \text{Rad}(A(D_1, S_1)).
\]

It follows that \( p_{1,1}(\text{Rad}(\hat{A}_f)) = e_1 \text{Rad}(\hat{A}_f)e_1 \) is a subset of \( \text{Rad}(A(D_1, S_1)) \). We now demonstrate the reverse inclusion.

If the maximal ideal of \( R \) is generated by \( \pi_R \), then the maximal ideal of \( \hat{R} \) is also generated by \( \pi_R \).

Because \( \hat{A}_f \) is a finitely generated \( \hat{R} \)-module, we have \( \pi_R \hat{A}_f \subseteq \text{Rad}(\hat{A}_f) \) by Theorem 6.15 of Reiner [10] or Corollary 5.9 of Lam [8].

Let \( \hat{f} \) denote \( f \) followed by the inclusion of \( A_f \) in \( \hat{A}_f \), and let

\[
I'_\sigma := \prod_{\{M \subseteq S \otimes \hat{R} | f(\sigma, \sigma^{-1}) / \in M\}} M
\]

be the product of the maximal ideals of \( S \otimes \hat{R} \) whose generators are missing from the factorization of \( \hat{f}(\sigma, \sigma^{-1}) \). Define also

\[
J' := \sum_{\sigma \in G} I'_\sigma x_\sigma.
\]

Then Lemma 4.2 demonstrates that \( J' \) is an ideal which is nilpotent modulo the product \( \prod M \) of all maximal ideals of \( S \otimes \hat{R} \). But this ideal product is nilpotent modulo \( \pi_R \hat{A}_f \), and \( \pi_R \hat{A}_f \subseteq \text{Rad}(\hat{A}_f) \). Therefore, \( J' \subseteq \text{Rad}(\hat{A}_f) \). Using Theorem 2.4, we see that

\[
\text{Rad}(A(D_1, S_1)) = \sum_{d \in H_1} e_1 S \pi_{M_1} x_d + \sum_{d \in D_1 - H_1} e_1 S x_d
\]

\[
= e_1 \sum_{\sigma \in G} \left( \prod_{\{M \subseteq S \otimes \hat{R} | f(\sigma, \sigma^{-1}) / \in M\}} M \right) x_{\sigma} e_1
\]

\[
= e_1 J' e_1,
\]

which is a subset of \( e_1 \text{Rad}(\hat{A}_f)e_1 = p_{1,1}(\text{Rad}(\hat{A}_f)) \). \( \square \)

We are ready to prove our final result: under the hypothesis that there is a “nice enough” set of coset representatives for each \( G/D_i \), the hereditariness of \( A_f \) is determined by the hereditariness of \( A(S_1, D_1) \).

Theorem 4.14. Suppose that, for every maximal ideal \( M_i \) of \( S \), there is a set of representatives \( D_i \sigma_1, \ldots, D_i \sigma_r \) for the cosets of \( D_i \) in \( G \) (so \( G \) is the disjoint union \( \bigcup_j D_i \sigma_j \) satisfying \( f(\sigma_j, \sigma_j^{-1}) / \in M_i \) for all \( j \). Then the following are equivalent:

1. \( \forall M_1 (f(d_1, d_2)) \leq 1 \) for some maximal ideal \( M_1 \) of \( S \) and for every \( d_1 \) and \( d_2 \in D_1 \). Equivalently, \( A(D_1, M_1) \) is hereditary.
2. \( \forall M_1 (f(d_1, d_2)) \leq 1 \) for every maximal ideal \( M_1 \) of \( S \) and for every \( d_1 \) and \( d_2 \in D_i \). Equivalently, each \( A(D_i, M_i) \) is hereditary.
3. \( A_f \) is a hereditary order.

Proof. Put \( \hat{A}_f := A_f \otimes \hat{R} \). Then Theorem 3.1 says that

\[
A(D_1, S_1) \cap \text{Rad}(\hat{A}_f) \subseteq \text{Rad}(A(D_1, S_1)).
\]

It follows that \( p_{1,1}(\text{Rad}(\hat{A}_f)) = e_1 \text{Rad}(\hat{A}_f)e_1 \) is a subset of \( \text{Rad}(A(D_1, S_1)) \). We now demonstrate the reverse inclusion.

If the maximal ideal of \( R \) is generated by \( \pi_R \), then the maximal ideal of \( \hat{R} \) is also generated by \( \pi_R \).

Because \( \hat{A}_f \) is a finitely generated \( \hat{R} \)-module, we have \( \pi_R \hat{A}_f \subseteq \text{Rad}(\hat{A}_f) \) by Theorem 6.15 of Reiner [10] or Corollary 5.9 of Lam [8].

Let \( \hat{f} \) denote \( f \) followed by the inclusion of \( A_f \) in \( \hat{A}_f \), and let

\[
I'_\sigma := \prod_{\{M \subseteq S \otimes \hat{R} | f(\sigma, \sigma^{-1}) / \in M\}} M
\]

be the product of the maximal ideals of \( S \otimes \hat{R} \) whose generators are missing from the factorization of \( \hat{f}(\sigma, \sigma^{-1}) \). Define also

\[
J' := \sum_{\sigma \in G} I'_\sigma x_\sigma.
\]

Then Lemma 4.2 demonstrates that \( J' \) is an ideal which is nilpotent modulo the product \( \prod M \) of all maximal ideals of \( S \otimes \hat{R} \). But this ideal product is nilpotent modulo \( \pi_R \hat{A}_f \), and \( \pi_R \hat{A}_f \subseteq \text{Rad}(\hat{A}_f) \). Therefore, \( J' \subseteq \text{Rad}(\hat{A}_f) \). Using Theorem 2.4, we see that

\[
\text{Rad}(A(D_1, S_1)) = \sum_{d \in H_1} e_1 S \pi_{M_1} x_d + \sum_{d \in D_1 - H_1} e_1 S x_d
\]

\[
= e_1 \sum_{\sigma \in G} \left( \prod_{\{M \subseteq S \otimes \hat{R} | f(\sigma, \sigma^{-1}) / \in M\}} M \right) x_{\sigma} e_1
\]

\[
= e_1 J' e_1,
\]

which is a subset of \( e_1 \text{Rad}(\hat{A}_f)e_1 = p_{1,1}(\text{Rad}(\hat{A}_f)) \). \( \square \)

We are ready to prove our final result: under the hypothesis that there is a “nice enough” set of coset representatives for each \( G/D_i \), the hereditariness of \( A_f \) is determined by the hereditariness of \( A(S_1, D_1) \).

Theorem 4.14. Suppose that, for every maximal ideal \( M_i \) of \( S \), there is a set of representatives \( D_i \sigma_1, \ldots, D_i \sigma_r \) for the cosets of \( D_i \) in \( G \) (so \( G \) is the disjoint union \( \bigcup_j D_i \sigma_j \) satisfying \( f(\sigma_j, \sigma_j^{-1}) / \in M_i \) for all \( j \). Then the following are equivalent:

1. \( \forall M_1 (f(d_1, d_2)) \leq 1 \) for some maximal ideal \( M_1 \) of \( S \) and for every \( d_1 \) and \( d_2 \in D_1 \). Equivalently, \( A(D_1, M_1) \) is hereditary.
2. \( \forall M_1 (f(d_1, d_2)) \leq 1 \) for every maximal ideal \( M_1 \) of \( S \) and for every \( d_1 \) and \( d_2 \in D_i \). Equivalently, each \( A(D_i, M_i) \) is hereditary.
3. \( A_f \) is a hereditary order.
Proof. (2) ⇒ (1) This is clear.

(3) ⇒ (2) This follows directly from 4.6 and 2.15.

(1) ⇒ (3) The order $A_f$ will be hereditary if and only if $A_f \otimes \hat{R}$ is hereditary ($\hat{R}$ is a faithfully flat $R$-module and $A_f$ is a Noetherian ring, cf. Reiner [10], 2.21-2.22, 3.29, and the remarks preceding 3.29). To see that $A_f \otimes \hat{R}$ is hereditary, we will show that the Jacobson radical is principally generated. From Lemma 4.13, we have

$$p_{1,1}(\text{Rad}(A_f \otimes \hat{R})) = \text{Rad}(A(D_1, S_1)),$$

which is a principally generated ideal of $A(D_1, S_1)$ by Theorem 2.15. By Proposition 4.12, $p_{1,1}(\text{Rad}(A_f \otimes \hat{R}))$ generates $\text{Rad}(A_f \otimes \hat{R})$, so $\text{Rad}(A_f \otimes \hat{R})$ is also principally generated. The result follows by Lemma 2.16. □

Remark 4.15. The hypothesis of 4.14 requires the existence of a suitable set of representatives for the cosets of $D_1$ in $G$. The extent to which this hypothesis is restrictive or necessary is unclear.

Haile and Williamson both provide examples of hereditary crossed products satisfying the hypotheses of Theorem 4.14:

1. Haile gives several examples of weak cocycles in Chapter 4 of [2] in the case that $S/R$ is unramified. When $S/R$ is unramified, the existence of the special set of representatives for the cosets of $D_1$ in $G$ is equivalent to $A_f$ having a unique maximal ideal (that is, $A_f$ is primary, cf. Theorem 3.2 of [2]). If $A_f$ is both hereditary and primary, then $A_f$ is a maximal order in $\Sigma_f$ by results of Harada [4].

2. If $S/R$ is tamely ramified and $A_f$ is a classical crossed product order ($f$ is unit-valued), then $A_f$ is hereditary (Williamson [11] Theorem 1.6). Because $f$ is unit-valued, the hypothesis of Theorem 4.14 is satisfied by any set of representatives for the cosets of $D_1$ in $G$.

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References