General integral equations of thermoelasticity in micromechanics of composites with imperfectly bonded interfaces

Valeriy A. Buryachenko *

Civil Engineering Department, University of Akron, Akron, OH 44325-3901, USA
IllinoisKosstara LLC, 60 Hazelwood Drive, Champaign, IL 61820, USA

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ABSTRACT

One considers a linear thermoelastic composite medium, which consists of a homogeneous matrix containing a statistically inhomogeneous random set of heterogeneities with various interface effects and subjected to essentially inhomogeneous loading by the fields of the stresses, temperature, and body forces (e.g., for a centrifugal load). The general integral equations connecting the stress and strain fields in the point being considered and the surrounding points are obtained for the random and deterministic fields of inclusions. The method is based on a centering procedure of subtraction from both sides of a new initial integral equation their statistical averages obtained without any auxiliary assumptions such as the effective field hypothesis (EFH), which is implicitly exploited in the known centering methods. The new initial integral equation is presented in a general form of perturbations introduced by the heterogeneities and taking into account both the spring-layer model and coherent imperfect one. Some particular cases, asymptotic representations, and simplifications of proposed equations as well as a model example demonstrating the essence of two-step statistical average scheme are considered. General integral equations for the doubly and triply periodical structure composites are also obtained.

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1. Introduction

A growing recognition that the properties of composite materials (CMs) essentially depend on the interface matrix/fillers phenomena reflects the explosive character of the progress in modern nano- and micromechanics caused by the development of image analyses and computer-simulation methods on one hand as well as advanced experimental techniques (such as X-ray tomography and electron microscopy) and improved materials processing (prescribed structure controlled by processing) on the other. Interfacial bonding conditions play an important role that control key properties of the composite materials (CM) such as stiffness, strength and fracture behavior. In the context of classical elasticity the interface between constituent phases are assumed to be perfect that means continuity of both the displacement and traction vector across interface. However, in many cases imperfect interfacial bonding may exist in CMs. These situations of imperfect interfaces include the following cases: unilateral contact of two adjacent parts, weak interfacial layer due to imperfect bonding between the two phases; inter-diffusion and/or chemical interaction zones (with properties varying through the thickness and/or along the surface) at the interface between the two phases. Significance of interface effects are drastically increased for nanocomposites when surface to volume ratio becomes appreciable and the coupled deformation-dependent interfacial energy can not be negligible with respect to the bulk energy. In the literature, three kinds of model are often used to simulate the properties of interface regions in CM. The first kind of model can be referred to as interface models in which displacement discontinuities are assumed to exist at an interface. Interface models include the dilatation-like model, free sliding model, linear-spring interface model and its nonlinear generalization such as cohesive zone model, where interfaces are assumed to be comprised of zero thickness nonlinear springs with a specific traction–displacement law. The second kind of model (coherent interface model) is characterized by traction discontinuities at the interface where additional interface stresses appear due to the different coordination numbers of the surface/interface atoms, different bond lengths, angles, and a different charge distribution (see e.g. Ibach, 1997; Gurtin et al., 1998). This interface is assumed to be “attached” to both the heterogeneity and matrix without any discontinuity of displacement across it. The third kind of model is the interphase model which describes the interface region as a layer, called an interphase, between the inhomogeneity and matrix (see for references Buryachenko, 2007a). The elastic moduli of the interphase are different from those of the matrix and inhomogeneity, and can be homogeneous or variable. Perfect
bonding is generally assumed to prevail at both the matrix/interphase and interphase/inhomogeneity interfaces.

We shortly consider the basic results of micromechanics for one heterogeneity in the infinite homogeneous matrix (see for references Mura et al., 1996; Buryachenko, 2007a; Zhou et al., 2013). For elastically isotropic ellipsoidal inclusion perfectly bonded with an elastically isotropic matrix, Eshelby (1961) showed that the field in the ellipsoid is polynomial if the applied field is polynomial. The simplest case of the first kind of model associates the material displacement discontinuities discussed above with Somigliana (1886) dislocations describing the displacements given by \( \mathbf{b}(\mathbf{s}) \) at the interface \( \mathbf{s} \in \Gamma \). Asaro (1975) proved that for the ellipsoidal transformed region and polynomial \( \mathbf{b}(\mathbf{s}) \) in the spatial variables, the internal strain within the region caused by the constraint of the matrix is also polynomial. Grain boundary sliding in polycrystalline and granular materials can be observed even at room temperature, and Mura and Furuhashi (1984), Mura et al. (1985) and Jasiuk et al. (1987), among others solved the problem for ellipsoidal inclusion with a frictionless sliding condition at the interface, while Hatami-Marbini and Shodja (2007) have considered the thermoelastic problem for a functionally graded coated inhomogeneity with sliding interfaces. Numerous researchers (see, e.g., Hashin, 1991, 2002; Dvorak and Benveniste, 1992; Huang et al., 1993; Zhong and Meguid, 1997) have analyzed a more realistic model, so called linear spring imperfect interface model, where the resistance of the interface is accommodated by assuming that the discontinuous components of displacement are proportional to the corresponding traction field at the interface. In a cohesive law, the traction vector assumed to be continuous is a non-linear function of the displacement jump (bilinear, trapezoidal, exponential and polynomial cohesive laws); linear approximation of this function at the origin leads to the spring-layer model. The cohesive model originated by Barenblatt (1962) in fracture mechanics (see also Needleman, 1990; Ortiz and Pandolfi, 1999) has received wide development in micromechanics of CM (see, e.g., Tvergaard, 1990; Tan et al., 2007a,b; Othmani et al., 2011). The linear coherent interface model, which can be viewed as dual with respect to the linear spring-layer model, specifies that the displacement vector is continuous across an interface whereas the stress vector suffers a jump across the same interface, which must comply with the generalized Young–Laplace equation, and 2D interface constitutive equation (see, e.g., Gurtin and Murdoch, 1975; Povstenko, 1993; Ibach, 1997; Gurtin et al., 1998; where additional references can be found). Additional surface stress appears due to the difference of the equilibrium lattice spacing in the surface and the bulk that manifested for solids with large interface surface-to-volume ratio as for materials containing nanoparticles and nanovoids (see for references, e.g., the review articles by Maranganti and Sharma, 2007; Wang et al., 2011). Recently the coherent model was applied for analysis of nanoscale structures including the stretching, bending and torsion of rods (Miller and Shenoy, 2000; Shenoy, 2002). The classical Eshelby formalism was extended for inclusion/inhomogeneity problems to the nanoscale by taking the linear coherent model into account by Sharma and Ganti (2004), Duan et al. (2005), He and Li (2006), Chen et al. (2007) and Sharma and Wheeler (2007). Kushch et al. (2011) have obtained a complete and accurate solution for the problem of multiple interacting spherical inhomogeneities with coherent interface by the use a vectorial spherical harmonics-based analytical technique developed. The presence of interface stresses results in a size-dependent of the elastic response of nanocomposites, consistent with some experimental observations (see, e.g., Wong et al., 1997). It is interesting that both two dual basic models of the imperfect interface (spring-layer and coherent ones) were at the beginning proposed as phenomenological ones and then derived rigorously as a limiting asymptotic approach of interphase model (see for references and details Benveniste, 2006; Quang and He, 2008). Namely, when a linearly elastic interphase between two linearly elastic bodies is very thin, the linear spring-layer model or the linear coherent model can be considered as a limiting case of the interphase elastic properties which are much softer or stiffer, respectively, than each of the bodies (otherwise, an imperfect interface “intermediate” model is more appropriate).

The prediction of the behavior of CM in terms of the mechanical properties of constituents and their microstructure is a central problem of micromechanics, which is evidently reduced to the estimation of stress fields in the constituents. Appropriate, but by no means exhaustive, references for the estimation of effective elastic moduli of statistically homogeneous media are provided by the reviews Shermegor (1977), Mura (1987), Nemat-Nasser and Hori (1993), Torquato (2002), Milton (2002) and Buryachenko (2007a). It appears today that variants of the effective medium method (Kröner, 1958; Hill, 1965) and the Mori–Tanaka method (MTM, Mori and Tanaka, 1973; Benveniste, 1987) are the most popular and widely used methods. More recently the multiparticle effective field method (MFEM) was put forward and developed by the author (see for references Buryachenko, 2007a). The popular approaches of analysis of CMs with imperfect interface properties are usually based on the coupling of two problems. At first the problem for one heterogeneity inside an infinite matrix subjected to the homogeneous remote stresses is solved for the different boundary bonded laws of both the heterogeneity and matrix (see, e.g., the references above). The second problem is incorporating the mentioned solution into one of the known micromechanical schemes, e.g., in the MTM which is totally dominating in micromechanics of CMs with the accepted imperfectly bonded interfaces (see, e.g., Tan et al., 2007a,b; Duan et al., 2007; Chen et al., 2007; Sharma and Wheeler, 2007 and other references mentioned above). However, all mentioned methods of micromechanics are based on the effective field hypothesis (EFH) by Mossotti (1850) critically analyzed in the resent publications by Buryachenko (2010a,b, 2011a,b) and Buryachenko and Brun (2011, 2012a,b, 2013). Moreover, a concrete average scheme of micromechanics is initially determined by both the EFH and additional ones used for the approximate solution of exact initial integral equations connecting the random stress fields at the point being considered and the surrounding points.

The main goal of the current paper is obtaining such general integral equation taking into account the actual microstructure and possible imperfectly bonded interfaces of CMs subjected to essentially inhomogeneous mechanical, body force, and temperature loading. Unfortunately, the starting assumptions made in the majority of studies, namely, that the structure of the composite media as well as the random fields of stresses are statistically homogeneous and therefore are invariant with respect to the translation, are incorrect. For example, due to some production technologies, the inclusion concentration may be a function of the coordinates (see, e.g., Conlon and Wilkinson, 1996; Plankenhans et al., 1996,1997). The accumulation of damage also occurs locally in stress-concentration regions, for example, at the tip of a macroscopic crack (see, e.g. Hori and Nemat-Nasser, 1987). Furthermore, in layered composite shells, the location of the fibers is random within the periodic layers, and the micromechanics equations have almost periodic coefficients. Finally, functionally graded materials (FGMs) have been the subject of intense research efforts from the mid–1980s when this term was originated in Japan in the framework of a national project to develop heat-shielding structural materials for the future Japanese space program. FGMs is a composite consisting of two or more phases which is fabricated with a spatial variation of its composition that may improve the structural response (see, e.g., Markworth et al., 1995; Mortensen and Suresh, 1995). Moreover, FGMs on frequent occasions are sub-
jected to essentially inhomogeneous loading by fields of the stresses, temperature, and body forces (e.g., for a centrifugal load) when the length-scale hypothesis (2.21) is violated. The situation becomes even more complicated with consideration of diversified imperfectly bonded interface conditions.

The outline of the paper is as follows. In Section 2 we present the basic field equations of linear thermoelasticity, notations, and statistical description of the composite microstructure. The different cases of imperfectly bonded interface conditions are considered. In Section 3 the general integral equations connecting the stress and strain fields in a point with the stress and strain field in the surrounding points are obtained for the random fields of heterogeneities. This approach is based on a centering procedure of subtracting from both sides of a new initial integral equation their statistical averages obtained without any auxiliary assumptions such as the EFE, which is implicitly exploited in the known centering methods. The new initial integral equation is presented in a general form of perturbations introduced by the heterogeneities and taking into account both the spring-layer model and coherent imperfect one. In Section 4 some particular cases, asymptotic representations, and simplifications as well as a model example demonstrating the essence of two-step statistical average scheme are considered. General integral equations for the double and triple periodical structure composites are obtained in Section 5.

2. Preliminaries

2.1. Basic equations

Let a linear elastic body occupy an open simply connected bounded domain \( \Omega \subset \mathbb{R}^d \) with a smooth boundary \( \Gamma_\Omega \) and with an indicator function \( W \) and space dimensionality \( d (d = 2 \text{ and } d = 3) \) for 2-D and 3-D problems, respectively. The domain \( \Omega \) contains a homogeneous matrix \( \nu_0 \) and, in general, a statistically inhomogeneous field \( \nu = (\nu) \) of heterogeneity \( \nu \) with indicator functions \( V_i \) and bounded by the closed smooth surfaces \( \Gamma_i (i = 1, 2, \ldots) \) defined by the relations \( \Gamma_i(x) = 0 \) (\( x \in \Gamma_i \)), \( \Gamma_i(x) > 0 \) (\( x \in \nu_i \)), and \( \Gamma_i(x) < 0 \) (\( x \notin \nu_i \)). It is assumed that the heterogeneities can be grouped into components (phases) \( \nu_i (q = 1, 2, \ldots, N) \) with identical mechanical and geometrical properties (such as the shape, size, orientation, and microstructure of heterogeneities). For the sake of definiteness, in the 2-D case we will consider a plane-strain problem. Initially no restrictions are imposed on the elastic symmetries of the phases or on the geometry of the heterogeneities.\(^1\)

We will consider the local basic equations of thermoelasticity of composites

\[
\sigma(x) = -f(x), \quad \sigma(x) = L(x)\epsilon(x) + \beta(x), \quad \epsilon(x) = M(x)\sigma(x) + \beta(x),
\]

\[
\epsilon(x) = \nabla \delta u, \quad \nabla \times \epsilon(x) \times \nabla = 0,
\]

where \( \otimes \) and \( \times \) are the tensor and vector products, respectively, and the operator \( a \otimes b = \otimes (a \otimes b + b \otimes a) / 2 \) where \( \otimes \) denotes matrix transposition. The body force tensor \( f \) can be generated by either gravitational loads or a centrifugal load. \( L(x) \) and \( M(x) = E(x)^{-1} \) are the known phase stiffness and compliance fourth-order tensors, \( \beta(x) \) and \( \beta(x) = -L(x)\beta(x) \) are second-order tensors of local eigenstresses and eigenstresses, and the common notation for contracted products has been employed: \( \{L\} = \frac{4\alpha}{d} \mathbf{I}_{d \times d} \) (\( i, j, k, l = 1, \ldots, d \)). In particular, for isotropic constituents the local stiffness tensor \( L(x) \) is given in terms of the local bulk modulus \( k(x) \) and the local shear modulus \( \mu(x) \) and the local eigenstress \( \beta(x) \) is given in terms of the bulk component \( \beta_0(x) \) by the relations:

\[
L(x) = (2k\mu) \delta (x) + 2\mu(x)N_x, \quad \beta(x) = \beta_0(x) \delta.
\]

(2.4)

\( N_i = \delta \otimes \delta/d; \quad N_i = 1 - N_i \), \( d = 2 \) or 3; \( \delta \) and \( I \) are the unit second-order and fourth-order tensors. For the fiber composites it is the plane-strain bulk modulus \( k_3 \) and the 3-D bulk modulus \( k_{32} \) that play the significant role: \( k_{32} = k_{31} + \mu_3/3, \mu_3 = \mu_3 \). For all material tensors \( g \) (\( L, M, \alpha, \beta, f \)) the notation \( g_i(x) = g(x) - g = g^{(i)}(x) \) \( x \in \nu^{(i)}, m = 0, 1, \ldots, N \) is used and one introduces a homogeneous “comparison” body with homogeneous moduli \( L \), and with the inhomogeneous deterministic transformation field \( x'(x) \) and body force \( F(x) \). The introduction of jumps of material properties allows one to define the stress \( \tau \) and strain \( \eta \) polarization tensors \( (x \in w) \)

\[
\tau(x) = T_1(x)\epsilon(x) + \gamma_1(x), \quad \eta(x) = M_1(x)\epsilon(x) + \beta_1(x),
\]

(2.5)

which are simply a notational convenience and vanish inside the matrix \( \tau(x) = \eta(x) = 0 \) \( x \in \nu^{(0)} \) \( \gamma \) \( L = L^{(0)} \). The upper index \( \nu \) indicates the components and the lower index \( i \) indicates the individual heterogeneities: \( \nu_i = \nu \setminus \nu_i \). The notation \( \eta = \eta^{(0)} \) \( \nu_i = \nu \setminus \nu_i \) respectively, equals 1 at \( x \in \nu^{(i)} \) and 0 otherwise, \( (m = 0; k = 1, 2, \ldots; N; i = 1, 2, \ldots) \). \( V_i(x) = \delta(x - x_i) \) is the delta function of random set of heterogeneities centers (also called the density field, see Stratonovich, 1963; Ponte Castañeda and Willis, 1995).

The interfaces between the constituent phases of CM are classically assumed to be perfect. If the phases are perfectly bonded, the displacements and the traction components are continuous across the interphase boundaries, i.e.

\[
\sigma \cdot n = 0, \quad \mathbf{u} = 0
\]

(2.6)

on the interface boundary \( \Gamma = \Gamma_i (i = 1, \ldots) \) (assumed to be sufficiently smooth) where \( n \) is the outward normal vector on \( \Gamma \) from \( \nu \) to \( \nu^{(0)} \) and \( \Gamma_i \) is the jump operator. The traction \( t(x) = \sigma(x)n(x) \) acting on any plane with the normal \( n(x) \) through the point \( x \) can be represented in terms of displacements \( t(x) = t(n, \nabla u(x)) + z(n, u(x)) \), where \( t(n, \nabla u(x)) = t_{\text{out}}(n, \nabla u(x)) / \partial \delta t \) is the conormal derivative operator.

The boundary conditions at the interface boundaries will be considered together with the mixed boundary conditions on \( \Gamma_0 \) with the unit outward normal \( n^{(i)} \nabla u^{(i)}(x), \quad x \in \Gamma_0^{(i)}, \quad \sigma^{(i)}(x)n^{(i)}(x) = t^{(i)}(x), \quad x \in \Gamma_0^{(i)}, \quad (2.7) \)

\[
\sigma^{(i)}(x)n^{(i)}(x) = t^{(i)}(x), \quad x \in \Gamma_0^{(i)},
\]

(2.8)

where \( \Gamma_0 \) and \( \Gamma_0^{(i)} \) are prescribed displacement and traction boundaries such that \( \Gamma_0 \cap \Gamma_0^{(i)} = \emptyset \). \( \Gamma_0^{(i)} \) \( \Gamma_0 \) \( \Gamma_0 \) \( \emptyset \) \( \mathbb{R}^{(i)}(x) = \text{const} \), \( x \in \Gamma_0^{(i)} \). \( t^{(i)}(x) = \sigma^{(i)}(x)n^{(i)}(x) = \text{const} \), \( x \in \Gamma_0 \). \( (2.10) \)

\[
\frac{\partial t^{(i)}(x)}{\partial \delta t} = \frac{\partial \sigma^{(i)}(x)n^{(i)}(x)}{\partial \delta t} \text{ const}, \quad x \in \Gamma_0^{(i)}, \quad \sigma^{(i)}(x) \text{ are the given constant symmetric tensors of the macroscopic strain and stress, respectively. We will consider the interface problem when the body occupies the interior domain with respect to \( \Gamma_0 \).}

2.2. Imperfect interface conditions

The assumption of perfect interfaces (2.6) is not appropriate in a lot of cases. In the first kind of imperfect interface models, the

\(^1\) It is known that for 2-D problems the plane-strain state is only possible for material symmetry no lower than orthotropic (see e.g. Lekhntski, 1963) that will be assumed hereafter in 2-D case.
displacement components are discontinuous $[\mathbf{u}(s)] = 0$ at the surface $s \in \Gamma^\epsilon_1 \subset \Gamma_1$ and the strain (2.3.1) contains a delta function at the interface $\Gamma_1^\epsilon$ (with zero thickness) due to differentiation of discontinuous function $\mathbf{u}(s)$ on $\Gamma_1^\epsilon$

$$
\varepsilon(x) = \nabla \otimes \mathbf{u}(x) + [\mathbf{u}(x)] \otimes \mathbf{n}(x) \delta_{\Gamma_1^\epsilon},
$$

(2.11)

where $\mathbf{n}(s)$ is the normal vector on $\Gamma_1^\epsilon$ (assumed to be sufficiently smooth) defined by the equation $n_{\Gamma_1^\epsilon}(x) = 0$, and $\delta_{\Gamma_1^\epsilon}$ is the Dirac distribution on $\Gamma_1^\epsilon$. Here we used the properties of the distributional derivatives of $\mathbf{V}(x)$ (see for details, e.g., p. 127 in Hsiao and Wendland, 2008)

$$
\nabla \mathbf{V}(x) = -\mathbf{n}(x) \delta_{\Gamma_1^\epsilon},
$$

(2.12)

with the Dirac distribution $\delta_{\Gamma_1^\epsilon}$ on $\Gamma_1^\epsilon$ defined by

$$
(\delta_{\Gamma_1^\epsilon}, \phi)_{\mathcal{G}} = \int_{\Gamma_1^\epsilon} \delta_{\Gamma_1^\epsilon}(x) \phi(x) dx := \int_{\Gamma_1} \phi(x) ds
$$

(2.13)

for any test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

The normal $\mathbf{n}(s)$ is typically presented in the form of two normals $\mathbf{n}(s) = \mathbf{n}_1(s)$ and $\mathbf{n}(s) = -\mathbf{n}_1(s)$ at the surfaces $\Gamma^\epsilon_1$ and $\Gamma_1^\epsilon$ ($\Gamma_1^\epsilon = \Gamma_1^\epsilon_1 \cup \Gamma_1^\epsilon_2$), respectively, and the last item in Eq. (2.6) is conventionally represented as $[\mathbf{u}(s)] \otimes \mathbf{n}_{\Gamma_1^\epsilon} = [\mathbf{u}(s)] \otimes \mathbf{n}(s)$. The interface conditions for the stress field $\sigma' = \mathbf{P}' \mathbf{P}$ is also continuous $[\sigma'] = 0$ ($s \in \Gamma_1^\epsilon$) while the strain field $\varepsilon$ is in general discontinuous across $\Gamma_1^\epsilon$. The concept of surface or interface elasticity is introduced which is excluded in the classical elasticity theory (2.1)–(2.3) and governed by 2D constitutive equation at the interface $\Gamma_1^\epsilon$

$$
\sigma' = \tau_0 + 2(\mu_0 - \tau_0)\varepsilon + (\lambda_0 + \tau_0)\text{Tr}(\varepsilon)\mathbf{I},
$$

(2.18)

where the considered isotropic interface is characterized by surface Lamé constants $\lambda_0, \mu_0$, and surface tension $\tau_0$. If $\Gamma$ and $\mathbf{T}$ represent the $2 \times 2$ unit tensor and the trace, respectively. Coupling of the bulk and surface tensors is performed through the static equilibrium at the interface $\Gamma_1^\epsilon$ governed by the generalized Young–Laplace equation

$$
[\sigma'] \cdot \mathbf{n} + \nabla \cdot \sigma' = 0
$$

(2.19)

where the action of the surface gradient $\nabla$ on a vector $\mathbf{v}$ is defined through the usual 3D gradient operator $\nabla \mathbf{v} = \nabla \mathbf{v}$. In the case of absent surface terms denoted by the upper index $i$, Eq. (2.19) reduces to the usual traction continuity equation of classical elasticity (2.61).

### 2.3. Statistical description of the composite microstructure

It is assumed that the representative macrodomain $\omega$ contains a statistically large number of realizations $\omega$ (providing validity of the standard probability technique) of heterogeneous $\omega_i \in \omega^{(k)}$ of the constituent $\omega^{(i)}$ ($i = 1, 2, \ldots ; k = 1, 2, \ldots, N$). A random event $\omega$ belongs to a sample space $\mathcal{A}$, over which a probability density $p(\mathbf{x}, \omega)$ is defined (see, e.g., Willis, 1981). For any given $\mathbf{x}$, any random function $g(\mathbf{x}, \omega)$ (e.g., $g = V, V^{(k)}, \sigma, \epsilon$) is defined explicitly as one particular member, with label $\omega$, of an ensemble realization. Then, the mean, or ensemble average is defined by the angle brackets enclosing the quantity $g$

$$
\mathbf{g}(\mathbf{x}) = \int_A g(\mathbf{x}, \omega) p(\mathbf{x}, \omega) d\mathbf{x}.
$$

(2.20)

No confusion will arise below in notation of the random quantity $g(\mathbf{x}, \omega)$ if the label $\omega$ is removed. One treats two material length scales (see, e.g., Torquato, 2002): the macroscopic scale $L$, characterizing the extent of $\omega$, and the microscopic scale $a$, related with the heterogeneities $\omega_i$. Moreover, one supposes that applied field varies on a characteristic length scale $A$. The limit of our interests for both the material scales and field one is

$$
L \gg A > a.
$$

(2.21)

All the random quantities under discussion are described by statistically inhomogeneous random fields. For the alternative description of the random structure of a composite material let us introduce a conditional probability density $\phi(t, \mathbf{x}|\mathbf{t}', \mathbf{x}_1, \ldots$.
In the domain $\psi$, which is a probability density for finding a heterogeneity of type $i$ with the center $x$ in the domain $\nu_i$ given that the fixed heterogeneities $\nu_1, \ldots, \nu_n$ are centered at $x_1, \ldots, x_n$ (see, e.g., Willis, 1978). The notation $\psi(t, x; x_1, x_2, \ldots, x_n, \nu_i)$ denotes the case $x = x_1, \ldots, x_n$. In the case of statistically inhomogeneous media with homogeneous matrix (for so-called Functionally Graded Materials (FGM)), the conditional probability density is not invariant with respect to translation: $\psi(t, X + x; x_1, x_2, \ldots, x_n, \nu_i) \neq \psi(t, x; x_1, x_2, \ldots, x_n, \nu_i)$, where $X$ is the microstructure functions depend upon their absolute positions. In particular, a random field is called statistically homogeneous in a narrow sense if its multi-point statistical moments of any order are shift-invariant functions of spatial variables. Of course, $\psi(t, x; x_1, x_2, \ldots, x_n, \nu_i)$ at the point $x$ and $\psi(t, x; x_1, x_2, \ldots, x_n, \nu_i)$ is the concentration, i.e. volume fraction, of the component $v$ intersected with the boundary $\nu = (x)$ lying inside are centered at $x_1, \ldots, x_n$ with indicator function $\chi_i$ if a fixed ellipsoid $\nu_i$ is shrinking to the point $x$. Then we can get a relation between the mentioned averages $[x = (x_1, x_2, \ldots, x_n)]$:

$$c^{(i)}(x)[g^{(i)}(x)] = \int_{v_i(x)} \rho^{(i)}(y)[g(v_i(x), y)](x, y - x) \, dy. \quad (2.24)$$

Formula (2.24) is valid for any material inhomogeneity of inclusions of any concentration in the macrodomain $w$ of any shape (if $v_i(x) \subset w$). In this relation (2.24), the total probability is expressed in term of conditional probabilities over portioned probability space. Obviously, the general Eq. (2.24) is reduced to Eq. (2.24) for both the statistically homogeneous media subjected to homogeneous boundary conditions and statistically homogeneous fields $g$ (e.g., $g = \sigma, \epsilon$). However, in a general case $g(v_i(x, y)) \equiv f(x, y)g(v_i(x)) [g(v_i(x), y)]$ is a statistically homogeneous field and $f(x, y)$ is a function of $x, y$. Eq. (2.24) is not reduced to Eq. (2.24). It should be mentioned that for statistically homogeneous media fields $g^{(i)}(x)[g^{(i)}(x)] = \text{const}$ and $v_i(x) \equiv v_i; \text{const}$, while in general $v_i(x, z) \equiv g^{(i)}(x, z) \neq \text{const}$. At the micro-coordinate $z \in v_i$. Because of this, no confusion will arise below in notations of the average $g^{(i)}(x)$ and conditional average $g^{(i)}(x)$ (at the fixed inclusion $v_i$) which are functions of macro-coordinate (with resolution equal to $L$) and micro-coordinate $z \in v_i$ used in the case of fixed inclusion $v_i$, respectively.

3. General integral equation

In this section we show a revised version of the general integral equation obtained in Buryachenko (2010a) in a more general and illustrative form. We start from the solution of a single heterogeneity inside a finite homogeneous domain in the form suitable to obtain the new integral equations.

3.1. Stress and strain distributions for one heterogeneity inside macrodomain $w$

Let us assume that the domain $w$ contains one heterogeneity $v_1 \subset w$. Substituting (2.22) and (2.3) into the equilibrium Eq. (2.1) leads to a differential equation with respect to the displacement $u$

$$\nabla \cdot [L(x) \nabla \sigma(u(x))] + x^{(r)}(x) = 0 \quad (3.1)$$

with a fictitious random “body-force” in the right-hand side of the equation. Then Eq. (3.1) can be reduced to a symmetrized integral form:

$$\varepsilon(x) = \varepsilon^0(x) + \nabla \frac{\mathcal{S}}{w} \int_{w} G(x, y) \, dy \quad (3.2)$$

where $G(x)$ is the infinite body Green’s function of the Navier equation with homogeneous elastic modulus tensor $L^*$, defined by

$$\nabla \cdot [L^*(x) \nabla G(x, y)] = -\delta(x - y) \quad (3.3)$$

of order $O(\|x\|^{-4}d(x))$ as $|x| \to \infty$; the symbols $\mathcal{S}$ are replaced by $\odot$ in Eqs. (3.1)-(3.3) due to the symmetry of $L^*$ and $L$. In Eq. (3.2), $\varepsilon^0(x)$ is the strain corresponding to a homogeneous domain $w$ with modulus tensor $L^*$ and which is described by the boundary integral equation (BIE, see, e.g., Brebbia et al., 1984; Ballas et al., 1989) on the boundary $\Gamma_0$ of the domain $w$

$$\varepsilon^0(x) = \int_{\Gamma_0} \nabla \odot \sigma^0(x - s) \, \Gamma_0(s) + \nabla \odot T^0(x, s) \psi_0(s) \, ds \quad (3.4)$$

where $\sigma^0(x)$ is the force density.
which confirms with the stress field $\sigma^0(x) = L^\prime \theta(x) + \sigma^0(x); \theta(x)$ is the modified traction on the surface $\Gamma_0$ with the outer normal $n^0(s)$ ($s \in \Gamma_0$). The tensor of the “fundamental traction” $T$ on $\Gamma_0$ (called also Kupradze tensor) associated with the tensor of “fundamental displacement” $G$ is given by

$$T_v(x, s) = L_{qpv} \frac{\partial G_p}{\partial q}(x - s).$$

It should be mentioned that Eqs. (3.1) and (3.2) are well known for the ideal interface $\Gamma_v$ (2.6) and presented for an illustrative demonstration of corrections of these equations for the discontinuity of the displacement or/and stress fields at the interface $\Gamma_v$. The case of discontinuities of $u(x)$ and $\sigma(x)$ at the surfaces $\Gamma^0_v$ and $\Gamma^\prime_v (\Gamma_v = \Gamma^0_v \cup \Gamma^\prime_v)$, $\Gamma^\prime_v \cap \Gamma^0_v = \emptyset$, respectively. Eq. (3.1) should be considered in the class of distributions with additional Dirac distribution concentrated on the surfaces (2.13) of each jump discontinuity $\Gamma^0_v$ and $\Gamma^\prime_v$ defined by the equations $\gamma^0_v(x) = 0$ and $\gamma^\prime_v(x) = 0$, respectively:

$$\nabla \cdot [\Delta^0 \{\nabla (\nabla u(x) + \theta(x)) + T(x)\} + \gamma^0_v]$$

$$= \nabla \cdot [\Delta^0 \{\nabla (\nabla u(x) + \theta(x)) - \gamma^0_v\} - \sigma(x) \cdot \nabla(x)\delta \gamma^0_v$$

$$+ \nabla \cdot [\Delta^0 \{\nabla (\nabla u(x) + \theta(x)) + \gamma^0_v\} + \sigma(x) \cdot \nabla(x)\delta \gamma^0_v],$$

where one usually assumes that either $\Gamma^0_v = \Gamma_v$, $\Gamma^\prime_v = \emptyset$ or $\Gamma^0_v = \emptyset$. $\Gamma^\prime_v = \emptyset$. (3.7)

By rearranging Eq. (3.6) into a linear form and applying Gauss’ theorem, we obtain

$$c(x) = c^0(x) + \int_u U(x-u)\gamma(y)dy + \int_u \nabla \cdot \frac{\nabla G(x-y)f_1(y)}{\nabla} dy$$

$$- \int_{\Gamma^0_v} \nabla \gamma^0_v \Gamma^0_v(x-s)\cdot u(s)ds$$

$$- \int_{\Gamma^\prime_v} \nabla \gamma^\prime_v \Gamma^\prime_v(x-s)\cdot u(s)ds,$$

where we have also used the rule $\nabla \cdot \frac{\nabla G(x-y)}{\nabla} = -\nabla \cdot \frac{\nabla G(x)}{\nabla}$; the volume integral kernel $U_{\psi \varphi}(x) = \nabla \cdot \frac{\nabla G(x)}{\nabla}$ is an even homogeneous generalized function of degree $-d$. The surface integrals over $\Gamma^0_v$ and $\Gamma^\prime_v$ (3.8) coincide with the corresponding integrals considered by Asaro (1975) and Sharma and Ganti (2004), respectively.

To express Eq. (3.8) in terms of stresses, we use the identities:

$$L_{1} (\sigma - \beta) = -L_{M} \sigma, \quad \psi = [M^{\sigma} + \beta^\prime] + [M_{1 (\sigma - \beta)].$$

Substituting (3.9) into the right-hand side and the left-hand side of (3.8), respectively, and contracting with the tensor $L_1$ gives the integral equation for stresses

$$\sigma(x) = \sigma^0(x) + \int_u \Gamma(x-y)\eta(y)dy + \int_u L \nabla \cdot \frac{\nabla G(x-y)f_1(y)}{\nabla} dy$$

$$- \int_{\Gamma^0_v} L \nabla \Gamma^0_v(x-s)\cdot u(s)ds - \int_{\Gamma^\prime_v} L \nabla \Gamma^\prime_v(x-s)\cdot [\sigma(s)] \cdot u(s)ds,$$

where the integral operator kernel, $\Gamma(x-y) = -L \cdot \frac{\nabla (\nabla G(x-y))}{\nabla} \cdot \frac{\nabla G(x)}{\nabla}$ is called the Green stress tensor, see Kröner (1977, 1990).

Analysis of Eqs. (3.8) and (3.10) generates an introduction of general notions of the perturbations of the strain $L^\prime_v(x-x_v, \tau)$, the stress $L^\prime_v(x-x_v, \eta)$, and the displacement $L^\prime_v(x-x_v, \tau)$

$$L^\prime_v(x-x_v, \tau) = \gamma(x) - \delta^0(x),$$

$$L^\prime_v(x-x_v, \eta) = \sigma(v) - \sigma^0(x),$$

$$L^\prime_v(x-x_v, \tau) = \gamma(x) - \gamma^0(x),$$

which produce the perturbation of the fields $\gamma(x)$, $\sigma(x)$, and $u(x)$, respectively, in the point $x$ due to the insertion of the heterogeneity center into the point $x_\tau, \tau$ and $\eta$ are the symbolic notations of dependence of $L^\prime_v(x-x_v, \tau)$, $L^\prime_v(x-x_v, \eta)$, and $L^\prime_v(x-x_v, \tau)$ on $\tau(x), \eta(x) (x \in \Gamma_v)$, respectively, $\gamma_v(x \in \Gamma_v), \eta_v(x \in \Gamma_v)$, and $\tau_v(x \in \Gamma_v)$. Obviously, the perturbators $L^\prime_v(x-x_v, \tau)$, $L^\prime_v(x-x_v, \eta)$, and $L^\prime_v(x-x_v, \tau)$ are generalizations of the relevant multiplied Green’s functions with some polarizations of a point defect to the impacts of heterogeneity $\Gamma_v$ of the finite size with possible imperfection of bounding conditions.

The mentioned perturbators can be found by any available numerical method, such as e.g. the volume integral equation (VIE), boundary element method (BEM), FEM, hybrid FEM–BEM, multipole expansion method, complex potential method among others (see for references Buryachenko, 2007a; Ghosh, 2011; Liu et al., 2011). Each method has advantages and disadvantages and it is crucial for the analyst to be aware of their range of applications (see for references Buryachenko, 2010c; Buryachenko and Brun, 2011a, 2012a, 2013).

We obtained representation of the perturbators by the hybrid VIE–BIE (3.8) and (3.10)

$$L^\prime_v(x-x_v, \tau) = \int_u u(x-u)\gamma(y)dy + \int_u \nabla \cdot \frac{\nabla G(x-y)f_1(y)}{\nabla} dy$$

$$- \int_{\Gamma^0_v} \nabla \gamma^0_v \Gamma^0_v(x-s)^{\prime}
\cdot u(s)ds - \int_{\Gamma^\prime_v} \nabla \gamma^\prime_v \Gamma^\prime_v(x-s)^{\prime}
\cdot [\sigma(s)] \cdot u(s)ds,$$

(3.14)

$$L^\prime_v(x-x_v, \eta) = \int_u \Gamma(x-y)\eta(y)dy + \int_u L \nabla \cdot \frac{\nabla G(x-y)f_1(y)}{\nabla} dy$$

$$- \int_{\Gamma^0_v} L\nabla \Gamma^0_v(x-s)^{\prime}
\cdot u(s)ds - \int_{\Gamma^\prime_v} L \nabla \Gamma^\prime_v(x-s)^{\prime}
\cdot [\sigma(s)] \cdot u(s)ds,$$

(3.15)

$$L^\prime_v(x-x_v, \tau) = \int_u \nabla \Gamma(x-y)\eta(y)dy + \int_u G(x-y)f_1(y)dy$$

$$- \int_{\Gamma^0_v} \nabla \cdot \Gamma^0_v(x-s)^{\prime} \cdot u(s)ds - \int_{\Gamma^\prime_v} \nabla \cdot \Gamma^\prime_v(x-s)^{\prime} \cdot [\sigma(s)] \cdot u(s)ds,$$

(3.16)

The perturbators $L^\prime_v(x-x_v, \tau), L^\prime_v(x-x_v, \eta)$, and $L^\prime_v(x-x_v, \tau)$ can be found from joint solutions of Eqs. (3.11)-(3.16) and the contact conditions at the interfaces $\Gamma^0_v$ and $\Gamma^\prime_v$. These contact conditions can be described by the limited cases (2.14)-(2.19) where only the displacement jump or traction jump is presented. However, a generalization of these conditions to a formalism which is able to take into account both previous schemes at the same time is of strong practical importance (see, e.g., application of this approach to a transport phenomena, Pavanello et al., 2012). In any way, the perturbators in the form defined by Eqs. (3.11)-(3.16) are also valid for subsequent applications.

For simplicity we will consider only internal points $x \in w$ of the macrodomain $w$ at sufficient distance from the boundary

$$a \ll \vert x - s \vert, \forall s \in \Gamma_v.$$

The Cauchy data $[u(s), \sigma(s)]$ at the smooth surface $s \in \Gamma_0$ can be found from the conventional BIE taking the limit $x \to \Gamma_0$

$$k(s)u(s) = \int_{\Gamma_0} [G(s - \xi)u(\xi) - T^*(\xi, \eta)u(\xi)]d\xi + L^\prime_v(s)$$

(3.18)

at the boundary conditions (2.7) and (2.8); here $k(x)$ is the free term coefficient (see, e.g., Ballas et al., 1989).
It should be mentioned that Eqs. (3.11) and (3.12) with the integ-
ral representation of the perturbators \( L_i^\gamma(x - x_i, \tilde{\tau}) \) (3.14) and
\( L_i^\gamma(x - x_i, \tilde{\eta}) \) (3.15) can be obtained one from the other by the
use of identities (\( x \in \mathbb{W} \))
\[
e = M'\sigma + \beta^c + \eta, \quad \tau = -L'\eta. \tag{3.19}
\]
Indeed, substitutions (3.19)\(_1\) and (3.19)\(_2\) into Eqs. (3.12) and (3.14),
respectively, lead to (3.11) and (3.15). In so doing, the strain pertur-
ba\( L_i^\gamma(x - x_i, \tilde{\tau}) \) and the stress perturbator \( L_i^\gamma(x - x_i, \tilde{\eta}) \) are
linked by the relation
\[
\tau(x) = \tau_s(x) + \int_w \frac{\nabla}{c}|\nabla G(x - s)| \tau^{(i)}(s, \eta|x|x)|n(s)||ds, \tag{3.21}
\]
where \( \tau_s(x) \) and \( \sigma_s(x) \) can be defined through Eq. (3.4) in which
the Cauchy data \( u_0(x, \eta|x|x)|n(s)|\) at \( s \in \Gamma_0 \) depend on the total
impacts of all perturbators \( L_i^\gamma(x - x_i, \tilde{\tau}) \) (analogously to Eq. (3.14))
presented in the form of the volume integrals (3.8). The parameters
\[
\tau^{(i)}(s, \eta|x|x)| = \tau(x) + L_i^\gamma(x - x_i, \tilde{\tau})|n(s)||ds, \tag{3.22}
\]
are introduced in the surface integrals in (3.21) and (3.22) by using
Gaus's theorem in Eq. (3.8) and taking into account the possibility of
intersection of some part of heterogeneities \( \psi \) with the macro-
boundary \( \Gamma_0 \) where \( \tau^{(i)}(s, \eta|x|x)| = 0 \) \( s \in \Gamma_0 \). In a particular
\( \tau^{(i)}(s, \eta|x|x)| = 0 \) \( s \in \Gamma_0 \). In a particular
case of floating heterogeneities with \( \eta(x, \tilde{\eta}|x|x)| = 0 \) \( s \in \Gamma_0 \),
the boundary integrals in Eqs. (3.17) and (3.18) vanish.

It should be mentioned that the equations for both the strains
(3.21) and stresses (3.22) are equivalent and can be obtained one from another by use of identities (3.19) and (3.20). This equiva-
ence leads to the equivalence of the dual effective laws with reciproc-
ally inverse effective properties for stiffness and compliance.

Now, we apply the centering method initially proposed in a partic-
ular form by Shermergör (1977) for statistically homogenized media
subjected to homogeneous boundary conditions. However, even in the mentioned simplified cases, the centering method
has an advantage with respect to the method by O'Brian (1979)
using the Gaus theorem applied to the asymptotic representation
\( \tau(x) = \eta(x) \) of the boundary integral corresponding to the
boundary integral in Eq. (3.21). This centering procedure subtracting
from both sides of the mentioned integral equation their statistical
averages obtained without any auxiliary assumptions (such as, e.g.,
EHF) leads to the following new equation for statistically inhom-
geneous media
\[
e(x, z) = \eta(x) + \int_w \tau^{(i)}(x - x_i, \tilde{\tau}) V^{(i)}(x, z) - \left(L_i^\gamma(x - x_i, \tilde{\tau})\right)\eta(x)|dx_i + L_i^\gamma, \tag{3.25}
\]
\[
\sigma(x, z) = \sigma(x) + \int_w \tau^{(i)}(x - x_i, \tilde{\eta}) V^{(i)}(x, z) - \left(L_i^\gamma(x - x_i, \tilde{\eta})\right)\eta(x)|dx_i + L_i^\gamma. \tag{3.26}
\]
In the right-hand-side of Eqs. (3.25) and (3.26), assuming traction
boundary conditions \( \Gamma_0 \equiv \Gamma_r \) (2.8), the integrals over the external surface \( \Gamma_0 \)

\[
L_i^\gamma = \int_{\Gamma_0} \left[ \nabla \cdot G(x - s) \frac{\tau^{(i)}(s, \eta|x|x)|n(s)|}{s} \right] n(s)|ds
\]
\[
- \int_{\Gamma_0} \left[ \nabla \cdot T\tau^{(i)}(s, \eta|x|x)|n(s)| - \left(\nabla \cdot T\tau^{(i)}(s, \eta|x|x)|n(s)|\right) \right] n(s)|ds. \tag{3.27}
\]
\[
L_i^\gamma = -L_i^\gamma \tag{3.28}
\]
vanish at the sufficient distance \( x \) (assumed hereafter) from the boundary \( \Gamma_0 \) (3.17), when the validity of separation of length scale
Eq. (2.21) holds. The first integrals in Eqs. (3.27) and (3.28) are the
centered boundary integral terms in Eqs. (3.21) and (3.22),
respectively. The second boundary integrals in Eqs. (3.27) and (3.28) are in
fact the centered items \( \phi(x, \tilde{\eta}|x|x)| - \phi(x) \) \( \sigma(x, \tilde{\eta}|x|x)| - \sigma(x) \) (see
Eq. (3.4)), respectively, which can be different from zero in a "boundary layer" region close to the surface \( s \in \Gamma_0 \) where boundary
data \( u_0(s, \eta|x|x)|n(s)|\) not prescribed by the boundary conditions depend
on perturbations introduced by all inhomogeneities, and, therefore
\( \omega(x) = \omega(x), \sigma(x) = \sigma(x, \tilde{\eta}|x|x)|n(s)|\) in Eqs. (3.27) and (3.28)
can be proved analogously to analysis by Bury-
chenko and Brun (2012a,b).

Thus, the last items \( L_i^\gamma \) and \( L_i^\gamma \) in Eqs. (3.25) and (3.26),
respectively, can be omitted. However, the remaining integrals in Eqs.
(3.25) and (3.26) depend only on the perturbators \( L_i^\gamma(x - x_i, \tilde{\tau}) \)
and \( L_i^\gamma(x - x_i, \tilde{\eta}) \) presenting notations of the numerical solutions
(3.11) and (3.12) which are currently general and not related with
a concrete numerical method. We only need to know that \( \tau^{(i)}(s, \eta|x|x)| = 0 \) \( s \in \Gamma_0 \) at the infinity
\( \infty \) \( s \in \Gamma_0 \). The volume integrals in (3.25) and (3.26) con-
verge absolutely for both the statistically homogeneous and
inhomogeneous random set of heterogeneities. Indeed, even for the
FGMs, the integrands in the square brackets in Eqs. (3.25) and (3.26)
are of order \( O(|x - x_i|^{-d}) \) as \( |x - x_i| \rightarrow \infty \), and the inte-
grals in Eqs. (3.25) and (3.26) converge absolutely. For no long-
range order assumed, the function \( \phi(x, \tilde{\eta}|x|x)|n(s)| - \phi(x) \) decays at infinity sufficiently rapidly and guarantees an absolute conver-
gence of the integrals involved. Therefore, for \( x \in \mathbb{W} \) far enough from the boundary \( \Gamma_0 \) (3.17) (that vanishes the integrals in Eq.
(3.21)), the right-hand-side integrals in (3.25) and (3.26) do not dep-
d on the shape and size of the domain \( w \), and they can be re-
calculated by the integrals over the whole space \( \mathbb{R}^d \). With this
assumption Eqs. (3.25) and (3.26) take the form
\[
e(x, z) = \eta(x) + \int_w \tau^{(i)}(x - x_i, \tilde{\tau}) V^{(i)}(x, z) - \left(L_i^\gamma(x - x_i, \tilde{\tau})\right)\eta(x)|dx_i + L_i^\gamma, \tag{3.29}
\]
\[
\sigma(x, z) = \sigma(x) + \int_w \tau^{(i)}(x - x_i, \tilde{\eta}) V^{(i)}(x, z) - \left(L_i^\gamma(x - x_i, \tilde{\eta})\right)\eta(x)|dx_i + L_i^\gamma. \tag{3.30}
\]
3.3. Infinite coupled system of general integral equations

Let the inclusions \(v_1, \ldots, v_n\) be fixed and we define two sorts of effective fields for the strains \(\tilde{\sigma}(x), \tilde{\varepsilon}_{1\ldots n}(x)\) and stresses \(\tilde{\sigma}(x), \tilde{\varepsilon}_{1\ldots n}(x)\) (\(i = 1, \ldots, n; x \in v_1, \ldots, v_n\)) by rearranging of Eq. (3.29) and (3.30) in the following forms (an analogous particular case of these manipulations approach is given in Buryachenko, 2007a):

\[
\tilde{\varepsilon}(x) = \tilde{\varepsilon}(x) + L_i(x, x, \tilde{\tau}),
\]

\[
\tilde{\varepsilon}_{1\ldots n}(x) = \tilde{\varepsilon}_{1\ldots n}(x) + \sum_{j=1}^n L_{ij}^H(x - x_j, \tilde{\tau}),
\]

\[
\tilde{\varepsilon}_{1\ldots n}(x) = \langle \varepsilon(x) \rangle + \int [L_{ij}^H(x - x_j, \tilde{\tau})V^H(x_j; v_1, x_1; \ldots; v_n, x_n)]d\mathbf{x}_n
\]

\[
- \langle L_{ij}^H(x - x_j, \tilde{\tau}) \rangle d\mathbf{x}_n \quad (3.31)
\]

and

\[
\tilde{\varepsilon}_{1\ldots n}(x) = \langle \sigma(x) \rangle + \int \left[ L_{ij}^H(x - x_j, \tilde{\eta}) \right] V^H(x_j; v_1, x_1; \ldots; v_n, x_n) \nonumber \]

\[
- \langle L_{ij}^H(x - x_j, \tilde{\eta}) \rangle d\mathbf{x}_n
\]

\[ \quad (3.32) \]

respectively, for \(x \in v_i, i = 1, 2, \ldots, n\); here \(V^H(x_j; v_1, x_1; \ldots; v_n, x_n) = \sum_m \delta(x_j - x_{m,j}) - \sum_m \delta(x_j - x_{r,j})\) is a random delta function of heterogeneity centers \(x_{m,j}(m = 1, 2, \ldots)\) under the condition that \(x_j \neq x_{i,j} \neq x_{r,j}\) if \(i \neq j\) \((j = 1, \ldots, n)\). Then, considering some conditional statistical averages of the general integral Eqs. (3.29) and (3.30) leads to an infinite system of new integral equations \((n = 1, 2, \ldots)\):

\[
\langle \sigma(x) \rangle + \int \left\{ \left[ L_{ij}^H(x - x_j, \tilde{\tau}) \right] V^H(x_j; v_1, x_1; \ldots; v_n, x_n) \right\} d\mathbf{x}_n \quad (3.33)
\]

\[
\langle \sigma(x) \rangle + \int \left\{ \left[ L_{ij}^H(x - x_j, \tilde{\eta}) \right] V^H(x_j; v_1, x_1; \ldots; v_n, x_n) \right\} d\mathbf{x}_n \quad (3.34)
\]

Since \(x \in v_1, \ldots, v_n\) in the nth line of the systems (3.33) and (3.34) can take the values inside the inclusions \(v_1, \ldots, v_n\), the nth line actually contains \(n\) equations. It should be mentioned that the integrands in the nth line of the systems (3.33) and (3.34) behave identically for any finite \(n\). Therefore, we can prove absolute convergence of the integrals (3.33) and (3.34) analogously to the proof of the same properties of the integrals (3.29) and (3.30). The definitions of the effective fields \(\tilde{\varepsilon}(x), \tilde{\varepsilon}_{1\ldots n}(x)\) and \(\tilde{\sigma}(x), \tilde{\sigma}_{1\ldots n}(x)\) as well as their statistical averages \(\tilde{\varepsilon}(x), \tilde{\varepsilon}_{1\ldots n}(x)\) and \(\tilde{\sigma}(x), \tilde{\sigma}_{1\ldots n}(x)\) are nothing more than a notation convenience for different terms of the infinite systems (3.31), (3.33) and (3.32), (3.34) respectively.

4. Some particular cases, asymptotic representations, and simplifications

4.1. Particular cases

The subsequent analysis of Eqs. (3.33) and (3.34) can be done for the comparison medium with any elastic modulus \(L^F\), which necessarily leads to some additional assumptions for the structure of the strain fields in the matrix (see for details Chapter 8 in Buryachenko, 2007a). Eqs. (3.33) and (3.34) are much easier to solve when they contain the stress–strain fields only inside the heterogeneities. There are two fundamentally different approaches to ensuring it.

In the first one we postulate

\[ \mathbf{L}^F \equiv \mathbf{L}^0. \]

(4.1)

Then the integrands with the arguments \(y\) in Eqs. (3.34) and (3.34) vanish at \(y \in \mathbb{R}^0\). However, it does not remove the necessity of estimating the stress–strain distributions in the matrix in the general cases of both the inhomogeneous inclusions and inhomogeneous boundary conditions. Fortunately, this domain of the matrix is only located in the vicinity of a representative inhomogeneity \(v_0\) (see for details Buryachenko, 2010c).

In the second case we choose \(L^F\) quite arbitrarily, and analyze Eq. (3.30) (Eq. (3.29) can be considered analogously). Eq. (3.30) being exact for any \(\eta(x)\) can be simplified with the additional assumption that the strain polarization tensor in the matrix \(\eta(x)\) \((x \in \mathbb{R}^0)\) coincides with its statistical average in the matrix \(\langle \eta(x) \rangle \equiv \langle \eta \rangle_0(x)\) \((x \in \mathbb{R}^0)\)

(4.2)

In so doing, the assumption (4.1) is more restricted in the sense that the assumption (4.1) yields the assumption (4.2) (the converse is not true) and, moreover, in such a case the exact equality \(\eta(x) = \langle \eta \rangle_0(x)\) \((x \in \mathbb{R}^0)\) holds. For convenience in the presentation that follows, we will recast Eq. (3.30) in another form, for which we introduce the operation \(\mathbf{L}^F(x) = \lambda^F(x) - \langle \lambda \rangle_0(x)\) for the random function \(\lambda\) (e.g., \(\lambda = \sigma, \varepsilon, \tau, \eta, f\) with statistical average in the matrix \(\langle \lambda \rangle_0(x)\). Then, Eq. (3.34) at \(n = 1\) can be rewritten in the form

\[
\langle \sigma(x) \rangle + \int \left\{ \left[ L_{ij}^H(x - x_j, \tilde{\tau}) \right] V^H(x_j; v_1, x_1; \ldots; v_n, x_n) \right\} d\mathbf{x}_n \quad (3.33)
\]

\[
\langle \sigma(x) \rangle + \int \left\{ \left[ L_{ij}^H(x - x_j, \tilde{\eta}) \right] V^H(x_j; v_1, x_1; \ldots; v_n, x_n) \right\} d\mathbf{x}_n \quad (3.34)
\]

where in the case of the perturbation representation \(L_{ij}^H(x - x_j, \tilde{\eta})\) through Green's functions we need to replace in Eq. (3.15) \(\eta(y) - \langle \eta \rangle_0(x) = f_1(y) - f_1(x)\) and \(f_1(y) - f_1(x)\) in such a case, the perturbers \(L_{ij}^H(x - x, \tilde{\eta})\) only depend on the stress and displacement distributions inside both the heterogeneities and interfaces. However, for estimation of the effective fields \(\tilde{\sigma}(x, \tilde{\eta})\)
(which are necessary for assessment of effective properties) we need to evaluate the stress–strain distributions in the matrix in the vicinity of a moving inhomogeneity \( \nu \). For both assumptions (4.1) and (4.2), only in the case of asymptotic approximation of Eqs. (3.33) and (3.34) considered in Section 4.2, one can estimate the effective properties through the evaluation of the stress–strain fields only inside the heterogeneities and at the interfaces.

We will consider a particular case of the field \( X \) bounded in one direction such as a laminated structure of some real FGM (see Plankenshtainer et al., 1996, 1997). Then the surface integral \((3.21)\) and \((3.22)\) over a “cylindrical” surface (with the surface area proportional to \( \rho^{d-2} \), \( \rho = |x - s| \)) tends to zero with \( |x - s| \to \infty \) as \( \rho \to 1 \) simply because the generalized function \( \nabla G(x - s) \) is an even homogeneous function of order \( -d + 1 \). Therefore, for infinite media, the surface integrals \((3.21)\) and \((3.22)\) vanish, and Eqs. (3.21) and (3.22) can be rewritten as

\[
\begin{align*}
\epsilon(x, \tau) &= \epsilon_0(x, \tau) + \int_{w} \mathcal{L}_2^{s}(x - x_i, \tilde{\tau}) \nabla \epsilon_i(x_i, \tau) \, dx_i, \\
\sigma(x, \tau) &= \sigma_0(x, \tau) + \int_{w} \mathcal{L}_2^{s}(x - x_i, \tilde{\eta}) \nabla \sigma_i(x_i, \tau) \, dx_i,
\end{align*}
\]

(4.4)

(4.5)

Clearly in the considered case of \( X \) bounded in one direction, Eqs. (4.4) and (4.5) are exact, and the right-hand-side integrals in (4.4) and (4.5) converge absolutely. A particular case of asymptotic approximation (see Section 4.2) of Eq. (4.4) was used by Torquato and Lado (1992) and Torquato (2002) for the homogeneous boundary conditions (2.9) and for the inclusion field \( X \) with a constant concentration of inclusions within an ellipsoidal domain included in the infinite matrix. Although Eqs. (3.34) and (3.35) are more complicated than Eqs. (4.4) and (4.5), nevertheless they provide practical advantages because their integrands decay at infinity faster than the integrands involved in Eqs. (4.4) and (4.5).

For the ideal contact heterogeneities–matrix (2.6) and for the integral representation of perturbators (3.14)–(3.16), Eqs. (3.29), (3.30), (4.4) and (4.5) are reduced to corresponding equations proposed by Buryachenko (2010a,b).

### 4.2. Asymptotic approximations

Eqs. (3.29) and (3.30) contain the general representations for statistical averages of the perturbators such as, e.g., \((\mathcal{L}_2^{s}(x - x_i, \tilde{\tau})) \) and \((\mathcal{L}_2^{s}(x - x_i, \tilde{\eta})) \) for which we obtain some different particular cases of approximations of these statistical averages in Eq. (3.29), (3.33) and (3.30), (3.34) respectively. Such an analysis will be performed for Eq. (3.29) (Eq. (3.30) can be considered analogously) for the ideal interface conditions (2.6), no body forces acting and purely mechanical loading i.e. \((s \in \Gamma, x \in \omega)\)

\[
[\sigma(s)] \cdot n = 0, \quad [u(s)] = 0, \quad f(x) = 0, \quad \beta(x) = 0,
\]

(4.6)

when the integral representations of the perturbators (3.14) and (3.15) have the forms

\[
\begin{align*}
\mathcal{L}_2^{s}(x - x_i, \tilde{\tau}) &= \int U(x - y, \tau(y) \, dy, \\
\mathcal{L}_2^{s}(x - x_i, \tilde{\eta}) &= \int \Gamma(x - y) \eta(y) \, dy
\end{align*}
\]

(4.7)

leading Eqs. (3.31) and (3.32) to

\[
\begin{align*}
\epsilon(x) &= \langle \epsilon \rangle + \int \{U(x - y, \tau(y) - (\langle \tau \rangle) \, dy, \\
\sigma(x) &= \langle \sigma \rangle + \int \{\Gamma(x - y) \eta(y) - (\langle \eta \rangle) \, dy
\end{align*}
\]

(4.8)

respectively. The deterministic analog of the mentioned approximations can be presented in the following different forms

\[
\begin{align*}
\int U(x - y, \tau(y) \, dy &= \langle U \rangle \cdot \tau \, dy, \\
\int U(x - y, \eta(y) \, dy &= \langle U \rangle \cdot \eta \, dy
\end{align*}
\]

(4.9)

(4.10)

where \( \eta(y) \) is some deterministic function, \( \tau \) is some representative fixed heterogeneity, and the tensors

\[
\begin{align*}
T'(x_i - x_0) &= \left\{ \frac{(-\tau_i)}{\langle \tau \rangle} \right\} \cdot P(\langle \tau \rangle) \quad \text{for} \quad x \in \nu, \\
\left\{ \frac{(-\tau_i)}{\langle \tau \rangle} \right\} \cdot T(\langle \eta \rangle) \cdot U(x - y, \eta(y) \, dy \quad \text{for} \quad x \notin \nu,
\end{align*}
\]

(4.11)

have analytical representations for ellipsoidal inclusions for both the tensor\( \mathcal{P} = P(\langle \tau \rangle) = -\langle U \rangle \cdot \eta\rangle \, dy \) and \( \langle p \rangle \) for \( x \in \nu \). The most popular approximation (4.9) (which is simultaneously the most crude) was implicitly used by many authors (see for early references, e.g., Beran and McCoy, 1970) including sort of the centered Eq. (3.2) (see Zeller and Dederichs, 1970). A quantitative analysis of results obtained by the use of the representations (3.25) and (3.26) was performed by Buryachenko (2010c).

Substitution of the random analog (e.g., when \( p(x) \)) is replaced by \( \mathcal{G}(x, \tau, x, \eta(x, x)) \) of the approximation (4.9) into Eqs. (3.33) and (3.34) at \( n = 1 \) yields the equation (see for details Buryachenko, 2007a)

\[
\begin{align*}
\epsilon(x) &= \langle \epsilon \rangle + \int U(x - y, \tau(y) - (\langle \tau \rangle) \, dy, \\
\sigma(x) &= \langle \sigma \rangle + \int \Gamma(x - y) \eta(y) - (\langle \eta \rangle) \, dy.
\end{align*}
\]

(4.12)

Additional assumption of the statistical homogeneity of media subjected to the homogeneous boundary conditions (2.9) reduces Eq. (4.12) to the known ones

\[
\begin{align*}
\epsilon(x) &= \langle \epsilon \rangle + \int U(x - y, \tau(y) - (\langle \tau \rangle) \, dy, \\
\sigma(x) &= \langle \sigma \rangle + \int \Gamma(x - y) \eta(y) - (\langle \eta \rangle) \, dy.
\end{align*}
\]

(4.13)

Moreover, for a particular case of the field \( X \) bounded in one direction, Eqs. (4.4) and (4.5) are reduced to the equations

\[
\begin{align*}
\epsilon(x) &= \epsilon_0 + \int U(x - y, \tau(y) \, dy, \quad \sigma(x) = \sigma_0 + \int \Gamma(x - y) \eta(y) \, dy,
\end{align*}
\]

(4.14)

which were also widely used for statistically homogeneous media subjected to the homogeneous boundary conditions (see more detailed analysis of the incorrectness of Eq. (4.14) for statistically homogeneous composites by Buryachenko, 2001, 2010b)
a more general centering method which was generalized by Buryachenko (2001) to the FGM described by Eq. (4.12).

We will prove that for homogeneous boundary conditions (2.9) and statistically homogeneous media \( \langle n^i(x) = n^i \rangle \equiv \text{const.}, q = 1, \ldots, N \), the approximations (4.9) and (4.10) lead to an identical result reducing Eqs. (3.29) and (3.30) to (4.13). This statement holds if we prove that contributions made to Eq. (3.29) by the renormalizing terms (4.9) and (4.10) are identical for any macrodomain \( x \in w \) at the condition (3.17):

\[
\int_w U(x - y) \tau(x) dy \leq \sum_{q=1}^N \int_w \tau_q(y-x) \left| \frac{n^q(x)}{C_0} \right| dy_q.
\] (4.15)

For justification of the equality (4.15), it should be mentioned that for uniform distribution of inclusion centers \( x_k \), all volume of the domain \( w \) in the right-hand side of Eq. (4.15) is uniformly covered by the moving ellipsoids \( \nabla_q \). Then any point in the domain \( x \in w \) in the right-hand side integral (4.15) is covered by the same number k of the ellipsoids \( \nabla_q \) with homogeneous strain polarization tensor \( \tau(y) \equiv \{ \xi, \epsilon \} \), and, therefore, the integral over the covered domain \( w \) on the right-hand side of Eq. (4.15) is equal (within some probability factor) to k integrals over domain \( w \) in the left-hand side. Therefore, in the case \( \tau(y) \equiv \text{const.} \) inside moving inhomogeneity \( y \in \nabla_q \), both approximation (4.9) and (4.10) reduce Eq. (4.3) to the known one (4.13). However, a condition of homogeneity \( \tau(y) \equiv \text{const.} \) at \( y \in \nabla_q \) is fulfilled only for homogeneous ellipsoidal inhomogeneities in the framework of an additional hypothesis of effective field homogeneity according to which each inclusion is located inside a homogeneous so-called effective field (see also Buryachenko, 2010a,b). Moving away from the effective field hypothesis leads with necessity to inhomogeneity of the stress–strain fields inside the inhomogeneities that tend to results in different predictions of effective moduli based on Eqs. (3.2) and (3.18) even for both the statistically homogeneous media and homogeneous boundary conditions (see for details Buryachenko, 2010a,b). This difference is a result of the insensitivity of the renormalizing term \( U(x - y) / \tau(x) \) (obtained at the approximation (4.10)) in the asymptotic Eq. (4.13) to the details of heterogeneities of the stress–strain fields inside the inclusions, while a corresponding term \( U(x - y) / \tau(y) \) (which is exact and obtained without approximations neither (4.9) nor (4.10)) of Eq. (4.8) explicitly depends on the mentioned field inhomogeneity.

4.3. Effective field hypothesis and new background of micromechanics

It is interesting to establish a connection between Eq. (4.13) and the effective field hypothesis (EFH, H1) which is apparently the most fundamental, most prospective, and most exploited concept of micromechanics (see Buryachenko, 2007a where other references can be found). The notion of an effective field in which each particle of matter acts with a force \( \tau(x) \) leads with necessity to inhomogeneity of the stress–strain fields inside the inclusions (for references see Morse and Feshbach, 1953; Buryachenko, 2007a). The idea of this concept dates back to Mossotti (1850) and Clausius (in the dielectric context), Lorenz (in the refraction context), and Maxwell (in the conductivity context). Markov (1999) and Scaife (1989) presented comprehensive reviews of the 150 years history of this concept accompanied by some famous formulas with extensive references. Mossotti (1850) (especially Clausius) pioneered the introduction of the effective field concept as a local homogeneous field acting on the inclusions and differing from the applied macroscopic one. The EFH is usually formulated in the following form of a combination of two hypotheses:

(Hypothesis H1a). Each heterogeneity \( \nu_i \) has an ellipsoidal form and is located in the fields (3.31c) and (3.32c)

\[
\mathbf{z}(y) = \mathbf{z}(x_k), \quad \mathbf{\tau}(y) = \mathbf{\tau}(x_k) \quad (y \in \nu_i)
\] (4.16)

which are homogeneous over the inclusion \( \nu_i \).

(Hypothesis H1b). The perturbation introduced by the ellipsoidal heterogeneity \( \nu_i \) at the point \( y \neq \nu_i \) is defined by the relation

\[
\int U(x - y) \nu_i(x) \tau(x) dx = \nu_i \Gamma(y - x, \nu_i) \tau_i(x),
\]

\[
\Gamma(y - x) \nu_i(x) \tau(x) dx = \nu_i \Gamma(y - x, \nu_i) \tau_i(x).
\] (4.17)

where \( \Gamma(y - x, \nu_i) \) is expressed through \( \nabla_i \) as \( \Gamma(y - x, \nu_i) \) is represented through \( \mathbf{U} \) (4.11). For a homogeneous ellipsoidal inclusion \( \nu_i \) the standard assumption (4.16) (see e.g. Buryachenko, 2007a) yields the assumption (4.17), otherwise the formula (4.17) defines an additional assumption. Due to the arbitrary inhomogeneity of the heterogeneity \( \nu_i \), the assumption H1a for the ellipsoidal shape of \( \nu_i \) can be relaxed and we can assume any shape of \( \nu_i \) in the hypothesis H1a. In so doing, the ellipsoidal shape of heterogeneities \( \nu_i \) is essential for subsequent application of the hypothesis H1 (see for details Buryachenko, 2007a).

Among a few hypotheses used by Mossotti (1850), one of the most important ones was the quasi-crystalline approximation (closing hypothesis H2, see also its multiparticle generalization in Buryachenko (2007a)) proposed 100 years later by Lax (1952) in a modern concise form:

Hypothesis 2, H2, “quasi-crystalline” approximation. It is supposed that the mean value of the effective fields at a point \( x \in \nu_i \) does not depend on the effective fields inside surrounding heterogeneities \( \nu_i \neq \nu_i \):

\[
\langle \mathbf{z}_i(x) \rangle = \langle \mathbf{z}_j(x) \rangle, \quad \langle \mathbf{\tau}_i(x) \rangle = \langle \mathbf{\tau}_j(x) \rangle, \quad \mathbf{x} \in \nu_i. \] (4.18)

General case of the closing hypothesis taking \( n \) interacting heterogeneities is considered in Buryachenko (2007a). It should be mentioned that the closing hypotheses (H2 or other one) do not conceptually depend on the hypothesis H1 and can be applied in general case even if the hypothesis H1 is violated. To make additional progress, the hypothesis of “ellipsoidal symmetry” H3 proposed by Khoroshun (1972, 1974) (see also Willis, 1977; Khoroshun, 1978; Ponte Castañeda and Willis, 1995) for the distribution of inclusions is widely used:

Hypothesis 3, H3, “ellipsoidal symmetry”. The conditional probability density function \( \varphi(t_j, x_k) \) depends on \( x_k - x_l \) only through the combination \( t_j = |a_i^{(3)} - x_l - x_k| \):

\[
\varphi(t_j, x_k) = \hat{h}(t_j),
\] (4.19)

where the matrix \( a_i^{(3)} \) (which is symmetric in the indexes \( i \) and \( j \)) defines the ellipsoid excluded volume \( \nu_i = \{ x : |a_i^{(3)} - x_l - x_k| < 1 \} \).

All these concepts (4.16)–(4.19), which create the framework and classical background of modern statistical analytical micromechanics were transformed by the use of both the additional assumptions and sophisticated analytical and numerical tools to a few particular methods. However, Buryachenko (2010a,b,c) has proven that the EFH is a central hypothesis and other concepts play a satellite role in providing the conditions for application of the EFH. Indeed, the essence of the hypothesis H3 was analyzed by Ponte Castañeda and Willis (1995) (see also Buryachenko, 2007a) in the framework of the hypothesis H1. Buryachenko (2010c); Buryachenko and Brun (2011) demonstrated that the real destination of the hypothesis H3 is to provide the conditions for realizing hypothesis H1a rather than realizing a simplified solution of either Eq. (4.8) or (4.13). Abandoning the ellipsoidal symmetry hypothesis (4.19) necessarily leads to the inhomogeneity of the effective field \( \mathbf{\tau} \) (see for details Buryachenko, 2010c; Buryachenko and Brun, 2011) acting on the inclusion \( x \in \nu_i \) which is prohibited for acceptance of hypothesis H1b in the framework of Eq.
(4.13) is only valid although Eq. (4.8) is usable for any \( \Sigma \), independently on either accepting or rejecting of the hypothesis H2. From the other side, abandoning of the hypothesis H2 means taking into account binary (at least) interactions of heterogeneities that in general leads to the inhomogeneity of the effective field \( \Sigma \), (see for details Buryachenko, 2011a) yielding, as we know, a violation of conditions exploited for obtaining of Eq. (4.13) while Eq. (4.8) is valid independently on acceptance of the hypothesis H2. Thus, all mentioned so-called basic hypotheses (4.16)–(4.19) are not absolutely necessary and can be relaxed. The concept of the EFH (even if this term is not mentioned) in combination with subsequent assumptions (such as, e.g., H2 and H3) totally dominates (and creates the fundamental limitations) in all four groups of analytical micromechanics in physics and mechanics of heterogeneous media: model methods, perturbation methods, self-consistent methods (e.g., MTM, and the Method of Effective Field, MEF), and variational ones (see for references Buryachenko, 2007a). So Eq. (4.13) obtained seemingly independent on the EFH is in reality a result of application of a particular case of the EFH to the general integral Eq. (4.8). Indeed, we proved that Eq. (4.8) is reduced to Eq. (4.13) if the asymptotic assumption (4.10) for any shape of heterogeneities is accepted. Therefore, Eq. (4.8) is reduced to Eq. (4.13) in the framework of more restrictive hypothesis H1b (4.17) for ellipsoidal heterogeneities. Thus Eq. (4.8) and (3.29)–(3.34) obtained without using the EFH are really general and can be considered as a new background of micromechanics intrinsically free of any impact of the EFH.

It should be mentioned that the idea of representing of the new renormalized item \( \langle U(x − y)\rangle \equiv \{ \hat{g}(x) \} \) proposed by Buryachenko (2010a,b) through some perturbator does not constitute a fundamentally new approach. Buryachenko (2007b) has started from Eq. (4.13) and obtained a particular case of the perturbator (this term was not used) through the cumbersome manipulations in the framework of the hypothesis H1a; it was also mentioned that accepting the additional assumption for the effective field \( \Sigma \) reduces this renormalized item to the one proposed by Chen and Acrivos (1978). A principle novelty of the perturbator concept (3.11)–(3.13) is its generality and independence on concrete numerical method used for its evaluation as well as a sensitivity of the perturbators to inhomogeneity of both the effective field \( \Sigma \) (or \( \Sigma \)) and the polarization tensor \( \Sigma \) (or \( \Sigma \)) at \( x \in v_1 \) that means abandonment of the hypotheses H1a and H1b, respectively. From the other side, Buryachenko (2007a) proposed in Section 9.4. an opportunity of consider an inhomogeneity of the effective field \( \Sigma \) (or \( \Sigma \)) through nonellipsoidal shape of an excluded volume \( v_1, \) However, this prospective idea is essentially limited by its combination with Eq. (4.13) (rather then with Eq. (4.8)) and assumption of statistical homogeneity of both the material and field variables. All these limitation are overcome in the general integral Eqs. (3.29) and (3.30) expressed in terms of the perturbators which take into account the possible inhomogeneities of both the effective field \( \Sigma \) (or \( \Sigma \)) and the polarization tensor \( \Sigma \) (or \( \Sigma \)) at \( x \in v_1 \) as well as a possible imperfecton of bounded interfaces (2.15) and (2.16). However, Eqs. (3.29) and (3.30) were obtained at the scale conditions (2.21) and (3.17) which in fact means unboundedness of domain \( w = R^4. \) In so doing, Buryachenko and Kushch (2006) used a similar idea as a perturbator for estimating the statistical average of local residual stresses \( \sigma^{(i)}_n(x) \) in a half-space \( x \in R^4 = \{ x, x \in R^3, x^4 > 0 \} \) of a composite with circular inclusions and \( M(x) = 0, p(x) = p^{(i)}_n = \text{const} (x \in w). \) It would be interesting to combine the approaches of both the current paper and that of Buryachenko and Kushch’s (2006) to obtain a generalization of Eqs. (3.29) and (3.30) to the bounded domains.

4.4. About renormalized terms in Eqs. (4.8) and (4.12)

The introduction of two kinds of conditional averages \( g^{(i)}_d (x) (x \in w \subset w) \) and \( \hat{g}^{(i)}_d (z) (z \in v_1) \) allows us to present the essence of Eq. (3.26) more clearly. That is to say, the popular assumption

\[
J_i(x − y) = J_i(x − y), \quad \text{where} \quad J_i(x − y) = \langle U(x − y)\rangle.
\]

\[
J_i(x − y) = \langle U(x − y)\rangle (x),
\]

(4.20)

(i.e., \( g(x, y) = U(x − y)\)) is a statistically inhomogeneous function which is not invariant to translation \( x \rightarrow x + \epsilon \) reduces Eq. (4.8) to the classical one (4.13) for statistically homogeneous media \( n\gamma(x) = \text{const.} \) A popular justification (with an intuitive level of rigor) of Eq. (4.20) is that the deterministic function \( U(x − y) \) can be always carried out from the brackets (\( \langle \cdot \rangle \)) of statistical average that can be considered as an intuitive generalization of the corresponding basic linear property of an expectation described by an exact equation

\[
\langle A(\tau) \rangle = A(\langle \tau \rangle)
\]

(4.21)

for the constant variable \( A = \text{const} \) and the random one \( \tau. \) The estimation of \( J_i(x − y) \) (4.20) with \( \langle \tau(x)\rangle = \langle \tau(y) \rangle = \text{const} \) is straightforward while evaluation of \( J_i(x − y) \) (4.16) is less trivial and should use a two step scheme of averaging (2.24)

\[
J_i(x − y) = \left\langle \int_{\omega(x)} \hat{g}^{(i)}_d (r)\Sigma^{(i)}_d \int_{v_1} U(x − z)\tau(\nu, r, r − z) dz dr \right\rangle = \int_{\omega(x)} \hat{g}^{(i)}_d (r)\Sigma^{(i)}_d \int_{v_1} U(x − z)\tau(\nu, r, r − z) dz dr,
\]

(4.22)

\[
J_i(x − y) = U(x − y) \int_{\omega(x)} \hat{g}^{(i)}_d (r)\Sigma^{(i)}_d \int_{v_1} U(x − z)\tau(\tau, r, r − z) dz dr.
\]

(4.23)

where \( \nu_1 \) is a moving heterogeneity with the center \( r \), and \( \tau(\nu, r, r − z) \) denotes a random value of a polarization tensor \( \tau \) given that the fixed heterogeneity \( z \in v_1 \) is centered at \( r \). The second equality in Eq. (4.22) is determined by applying exact Eq. (4.21) to the conditional average of \( \tau(\nu, r, r − z) \) in a concrete point \( z \in v_1 \) where \( U(x − z) \) is constant while \( U(x − z) \) is a function of \( z \) if \( z \) passes through the cross-section of \( v_1 \). Thus, at this stage, the deterministic function \( U(x − y) \) can be carried out from the brackets of the conditional average at the fixed \( z \in v_1 \). The integral in the right-hand side of Eq. (4.23) was obtained as a result of a formal substitution \( U(x − z) = \mathbf{1} \to \langle \rangle \to U(x − y) \). For statistically homogeneous media \( n\gamma(x) = \text{const} \) subjected to the homogeneous boundary conditions (2.9), Eq. (4.23) is reduced to the obvious one \( J_i(x − y) = \delta^{(i)}(x − y) \). The exact Eqs. (4.22) and (4.23) depend on the variable \( \langle \nu, r, \rangle / (z − r) \) which is deterministic (not random). However, a forthcoming simplification of (4.22) related with the carrying out of \( U(x − z) \) from the two integral signs over \( r \) and \( z \) is possible only in the case if \( U(x − z) \equiv A = \text{const} \) for both \( z \in v_1 \) and \( \forall r \in v_1, \) This statement finally disproves a previous intuitive declaration mentioned before Eq. (4.21) that the deterministic function \( U(x − y) \) can always be carried out from the brackets (\( \langle \rangle \)) of statistical average. Thus we proved that the representation (4.20) is just an asymptotic approximation (which is fulfilled at \( x − y \to \infty \)) of the left hand side of (4.20) rather than an exact equality, while a very similar Eq.(4.21) is exact.

The quantitative estimation of the error of Eqs. (4.9) and (4.10) can in general be estimated numerically (see for details Buryachenko, 2010c). The quantitative estimation of an error introduced by approximation (4.20) into evaluation of the statistical
average of the effective field \( \langle \vec{\sigma}(x) \rangle (x \in \nu_i) \) (3.32)) was performed by Buryachenko and Brun (2012a,b) for degenerated case of random residual stresses \( M_i(x) = 0, \beta_i(x) = \beta_i^0 V(x) \) when \( \langle \eta(x, r) \rangle \) (r, z = r) = \( \beta_i^0 \rho_i^0 (z \in \nu_i) \), \( r \in \nu_i^0 (y) \) and \( \langle \rho_i \rangle = \beta_i^0 \rho_c(1) \) are exactly known and Eq. (3.30) can be solved with any prescribed numerical accuracy without any hypotheses (H1, H2, or H3). It was demonstrated that for a particular 2D case of aligned noncanonical inclusions, the approximation (4.20) leads to the error 40% at the estimation of the inhomogeneous effective field \( \langle \vec{\sigma}(x) \rangle (x \in \nu_i) \).

Thus, we proved that the violation of Eqs. (4.10) and (4.20) leads to the difference of Eqs. (4.8) and (4.13). Buryachenko (2010c, 2011a,c) and Buryachenko and Brun (2011, 2012a,b, 2013) have performed a comparative analysis of results obtained through the use of either the new Eq. (4.8) or old Eq. (4.13) for the perfect interface (2.6) and have established that the greater inhomogeneity (defined by either the noncanonical heterogeneity shape or nonlocal constitutive law) of the stress concentrator factor for a single heterogeneity inside infinite matrix subjected to the homogeneous boundary conditions (2.10) leads to a greater difference between the new and old approaches with the possible change of sign of predicted local statistical average stresses (consideration of binary interactions of heterogeneities contributes the additional difference). That is to say, Eq. (4.20) is a sort of test site for the discovery of fundamentally new results even that something seems to be only a formal trick for establishing the difference between some abstract Eqs. (4.8) and (4.13). Moreover, Buryachenko (2010c) described four wide classes of micromechanical problems (distinguished by either the constitutive law of constituents, interface properties, microstructure of heterogeneities, or their random arrangements in either the bounded or unbounded media subjected to the fields of different physical natures) where the use of the new background of micromechanics (4.8) can lead to the detection of fundamentally new effects that are impossible in the framework of the classical approach (4.13).

However, numerous results (see for references the Introduction for a single heterogeneity with imperfect interface (2.14)–(2.19) indicates on inhomogeneity (in general) of stress distribution even in an ellipsoidal inclusions loaded at infinity by a homogeneous boundary conditions of either (2.9) or (2.10). Therefore, it is expected that Eq. (4.10) is violated, that leads to a significant difference between results obtained by the use of Eqs. (3.29) and (4.13) (the additional availability of the surface interface integrals in Eq. (3.14) will make some further contribution into this difference). However, a more detailed consideration of particular problems mentioned is beyond the scope of the current paper and will be considered in subsequent publications. We mentioned in the last paragraph about a fundamental difference of results obtained by the use of the Eq. (4.8) or (4.13) due to the different renormalized terms (4.20) or (4.20b), respectively. In so doing, it is inferred from Eqs. (4.8) or (4.13) that the technical difficulties for solutions of these very similar Eqs. (4.8) and (4.13) are also comparable. Unfortunately, this expectation is not realized even for particular simplified problem in the framework of the hypothesis H2 for statistically homogeneous CM subjected to the homogeneous boundary conditions (2.10) (see for details Buryachenko, 2010c, 2011a,c; Buryachenko and Brun, 2011, 2012a,b, 2013). Indeed, in the case of Eqs. (4.13) and (4.20b), a correlation hole \( v_i^0 \) must be ellipsoidal, and the estimation of impact of the renormalized term (4.20) is reduced to evaluation of both the average tensor \( \langle \eta \rangle \) and analytical representation \( Q_i(v_i^0) = L^0(1 - S(v_i^0)) \) for the ellipsoidal correlation hole \( v_i^0 \). However, in the case of Eqs. (4.8) and (4.20b), a shape of \( v_i^0 \) has no restrictions and we need to estimate the perturbator \( L^0(x - x, \eta) \) (3.16) and (4.6) only not inside of the representative inclusion \( x \in \nu_i \) (compare with the estimation of \( \langle \eta \rangle \) in the classical approach (4.13) and (4.20) but also outside in some vicinity \( v_i^0 \) or \( v_i^0 \) for location of inclusion centers \( x \), where \( q(v_i, x_1, x_2, x_3, r) - n^{(k)} (v_i \subset \nu_i^0, k = 1, \ldots, N) \) is not negligible, and, an effect zone of stress perturbations produced by the heterogeneity \( v_i \) is limited by the Minkowski addition of domains \( x \subset v_i^0 \cong v_i^0 \oplus v_i \) rather than by the domain \( v_i^0 \) as in the MEF. However, due to the intrinsic feature of the FEM, the perturbator \( L^0(x - x, \eta) \) is found for both domains \( x \in \nu_i \) and \( x \in v_i^0 \setminus \nu_i \) in any way that defines some advantage of FEM with respect to the BEM where the concentration tensors at \( x \in \nu_i \) and \( x \in v_i^0 \setminus \nu_i \) should be additionally estimated after the found stress distributions at the heterogeneity boundary \( \partial v_i \).

4.5. Model example

We will consider a model numerical example admitting a simple analytical solution for a quantitative illustration of the assumption (4.16) rather than presenting some concrete micromechanical problem. Namely, one analyzes (see Fig. 1) a scalar version of Eq. (2.24) in a 1D space of statistically inhomogeneous distribution \( n^{(0)}(y) \neq \text{const.} \) of identical intervals (inclusions \( v_i = [-a, a] \)) of the length 2a with a random polarization tensor \( \tau(y) = \tau_0(V'Y(a) \text{ described by a stochastic process } \tau_0(y, z) = b \exp(-|y|) + \varepsilon \cos(\pi z/a + \xi) \) where the random variables \( \xi \) uniformly distributed on \([-\pi, \pi]\) and \( \varepsilon \) have the expectations \( M_0 = 0 \) and \( M_2 = \varepsilon^2 \), respectively. Then in the local coordinate system \( (\tau_y(y, z) \equiv \langle \tau_y(y) \rangle = b \exp(-|y|) + \varepsilon (z \in [-a, a]) \) while

\[
\langle \tau \rangle(y) = \int_{-a}^{a} n^{(0)}(r) \frac{1}{2a} \int_{-a}^{a} \langle \tau \rangle(r) dz dr.
\]

The case \( n^{(0)}(y) \neq \text{const.} \) describes a functionally graded system, while \( b \neq 0 \) corresponds to the macroscopically inhomogeneous loading. Then, according to a two-step estimation of statistical average (3.32), we get at \( |x - y| > 2a \) and \( U(x - y) = A + B/(a(x - y))^3 \)

\[
J_1(x - y) = \int_{y-a}^{y+a} n^{(0)}(r) \frac{1}{2a} \int_{-a}^{a} U(x - z) \langle \tau \rangle(r) dz dr.
\]

\[
J_2(x - y) = U(x - y) \int_{y-a}^{y+a} n^{(0)}(r) \langle \tau \rangle(q) dr.
\]

The locations of intervals \( \nu_i \) placed in the vicinity of the point \( y \) and involved in the integral evaluations (3.34) and (4.24) are schematically depicted in the enlarged part of Fig. 1. No confusion will arise due to the overlapping of these intervals \( \nu_i \) because they belong to different infinite realizations of the field \( x \) rather than to a single realization \( x \) of intervals \( \nu_i \) on the axis \( Ox \).

We see that both \( J_1(x - y) \) and \( J_2(x - y) \) are sensitive to some nonlocal effects and depend on \( \langle \tau \rangle(q) \) and \( n^{(0)}(r) \) in the vicinity \( |r - y| < \alpha \) of the point \( y \) rather than only on \( \langle \tau \rangle(q) \) and \( n^{(0)}(y) \) in the point \( y \). For demonstration of the fundamental difference between the averages \( J_1(x - y) \) and \( J_2(x - y) \), we consider a
degenerate case \( b = 0 \) and \( n^{(i)} = n = \text{const.} \) of the statistically homogeneous system when the stochastic process \( \tau(y,z) \) is a stationary one (but not ergodic). Then the analytical representations

\[
J_{1}(x-y) = gA + gBA^{4} \left[ \frac{1}{2} \left( \frac{x+y}{2} \right)^{2} - 4a^{2} \left( x - y \right) \right],
\]

\[
J_{2}(x-y) = gA + gBA^{4} (x-y)^{-1},
\]

make it possible to estimate a relative error of the approximation (4.16)

\[
\Delta \equiv \left| J_{1}(x-y) - J_{2}(x-y) \right| / J_{1}(x-y).
\]

As can be seen Eq. (4.16) is exactly fulfilled (\( \Delta = 0 \)) only for the constant \( U(x,y) = A (B = 0) \). In the other limiting case \( A = B = 0 \), the relative error \( \Delta = 4a^{2} / (x-y)^{2} \) indicates that the representation (4.16) is just an asymptotic approximation of \( \langle U(x,y) \rangle (y) \) as \( (x-y)/a \to \infty \); for example, \( \Delta = 25 \% \) and \( 4 \% \) at \( (x-y)/a = 4 \) and 10, respectively. The error \( \Delta \) (4.28) is defined by the inhomogeneity of \( U(x,y) \) in the domain \( y_{p} \in \{ x \in \mathbb{R} | y_{p} < 2a \} \) (see Fig. 1) rather than by the average value of \( U(x,y) \) in this domain \( y_{p} \). We can see in Eq. (4.25)(a) that the two-step average (2.24) and (4.25)(b) is performed at first by the use of the linear property of the expectation (4.21) with the estimation of the statistical average \( \langle \tau(y,z) \rangle \) in the local coordinate system \( z \in [-a, a] \). The second subsequent step of averaging of the weighted polarization amount \( \langle \tau(y,z) \rangle \) is accomplished over the volumes of the moving inclusions \( v_{p} \) with the centers \( y_{q} \in \{ y \in [-1, 1] + a \} \) when \( U(x,y) \) cannot be moved out of the ensemble average operator as was done in Eq. (4.25)(b). The considered example convincingly confirms the fact that the normalizing term \( U(x,y) \langle \tau(y,z) \rangle \) in the classical general Eqs. (3.2) and (3.6) is no more than an asymptotic approximation of the exact proposed one \( \langle U(x,y) \rangle (y) \) (3.18) in a long-range zone as \( (x-y) \to \infty \). However, the error of the approximation (4.16) is drastically increased in a short-range zone of a few wavelengths (especially if \( y \in \nu \)) where interaction of heterogeneities reaches its peak.

Thus we proved that although the equalities (4.20) intuitively look as Eq. (4.21), the relations (4.20)(a) are just asymptotic approximations of the left-hand sides of (4.20)(b) whereas Eq. (4.21) is exact. In this light, the general form of Eq. (3.29) presented in terms of perturbators has the additional interesting advantage with respect to Eq. (4.8) because there is no reason to the questionable error of the prediction by the MEF \( \langle f \rangle \) (4.32). Moreover, changing of a sign in the eigenstrain mismatch \( f(y) \neq 0 \) remains fixed \( \langle f \rangle = 0 \) (4.32) and changes the signs in the estimations \( \langle f \rangle \neq 0 \) (4.43) and \( \langle f \rangle \neq 0 \) (4.33). Furthermore, a small perturbation of \( f(y) \neq 0 \) can lead to any desirable sign of \( \langle f \rangle \) (4.32) while the signs of \( \langle f \rangle \) and \( \langle f \rangle \) are unaltered, and, therefore, the estimations (4.32) and (4.33) can have the different signers.

5. Doubly and triply periodical structure composites

5.1. Geometrical description and averaging of doubly and triply periodic structures

We now consider a composite medium with particle centers distributed at the nodes of some spatial lattice \( \Lambda \). Suppose \( e_{i} \) \( (i = 1, 2, 3) \) are linearly independent vectors, so that we can represent any node \( m \in \Lambda \):

\[
x_{m} = f_{1}(m_{1})e_{1} + f_{2}(m_{2})e_{2} + f_{3}(m_{3})e_{3}.
\]

where \( m = (m_{1}, m_{2}, m_{3}) \) are integer-valued coordinates of the node \( m \) in the basis \( e_{i} \), which are equal in modulus to \( e_{i} \), and \( f_{i}(m_{i}) = f_{i}(m_{i} + 1) \) \( \neq \text{const.} (i = 1, 2, 3) \). Thus, the lattice \( \Lambda \) can be considered as a single particular realization of the random field \( X(n) \) \( (n = 1, 2, 3, \ldots) \).

For triply periodic structures with linear-independent vectors of the principal period of \( \Lambda \subset \mathbb{R}^{3} \) determining a unit cell \( \Omega \) of volume \( \Omega = \mathbf{e}_{1} \cdot (\mathbf{e}_{2} \times \mathbf{e}_{3}) \), we can represent any node \( m \in \Lambda \) in the form

\[
x_{m} = m_{1}e_{1} + m_{2}e_{2} + m_{3}e_{3}.
\]

If, for example, the basis \( e_{i} \) is orthonormal, and the coefficients \( m = (m_{1}, m_{2}, m_{3}) \) are the integer set \( Z^{3} \), independent of one other, \( \Lambda \) defines a simple cubic (SC) packing; in the case where the coefficients \( m_{i} \) \( (i = 1, 2, 3) \) are either all even or odd, we have a body-centered cubic structure (BCC); a cubic face-centered structure (FCC) is obtained in the case where the coefficients \( m_{i} \) are either all even or two are odd, while the third is even. The method of assigning the lattice \( \Lambda \) is also possible where several nodes are located within the limits of a cell, and the coefficients \( m_{i} \) are the integer set \( Z^{3} \), independent of one another (see e.g. Kuznetsov, 1991). For doubly periodic structures

\[
x_{m} = m_{1}e_{1} + m_{2}e_{2} + f_{3}(m_{3})e_{3},
\]

(5.3)
where $f_j(m_1) - f_j(m_2 + 1) \neq \text{const.}$ in the plane $f(m_1) = \text{const.}$ The composite is reinforced by periodic arrays $\Lambda_m$ of inclusions in the direction of the $e_1$ axis and the $e_2$ axis. The type of lattice $\Lambda_m$ is defined by the law governing the coefficients in the equations (1.1, 2), and also by the magnitude and orientation of the vectors $e_1$ and $e_2$ (1.1, 2). In the functionally graded direction $e_1$, the inclusion spacing between adjacent layers may vary $f_j(m_1) - f_j(m_2 + 1) \neq \text{const.}$. For a doubly periodic array of inclusions in a finite ply containing $2m + 1$ layers of inclusions we have $f(m_1) \equiv 0$ at $|m_3| > m_2^2$; in the more general case of doubly periodic structures $f(m_1) \neq 0$ at $m_3 \to \pm \infty$. To make the exposition more clear we will assume that the basis $e_1$ is an orthogonal one and the axes $e_1$, $e_2$, $e_3$ are directed along axes of the global Cartesian coordinate system (these assumptions are not obligatory).

The composite material is constructed using the building blocks or cells: $w = \cup \Omega_m$, $\Omega_m \subset \Omega_m$. Hereafter the notation $\bar{f}(\bar{x})$ will be used for the average of the function $f$ over the cell $x \in \Omega$. In the center $x_0 \in \Omega$:

$$\bar{f}(x) = \int_{\Omega} f(y) dy, \quad x \in \Omega, \quad (5.4)$$

$n(x) = 1/\Omega \bar{f}$ is the number density of inclusions in the cell $\Omega$. Let $\xi$ be a “moving averaging” cell (or moving-window, see e.g. Graham-Brady et al., 2003) with the center $x$ and characteristic size $a = \sqrt{\xi}$, and let for the sake of definiteness $\xi$ be a random vector uniformly distributed on $\Omega$ whose value at $z \in \Omega$ is $\varphi_{\xi}(z) = 1/\xi$. Otherwise, we can define the average of the function $f$ with respect to translations of the vector $\xi$:

$$\bar{g}_s(x-y) = \frac{1}{\xi} \int_{\Omega} f(z-y) \, dz, \quad x \in \Omega, \quad (5.5)$$

Among other things, “moving averaging” cell $\xi$ can be obtained by translation of a cell $\Omega$ and can vary in size and shape during motion from point to point. Clearly, contracting the cell $\xi$ to the point $x$ occurs in passing to the limit $\bar{f}_s(x-y) = \bar{f}(x-y)$. To make the exposition more clear we will assume that $\xi$ results from $\Omega$ by translation of the vector $x - x_0$; it can be seen, however, that this assumption is not mandatory.

5.2. General integral equations

Recall that Eq. (3.21) is valid for concrete realization of the inclusion field $X$ which can be doubly periodic (5.2). In such a case, the function of the operation of a statistical average for random structure composites (see Section 3) becomes the volume average over the “moving averaging” cell $\xi$. In so doing, the transformation of Eq. (3.21) in the framework of the centering method is carried out by subtracting from both sides of Eq. (3.21) their average over the “moving averaging” cell $\xi$ (5.32):

$$\bar{e}(x) = \langle \varepsilon_{0,1} \rangle \bar{X} + \langle \varepsilon_{1} \rangle + \int_{\Omega} \{\bar{L}_{(1)}(x-x_{0}, \bar{\tau})\} dX_{0}$$

$$+ \int_{\Omega} \int_{\Omega} \{\nabla G(x-y) \tau^{(d)}(s) n(s)\} ds dX_0, \quad (5.6)$$

where $x_0$ passes through the lattice $\Lambda, x \in \Omega$, and one introduces a new centering operation over the “moving averaging” cell $x \in \Omega$ caused by translation of a cell $\Omega$: $\langle g(x-y) \rangle_{\Omega} = g(x-y) - g(x-y)_{\Omega}$. For the analysis of integral convergence in (5.6), we expand $\bar{L}_{(1)}(z-x_{0}, \bar{\tau})$ in a Taylor series about $x$ and integrate term by term over the cell $\xi$, then

$$\bar{L}_{(1)}(z-x_{0}, \bar{\tau}) = \bar{L}_{(1)}(x-x_{0}, \bar{\tau}) + (z-x_{0}) \nabla \bar{L}_{(1)}(x-x_{0}, \bar{\tau}) + \frac{1}{2} (z-x_{0}) \nabla \nabla \bar{L}_{(1)}(x-x_{0}, \bar{\tau}) + \ldots,$$

$$\int_{\Omega} \{\nabla G(x-y) \tau^{(d)}(s) n(s)\} ds.$$

A similar expansion can be performed for the tensor $\nabla G(x-y)$ that leads to

$$\int_{\Omega} \{\bar{L}_{(1)}(x-x_{0}, \bar{\tau})\} dX_{0} = \frac{1}{2} \int_{\Omega} \{\nabla G(x-y) \tau^{(d)}(s) n(s)\} ds.$$
6. Conclusion

It should be mentioned that the sketch of micromechanics of random structure composites can be subdivided on two schematic blocks (this classification as any other classification can not be perfect and reflects just one from possible points of views). The first one contains both the analytical and numerical solutions for one and a finite number of heterogeneities in the infinite homogeneous matrix. This first block (called the basis) is usually associated with computational mechanics becoming more and more ambitious because of the advances in modern computer software and hardware. On one hand, some models have been developed with the goal to which has minimize empirical elements and assumptions (e.g. for nonlinear and nonlinear phenomena). In many cases, the resolution of microscopic phenomena has lead to improved accuracy and offers the possibility of solving previously intractable problems. On the other hand, we are ambitious to attack increasingly large systems. Such methods are based on the wide exploitation of Monte Carlo simulation with forthcoming numerical analysis for each random realization of multiparticle interactions of microinhomogeneities taking into account both the complicated constitutive properties (coupled, nonlinear and nonlinear), imperfect interfaces (see, e.g. Introduction), and large number of interacting heterogeneities. However, at the present level of computer hardware and software, they are practical only for realizations containing no more then a few thousand inhomogeneities.

In parallel with computational micromechanics, the classical analytical micromechanics (the second block called a superstructure) represented by the model, perturbation, variational, and self-consistent methods presents; first of all, a set of both the hypotheses and tools for interactions of these hypotheses with the numerical results of the first block. We mentioned about the total domination in the second block of the EFH defining, in actual truth, the essence of classical analytical micromechanics. The EFH played a crucial progressive and encouraged role in the history of micromechanics and provided its current level of development. So, the basic hypotheses of the second block the EFH and quasicrystalline approximation (4.18) were proposed 160 and 60 years ago, respectively, while the basic result of the first block was obtained only in 1957 although the Eshelby (1957) theorem has a fundamental conceptual sense rather than only an analytical solution of some particular problem for the ellipsoidal homogeneous inclusion. The basic hypotheses H1–H3 were so much general that made it possible to realize a constantly increasing achievements of the first block. However, explosive character of the progress in the first block (especially in front of nanotechnology challenges) has lead to the opportunities of the second block rigidly bounded by the background by Mossotti (1850) with the hypotheses H1–H3 (the superstructure is found to be powerless for effective using of drastically increased productivity of the basis). As an attractive example of such a gap, we can mention a conventional incorporation of molecular dynamic simulation of nanofiber elastic moduli (implicitly assuming a local nature of continuum mechanics law at the nanoscale that is questionable) into Mori–Tanaka scheme based on very more strong hypothesis than H1 (see for detailed discussion Buryachenko (2011b,c) and Chapter 18 in Buryachenko (2007a)). This conflict is presently overcome in a conceptual sense very effectively by the new GIs (3.29) and (3.30) forming a new background of micromechanics that allow one to completely abandon the hypotheses H1 and H3 while the hypothesis H2 can be used for multiparticle generality. The researches can now forget about the basic restrictions of analytical micromechanics (such as, e.g., Eshelby tensor and the hypotheses H1 and H3) and use the numerical solutions for one and a few heterogeneities obtained by any available method of computational micromechanics.

In such a case the renormalizing terms \( \langle C_i^+(x - \mathbf{x}, \tau) | x \rangle \) and \( \langle C_i^-(x - \mathbf{x}, \eta) | x \rangle \) (see Eqs. (3.33), (3.34), (5.12), and (5.13)) explicitly depend on distributions \( \tau(x_i, y_i) \) and \( \eta(x_i, y_i) \) \( y_i \in \Omega_i \) respectively, while a fundamental deficiency of Eqs. (4.13) and (5.16), (5.17) is defined by the dependence of the renormalizing terms \( U(x - y) | \tau(y) \rangle \) and \( \Gamma(x - y) | \eta(y) \rangle \) obtained in the framework of the asymptotic approximation of the hypothesis H1b) only on the statistical averages \( \tau \) and \( \eta \). It is expected that the greater inhomogeneity of the field concentrator factor for a single heterogeneity inside infinite matrix leads to the greater difference between the new and old approaches (based on generalizations of either Eq. (3.29) or (4.13), respectively) with the possible change of sign of predicted local fields. Buryachenko (2010c, 2011a,c, 2012) and Buryachenko and Brun (2011, 2012a,b, 2013) have performed some particular comparative analysis of numerical results obtained by the use of either Eq. (4.8) or (4.13) for the perfect interface (2.6). Imperfection of interface conditions (2.14)–(2.19) leads in general to additional opportunity for the stress inhomogeneity inside the heterogeneities, and, therefore, a fundamental difference of results obtained in the framework of the new background (4.8) and old one (4.13) is expected. Moreover, the basic models of imperfect interface can be considered as the limiting asymptotic approaches of interphase models (see for references Section 1 and Section 2.2). More general cases of inhomogeneities of either the elastic moduli \( L(x) \neq \text{const.} \) or eigenstrains \( \mathbf{\mu}(x) \neq \text{const.} \) \( \mathbf{x} \in \Omega_i \) \( i = 1, 2, \ldots \) were considered by Buryachenko (2010c) and Buryachenko and Brun (2012a), respectively, who have demonstrated a significant difference between the new (3.29) and old (4.13) approaches at the estimation of the local stresses \( \sigma_i(x) \) \( \mathbf{x} \in \Omega_i \). The limiting case of imperfect interface (dislocation-like model) was considered in Section 4.6. However, the comparative numerical analysis for the other corresponding limiting cases (imperfect interface) (2.14)–(2.19) is beyond the scope of the current paper and will be considered in the subsequent publications.

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References


