A relation of Kauffman’s $f$-polynomials of virtual links

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Abstract

For an oriented virtual link diagram, Kauffman defined the $f$-polynomial. In this paper we give a relation of the $f$-polynomials of virtual link diagrams of a virtual skein triple. Then the $f$-polynomials for some one-component knots are computed by use of resolution trees of one-component knots.

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1. Introduction

An (oriented) virtual link diagram is a closed (oriented) 1-manifold generically immersed in $\mathbb{R}^2$ such that each double point is labeled to be either (1) a real crossing which is indicated as usual in classical knot theory or (2) a virtual crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Fig. 1 are called generalized Reidemeister moves. Two virtual link diagrams are said to be equivalent if they are related by a finite sequence of generalized Reidemeister moves. A virtual link is the equivalence class of a virtual link diagram; cf. [1,5,6].

Kauffman defined the $f$-polynomial $f_D(A) \in \mathbb{Z}[A, A^{-1}]$ of an oriented virtual link diagram $D$, which is also called the normalized bracket polynomial or the Jones polynomial; cf. [6]. It is a generalization of the normalized bracket polynomial of a classical link, and it is preserved under generalized Reidemeister moves.

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The following theorem holds in the category of virtual links as well as classical ones.

**Theorem 1** [6]. Let $(D_+, D_-, D_0)$ be a skein triple of oriented virtual link diagrams. Then we have

$$A^4 f_{D_+}(A) - A^{-4} f_{D_-}(A) + (A^2 + A^{-2}) f_{D_0}(A) = 0.$$

Here a skein triple $(D_+, D_-, D_0)$ means a triple of virtual link diagrams such that $D_-$ is obtained from $D_+$ by crossing change at a positive crossing point $p$, and $D_0$ is obtained from $D_+$ by smoothing $p$ as usual.

When we work in the virtual knot category, it is natural to consider another kind of triple. A **virtual skein triple** is a triple $(D_+, D_-, D_V)$ of virtual link diagrams such that $D_-$ is obtained from $D_+$ by crossing change at a positive crossing point $p$, and $D_V$ is obtained from $D_+$ by replacing $p$ with a virtual crossing.

In this paper, we give a relation of the $f$-polynomials of virtual link diagrams of a virtual skein triple. Using this relation, we can compute the $f$-polynomials for some knots without changing the number of components. In Section 3, we show how to compute the $f$-polynomial of the trefoil knot by use of a resolution tree of one-component knots.

### 2. Definitions and results

When a virtual link diagram $D'$ is obtained from another diagram $D$ by replacing a real crossing $p$ of $D$ with a virtual crossing, then we say that $D'$ is obtained from $D$ by **virtualizing** the crossing $p$.

Let $D$ be an unoriented virtual link diagram and let $p$ be a real crossing. If the virtual link diagram $D_1$ (or $D_2$ respectively) is mutually identical with $D$ but a neighborhood of $p$ indicated as in Fig. 2, it is said to be a virtual link diagram obtained from $D$ by doing **A-splice** (respectively **B-splice**) at $p$.
A state of \( D \) is a virtual link diagram obtained from \( D \) by doing A-splice or B-splice at each real crossing of \( D \). The Kauffman bracket polynomial of \( D \) is defined as the following:

\[
\langle D \rangle = \sum_S A^{\sharp(S)}(-A^2 - A^{-2})^{z(S)-1},
\]

where \( S \) runs over all states of \( D \), \( z(S) \) is the number of A-splice minus that of B-splice used for obtaining \( S \), and \( \sharp(S) \) is the number of loops of \( S \).

For an oriented virtual link diagram \( D \), the writhe \( \omega(D) \) is the number of positive crossings minus that of negative crossings of \( D \). The \( f \)-polynomial of \( D \) is defined by

\[
f_D(A) = (-A^3)^{-\omega(D)} \langle D \rangle.
\]

**Theorem 2** [6]. The \( f \)-polynomial is an invariant of a virtual link.

A pair \( P = (\Sigma, D) \) of a compact oriented surface \( \Sigma \) and a link diagram \( D \) on \( \Sigma \) is called an abstract link diagram (ALD) if \( |D| \) is a deformation retract of \( \Sigma \), where \( |D| \) is a graph obtained from \( D \) by replacing each crossing with a vertex. If \( D \) is oriented, \( P \) is said to be oriented. Unless otherwise stated, we assume that an ALD is oriented. An example of an ALD is illustrated in Fig. 3(a).

In [2] a map

\[
\phi: \{\text{virtual link diagrams}\} \rightarrow \{\text{ALDs}\}
\]

is defined so that each real crossing of a virtual link diagram corresponds to a crossing of the ALD and each virtual crossing is ignored as in Fig. 4. For a virtual link diagram \( D \), we
call $\phi(D)$ the ALD associated with $D$. For example, the ALD illustrated in Fig. 3(a) is the ALD associated with the virtual link diagram in Fig. 3(b). See [2] for details.

Let $P = (\Sigma, D)$ be an ALD. A checkerboard coloring of $P$ is a coloring of the components of $\Sigma - |D|$ by two colors, say black and white, such that any two components of $\Sigma - |D|$ that share an edge have different colors.

In [3], checkerboard colorability for a virtual link diagrams was defined by use of the ALD: A virtual link diagram $D$ admits a checkerboard coloring or it is checkerboard colorable if the associated ALD, $\phi(D)$, admits a checkerboard coloring. A checkerboard coloring of the ALD of Fig. 3(a) is shown in Fig. 3(c). Thus the virtual link diagram in Fig. 3(b) is checkerboard colorable. Note that not every virtual link diagram is checkerboard colorable and that checkerboard colorability is not preserved under generalized Reidemeister moves.

**Theorem 3** [4]. Let $(D_+, D_-, D_V)$ be a virtual skein triple of oriented virtual link diagrams. Suppose that $D_+$ (or $D_-$) is checkerboard colorable. Then

$$A^3 f_{D_+}(A) + A^{-3} f_{D_-}(A) - (A^3 + A^{-3}) f_{D_V}(A) = 0.$$ 

In this paper we consider a case where $D_+$ (or $D_-$) is almost checkerboard colorable.

**Definition.** A virtual link diagram $D$ is almost checkerboard colorable (a.c.c.) if $D$ is not checkerboard colorable and there exists a real crossing $p$ such that the virtual link diagram obtained from $D$ by virtualizing $p$ is checkerboard colorable. We call $p$ an anomalous crossing of $D$.

A virtual link diagram $D$ is called a virtual link diagram of type A (or type B respectively) with respect to a real crossing $p$ if the virtual link diagram obtained from $D$ by doing A-splice (respectively B-splice) at $p$ is checkerboard colorable.

**Proposition 4.** Let $D$ be an a.c.c. virtual link diagram with an anomalous crossing $p$ and let $D'$ be the virtual link diagram obtained from $D$ by crossing change at $p$. Then the following (i) or (ii) holds:

(i) $D$ is a virtual link diagram of type A with respect to $p$ and $D'$ is of type B with respect to $p$.

(ii) $D$ is a virtual link diagram of type B with respect to $p$ and $D'$ is of type A with respect to $p$.

Let $(D_+, D_-, D_V)$ be a virtual skein triple of oriented virtual link diagrams with respect to a real crossing $p$. Suppose that $D_+$ is a.c.c. such that $p$ is an anomalous crossing, namely,
D_+ is not checkerboard colorable and D_V is checkerboard colorable. By Proposition 4, D_+ is of type A and D_- is of type B with respect to p, or D_+ is of type B and D_- is of type A with respect to p.

**Theorem 5.** Let (D_+, D_-, D_V) be a virtual skein triple of oriented virtual link diagrams with respect to a real crossing p. Suppose that D_+ is a.c.c. such that p is an anomalous crossing, namely, D_+ is not checkerboard colorable and D_V is checkerboard colorable.

(i) If D_+ is of type A with respect to p, then we have

\[ Af_{D_+}(A) + A^{-1}f_{D_-}(A) - (A^1 + A^{-1})f_{D_V}(A) = 0. \]

(ii) If D_+ is of type B with respect to p, then we have

\[ A^5 f_{D_+}(A) + A^{-5}f_{D_-}(A) - (A^1 + A^{-1})f_{D_V}(A) = 0. \]

When we use the Kauffman bracket polynomials instead of f-polynomials, the formulae in Theorem 5 become a single formula. Theorem 5 is obtained from Theorem 6 by definition of the f-polynomial.

**Theorem 6.** Let D_A be an a.c.c. virtual link diagram with an anomalous crossing p and D_B (or D_V respectively) virtual link diagram obtained from D_A by crossing change at p (respectively by virtualizing p). If D_A is a virtual link diagram of type A (and hence D_B is of type B) with respect to p, then we have

\[ A^{-2}(D_A) + A^2(D_B) + (A^1 + A^{-1})(D_V) = 0. \]

3. Application

Let (D_+, D_-, D_V) be a virtual skein triple of oriented virtual link diagrams with respect to a real crossing p. When D_+ is checkerboard colorable, we have the formula in Theorem 3. (Note that D_+ is checkerboard colorable if and only if D_- is so.) When D_+ is not checkerboard colorable and D_V is checkerboard colorable, we have a formula in Theorem 5. Using these formulae, we can calculate the f-polynomials for some knots without changing the number of components. An example of the resolution tree of calculation of the f-polynomial of the trefoil knot by using Theorems 3 and 5 is shown in Fig. 5.

4. Proofs of Proposition 4 and Theorem 6

**Proof of Proposition 4.** Let D be an a.c.c. virtual link diagram with an anomalous crossing p as in Fig. 6(a) and D_V a virtual link diagram obtained from D by virtualizing p. (1) Suppose that the ALD associated with D_V admits a checkerboard coloring such that the neighborhood of the virtual crossing p is as in Fig. 6(b). Then the virtual link diagram obtained from D by doing A-splice at p admits a checkerboard coloring which is induced...
from that of $D$ as in Fig. 6(c). Then $D$ is a virtual link diagram of type A with respect to $p$. For $D'$, the splice is a B-splice. Hence $D'$ is of type B. (2) Suppose that $D_V$ admits a checkerboard coloring as in Fig. 6(d). Then $D$ is a virtual link diagram of type B with respect to $p$ and $D'$ is of type A by a similar argument. □

**Remark.** In the proof of Proposition 4, the ALD, $P = (\Sigma, \tilde{D}_V)$, associated with $D_V$ may admit checkerboard colorings such that the neighborhoods of $p$ are as in Fig. 6(b) and (d). Such a situation occurs if and only if the two curves of $\tilde{D}_V$ involving $P$ are lying on distinct connected components of $\Sigma$. Then both (i) and (ii) of Proposition 4 occur simultaneously.

Let $D$ be a virtual link diagram and let $|D|$ be the graph obtained from $D$ by replacing all real crossings with vertices. Virtual crossings are ignored. We say that $D$ admits an **alternate orientation** if each edge of $|D|$ can be oriented such that the orientations of the four edges around each vertex are as in Fig. 7. An example an alternate orientation of the virtual link diagram of Fig. 3(b) is shown in Fig. 8(a).
Lemma 7 [4]. Let $D$ be a virtual link diagram. $D$ admits an alternate orientation if and only if $D$ is checkerboard colorable.

When an alternate orientation is given to a checkerboard colorable virtual link diagram $D$, each loop of a state of $D$ admits an orientation induced from the alternate orientation of $D$ (see Fig. 9). An example of an orientation of a state of the virtual link diagram of Fig. 3 induced from the alternate orientation of Fig. 8(a) is shown in Fig. 8(b).

Proof of Theorem 6. Since $D_V$ is checkerboard colorable, it admits an alternate orientation. We fix an alternate orientation of $D_V$ as in Fig. 10(a). Then $D_A$ and $D_B$ are as in Fig. 11(b) and (c). Any state of $D_V$ admits an orientation induced from the alternate orientation of $D_V$. 
Let $S_1$ (or $S_2$ respectively) be the set of states of $D_V$ such that for a state $S \in S_1$ (respectively $S \in S_2$), there is a single loop (respectively there are two loops) passing through $p$. See Fig. 11(a) and (b). The disjoint union of $S_1$ and $S_2$ is the set of all states of $D_V$. Then we have

$$\langle D_V \rangle = \sum_{S \in S_1} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-1}$$

$$+ \sum_{S \in S_2} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-2}. $$

Any state of $D_A$ is obtained from some state of $D_V$ by changing the virtual crossing $p$ to a real crossing as in Fig. 10(b) followed by doing A-splice or B-splice at $p$. For $i = 1, 2$, let $S^A_i$ (or $S^B_i$ respectively) be the set of states of $D_A$ obtained from the states of $S_i$ by changing $p$ to a real crossing as in Fig. 10(b) followed by doing A-splice (respectively B-splice) at $p$ as in Fig. 11(c) and (e) (respectively (d) and (f)). Note that the set of all states of $D_A$ is the disjoint union of $S^A_1$, $S^B_1$, $S^A_2$, and $S^B_2$. Then we have

$$\langle D_A \rangle = \sum_{S \in S^A_1} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-1} + \sum_{S \in S^B_1} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-1}$$

$$+ \sum_{S \in S^A_2} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-1} + \sum_{S \in S^B_2} A^\natural(S) (-A^2 - A^{-2})^{\sharp(S)-1}. $$
The number of loops of any state of $D_A$ belonging to $S_A^1$ (or $S_B^1$ respectively) obtained from a state $S$ belonging to $S_1$ by doing $A$-splice (respectively $B$-splice) at $p$ is $\sharp(S) + 1$ (respectively $\sharp(S)$). Then we have

$$\sum_{S \in S_A^1} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} = \sum_{S \in S_1} A^{\sharp(S)+1}(-A^2 - A^{-2})^{\sharp(S)}$$

and

$$\sum_{S \in S_B^1} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} = \sum_{S \in S_1} A^{\sharp(S)-1}(-A^2 - A^{-2})^{\sharp(S)-1}.$$

Since the number of loops of any state of $D_A$ belonging to $S_A^2$ (or $S_B^2$ respectively) obtained from a state $S$ belonging to $S_2$ by doing $A$-splice (respectively $B$-splice) at $p$ is $\sharp(S) - 1$, we have

$$\sum_{S \in S_A^2} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} = \sum_{S \in S_1} A^{\sharp(S)+1}(-A^2 - A^{-2})^{\sharp(S)-2}$$

and

$$\sum_{S \in S_B^2} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} = \sum_{S \in S_1} A^{\sharp(S)-1}(-A^2 - A^{-2})^{\sharp(S)-2}.$$

Thus we have

$$\langle D_A \rangle = -A^3 \sum_{S \in S_1} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} + (A + A^{-1}) \sum_{S \in S_2} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-2}.$$

For $i = 1, 2$, let $\tilde{S}_i^A$ (or $\tilde{S}_i^B$ respectively) be the set of states of $D_B$ obtained from the states of $D_V$ belonging to $S_i$ by doing $A$-splice (respectively $B$-splice) at $p$ as in Fig. 11(d) and (f) (respectively (c) and (e)). By a similar argument we obtain

$$\langle D_B \rangle = -A^{-3} \sum_{S \in S_1} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-1} + (A + A^{-1}) \sum_{S \in S_2} A^{\sharp(S)}(-A^2 - A^{-2})^{\sharp(S)-2}.$$

Hence we have the formulae in the theorem. \qed

References