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Bounds for the infinity norm of the inverse for certain *M*- and *H*-matrices

L.Yu. Kolotilina*

St. Petersburg Branch of the V.A. Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russian Federation

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ABSTRACT

The paper presents new two-sided bounds for the infinity norm of the inverse for the so-called *PM*-matrices, which form a subclass of the class of nonsingular *M*-matrices and contain the class of strictly diagonally dominant matrices. These bounds are shown to be monotone with respect to the underlying partitioning of the index set, and the equality cases are analyzed. Also an upper bound for the infinity norm of the inverse of a *PH*-matrix (whose comparison matrix is a *PM*-matrix) is derived. The known Ostrowski, Ahlberg–Nilson–Varah, and Morača bounds are shown to be special cases of the upper bound obtained.

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1. Introduction and preliminaries

The problem of bounding the infinity norm of the inverse of a nonsingular matrix satisfying certain assumptions was considered in a number of publications (e.g., see [1,13,14,2,12,9,10,8,5]).

The present paper considers the problem of bounding $\|A^{-1}\|_\infty$ for the so-called *PM*- and *PH*-matrices A, which form subclasses of the classes of nonsingular *M*- and *H*-matrices, respectively, and are defined below.

Let $A = (a_{ij}) \in \mathbb{C}^{m\times m}, m \geqslant 1$, and let

* Tel.: +7 812 571 5754; fax: +7 812 310 5377.
E-mail address: liko@pdmi.ras.ru

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$$\langle m\rangle = \bigcup_{i=1}^{n} M_i, \quad 1 \leqslant n \leqslant m, \tag{1.1}$$

be a partitioning of the index set $\langle m\rangle = \{1, \ldots, m\}$ into disjoint nonempty subsets. Denote

$$A_{ij} = A[M_i, M_j], \quad i, j = 1, \ldots, n, \tag{1.2}$$

and represent A in the following block form:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}. \tag{1.3}$$

Throughout the paper, for a real $t \times s$ matrix $A = (a_{ij})$, we denote

$$r_i(A) = \sum_{j=1}^{s} a_{ij}, \quad i = 1, \ldots, t, \tag{1.4}$$

so that $r_i(A)$ stands for the ith row sum of the entries of A.

If $A \in \mathbb{C}^{m\times m}, m \geqslant 1$, then its comparison matrix $\mathscr{M}(A) = (m_{ij})$ is defined by the relations

$$m_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

A complex matrix A is said to be an H-matrix if $\mathscr{M}(A)$ is a nonsingular M-matrix. Recall that if A is an H-matrix, then (e.g., see [4, p. 131]) it is nonsingular. Furthermore, by the Ostrowski theorem [11] (also see [4, p. 131]), the inverse matrices A^{-1} and $\mathscr{M}(A)^{-1}$ are interrelated as follows.

Theorem 1.1. *Let $A \in \mathbb{C}^{m\times m}, m \geqslant 1$, be an H-matrix. Then*

$$|A^{-1}| \leqslant \mathscr{M}(A)^{-1}. \tag{1.5}$$

In (1.5) and throughout the paper, for $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ we set $|A| = (|a_{ij}|)$, and, in the real case, matrix and vector inequalities are understood componentwise.

Introduce the following definitions, basic for the present paper.

Given a matrix $A \in \mathbb{C}^{m\times m}, m \geqslant 1$, and a partitioning of the index set $\langle m\rangle = \{1, \ldots, m\}$

$$\langle m\rangle = \bigcup_{i=1}^{n} M_i, \quad 1 \leqslant n \leqslant m, \tag{1.6}$$

into disjoint nonempty subsets, represent A in the form (1.2) and (1.3) and define the following collection of $m_1 \times \cdots \times m_n$ aggregated matrices of order n:

$$A^{(i_1,i_2,\ldots,i_n)} = \begin{bmatrix} r_{i_1}(A_{11}) & r_{i_1}(A_{12}) & \cdots & r_{i_1}(A_{1n}) \\ r_{i_2}(A_{21}) & r_{i_2}(A_{22}) & \cdots & r_{i_2}(A_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ r_{i_n}(A_{n1}) & r_{i_n}(A_{n2}) & \cdots & r_{i_n}(A_{nn}) \end{bmatrix}, \quad i_k \in M_k, \ k = 1, \ldots, n. \tag{1.7}$$

Here, $m_i = |M_i|, i = 1, \ldots, n$.

We say that A is a *PM-matrix* (partitioned M-matrix) with respect to the partitioning (1.6) if A is a Z-matrix (i.e., its off-diagonal entries are nonpositive) and all the matrices $A^{(i_1,\ldots,i_n)}, i_k \in M_k, k = 1, \ldots, n$, defined in accordance with (1.7) are nonsingular M-matrices. Also we say that A is a *PH-matrix* (with respect to the partitioning (1.6)) if $\mathscr{M}(A)$ is a PM-matrix (with respect to the same partitioning).

Obviously, a matrix A is a PM-matrix (PH-matrix) with respect to the finest (pointwise) partitioning $\langle m\rangle = \bigcup_{i=1}^{m}\{i\}$ if and only if A is a nonsingular M-matrix (an H-matrix). On the other hand, for the coarsest partitioning $\langle m\rangle = M_1$ with $n = 1, A$ is a PH-matrix if and only if it is strictly diagonally dominant (sdd).

In [7], the following result was established.

Theorem 1.2. *If $A \in \mathbb{C}^{m\times m}$, $m \geqslant 1$, is a PH-matrix with respect to a partitioning $\langle m\rangle = \bigcup_{i=1}^{n} M_i$, $1 \leqslant n \leqslant m$, of the index set into disjoint nonempty subsets, then A is an H-matrix.*

In particular, for a *PM*-matrix, which is a *Z*-matrix by definition, Theorem 1.2 implies the following result.

Corollary 1.1. *If $A \in \mathbb{R}^{m\times m}$, $m \geqslant 1$, is a PM-matrix with respect to a partitioning $\langle m\rangle = \bigcup_{i=1}^{n} M_i$, $1 \leqslant n \leqslant m$, of the index set into disjoint nonempty subsets, then A is a nonsingular M-matrix.*

Thus, *PM*- and *PH*-matrices are nonsingular, and the problem of bounding their inverses naturally arises. This problem is considered in the present paper, which is organized as follows. Section 2 deals with *PM*-matrices. The first main result (Theorem 2.1) states that the infinity norm of the inverse of a *PM*-matrix satisfies the following two-sided bounds in terms of the aggregated matrices (1.7):

$$\min_{i_1,\ldots,i_n} \|[A^{(i_1,\ldots,i_n)}]^{-1}\|_\infty \leqslant \|A^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|[A^{(i_1,\ldots,i_n)}]^{-1}\|_\infty. \tag{1.8}$$

For an irreducible matrix *A*, the cases of equalities in (1.8) are also described.

The second result on *PM*-matrices (Theorem 2.2) states that the bounds (1.8) are monotone with respect to the underlying partitioning of the index set, i.e., the finer the partitioning the tighter the bounds. This result is based on the fact that if a matrix *A* is a *PM*-matrix with respect to a partitioning $\langle m\rangle = \bigcup_{i=1}^{n} M_i$, then it also is a *PM*-matrix with respect to every partitioning $\langle m\rangle = \bigcup_{i=1}^{n'} M_i'$, $n' \geqslant n$, that is finer than the original one.

Section 3 considers the case of *PH*-matrices. Based on Theorem 1.1, for a *PH*-matrix *A* from Theorems 2.1 and 2.2 we infer the upper bound

$$\|A^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|[\mathscr{M}(A)^{(i_1,\ldots,i_n)}]^{-1}\|_\infty \tag{1.9}$$

conjectured in [6], and also the monotonicity of this bound with respect to the underlying partitioning.

Section 4 compares the bounds (1.8) and (1.9) with some known results, obtained in [1,13,3,9,10,6,5].

We conclude this introduction with two relevant remarks. First, if A^{T} is a *PM*-matrix (*PH*-matrix), then the results established obviously yield two-sided bounds (an upper bound) for $\|A^{-1}\|_1$. Second, if both *A* and A^{T} are *PH*-matrices with respect to some partitionings of the index set, which may be different, then, in the same way as in [13], one immediately obtains an upper bound for the spectral norm of A^{-1}, i.e., a lower bound for the smallest singular value of the original matrix *A*.

2. Two-sided bounds for *PM*-matrices

The first main result of this paper is the following theorem.

Theorem 2.1. *If $A \in \mathbb{R}^{m\times m}$, $m \geqslant 1$, is a PM-matrix with respect to a partitioning $\langle m\rangle = \bigcup_{i=1}^{n} M_i$, $1 \leqslant n \leqslant m$, of the index set into disjoint nonempty subsets, then it is a nonsingular M-matrix, and its inverse satisfies the two-sided bounds*

$$\min_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty \leqslant \|A^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty, \tag{2.1}$$

where the minimum and maximum are taken over all $i_k \in M_k$, $k = 1,\ldots,n$. Furthermore, if A is irreducible, then either inequality in (2.1) is an equality if and only if

$$(A^{-1}e)_{i_k} = c_k \quad \text{for all } i_k \in M_k,\ \ k = 1,\ldots,n, \tag{2.2}$$

where $e = [1,\ldots,1]^{\mathrm{T}}$ is the unit vector of appropriate dimension; otherwise both inequalities in (2.1) hold strictly.

In order to prove Theorem 2.1, we follow [7] and introduce into consideration the $(m-1)\times(m-1)$ matrices $A_i^{(1)}$ and $A_i^{(2)}$ that are defined as follows:

$$A_i^{(1)} = \widehat{A}_{i+1}\begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-i-1} \end{bmatrix}, \quad A_i^{(2)} = \widehat{A}_i\begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-i-1} \end{bmatrix}. \tag{2.3}$$

Here and below, I_k is the identity matrix of order k, and for a matrix $B \in \mathbb{C}^{t\times s}$, with $t \geqslant 2$ and $s \geqslant 1$, by $\widehat{B}_j$ we denote its $(t-1)\times s$ submatrix obtained by deleting the jth row, $j = 1,\ldots,t$. The passage from A to the pair $A_i^{(1)}, A_i^{(2)}$ is referred to as aggregation of columns i and $i+1$ of A. Obviously, the matrices $A_i^{(1)}$ and $A_i^{(2)}$ form the collection of aggregated matrices associated with the partitioning $\langle m\rangle = \bigcup_{k=1}^{m-1} M_k$, where $M_k = \{k\}, k = 1,\ldots,i-1$; $M_i = \{i, i+1\}$; $M_k = \{k+1\}, k = i+1,\ldots,m-1$, and, in terms of the matrices (1.7), we have

$$A_i^{(1)} = A^{(1,\ldots,i,i+2,\ldots,m)}, \quad A_i^{(2)} = A^{(1,\ldots,i-1,i+1,\ldots,m)}.$$

Thus, by Corollary 1.1, if both $A_i^{(1)}$ and $A_i^{(2)}$ are nonsingular M-matrices, then A also is a nonsingular M-matrix.

The proof of Theorem 2.1 is based on the following lemma.

Lemma 2.1. *Let $A \in \mathbb{R}^{m\times m}, m \geqslant 2$, be a nonsingular M-matrix such that for a certain $i, 1 \leqslant i \leqslant m-1$, both matrices $A_i^{(1)}$ and $A_i^{(2)}$ defined in (2.3) are nonsingular M-matrices. Then*

$$\min_{k=1,2} \|(A_i^{(k)})^{-1}\|_\infty \leq \|A^{-1}\|_\infty \leqslant \max_{k=1,2} \|(A_i^{(k)})^{-1}\|_\infty. \tag{2.4}$$

Furthermore, if A is irreducible, then either inequality in (2.4) is an equality if and only if

$$(A^{-1}e)_i = (A^{-1}e)_{i+1}; \tag{2.5}$$

otherwise both inequalities in (2.4) are strict.

Proof. Permuting (if necessary) the rows and columns of A, we may assume, without loss of generality, that $i = 1$ and that

$$g_1 \leqslant g_2, \tag{2.6}$$

where we set $g = A^{-1}e$. Using (2.3) and (2.6) and taking into account that A is a Z-matrix, we derive the right-hand side inequality in (2.4) in the following way:

$$A_1^{(2)}\hat{g}_1 = \widehat{A}_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & I_{m-2} \end{bmatrix}\begin{bmatrix} g_2 \\ g_3 \\ \vdots \\ g_m \end{bmatrix} = \widehat{A}_1\begin{bmatrix} g_2 \\ g_2 \\ g_3 \\ \vdots \\ g_m \end{bmatrix} \leqslant \widehat{A}_1 g = \widehat{(Ag)}_1 = \hat{e}_1. \tag{2.7}$$

Note that the inequality in (2.7) stems from (2.6) and the fact that all the entries in the first column of $\widehat{A}_1$ are nonpositive. Since, by assumption, $A_1^{(2)}$ is a nonsingular M-matrix, we have $(A_1^{(2)})^{-1} \geqslant 0$, and (2.7) implies that

$$\hat{g}_1 \leqslant (A_1^{(2)})^{-1}\hat{e}_1, \tag{2.8}$$

whence, with account for (2.6), we obtain

$$\|A^{-1}\|_\infty = \|g\|_\infty = \|\hat{g}_1\|_\infty \leqslant \|(A_1^{(2)})^{-1}\hat{e}_1\|_\infty = \|(A_1^{(2)})^{-1}\|_\infty.$$

This proves the right-hand side inequality in (2.4).

In order to prove the left-hand side inequality in (2.4), we similarly deduce

$$A_1^{(1)}\hat{g}_2 = \widehat{A}_2 \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & I_{m-2} \end{bmatrix} \begin{bmatrix} g_1 \\ g_3 \\ \vdots \\ g_m \end{bmatrix} = \widehat{A}_2 \begin{bmatrix} g_1 \\ g_1 \\ g_3 \\ \vdots \\ g_m \end{bmatrix} \geqslant \widehat{A}_2 g = (\widehat{Ag})_2 = \hat{e}_2, \tag{2.9}$$

implying that

$$\hat{g}_2 \geqslant (A_1^{(1)})^{-1}\hat{e}_2. \tag{2.10}$$

By using (2.6) and (2.10), we obtain

$$\|A^{-1}\|_\infty = \|g\|_\infty \geqslant \|\hat{g}_2\|_\infty \geqslant \|(A_1^{(1)})^{-1}\hat{e}_2\|_\infty = \|(A_1^{(1)})^{-1}\|_\infty.$$

This completes the proof of inequalities (2.4).

In order to analyze the cases of equalities in (2.4), we assume that A is irreducible and that

$$\|A^{-1}\|_\infty = \|(A_1^{(2)})^{-1}\|_\infty. \tag{2.11}$$

Set

$$u = [u_2, \ldots, u_m]^{\mathrm{T}} = (A_1^{(2)})^{-1}e.$$

Then we have

$$e = A_1^{(2)}u = \widehat{A}_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & I_{m-2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} = \widehat{A}_1 \begin{bmatrix} u_2 \\ u_2 \\ u_3 \\ \vdots \\ u_m \end{bmatrix} = (\widehat{A\tilde{u}})_1, \tag{2.12}$$

where we denote

$$\tilde{u} = [u_2, u_2, u_3 \ldots, u_m]^{\mathrm{T}}.$$

From (2.12) it immediately follows that

$$A\tilde{u} = \begin{bmatrix} \alpha \\ 1 \\ \vdots \\ 1 \end{bmatrix} = e + (\alpha - 1)e_1, \tag{2.13}$$

where e_i is the ith column of the identity matrix I. Since A is an irreducible M-matrix, its inverse is positive, and $A^{-1}e_1$ is a positive vector. By (2.13), we have

$$\tilde{u} = A^{-1}e + (\alpha - 1)A^{-1}e_1 = g + (\alpha - 1)A^{-1}e_1.$$

Suppose $\alpha > 1$. Then $\tilde{u} > g$ and

$$\|\tilde{u}\|_\infty > \|g\|_\infty = \|A^{-1}\|_\infty,$$

which contradicts (2.11), because

$$\|\tilde{u}\|_\infty = \|u\|_\infty = \|(A_1^{(2)})^{-1}\|_\infty.$$

In a similar fashion, we ascertain that the case $\alpha < 1$ is impossible as well. Thus, if equality (2.11) holds true, then $\alpha = 1$, i.e., $\tilde{u} = A^{-1}e = g$, which implies that $g_1 = g_2$.

Conversely, if $g_1 = g_2$, then $\hat{g}_1 = \hat{g}_2$, and both (2.7) and (2.9) are strings of equalities, implying that $(A_1^{(2)})^{-1}e = (A_1^{(1)})^{-1}e$, whence both inequalities in (2.4) are equalities.

The case $\|A^{-1}\|_\infty = \|(A_1^{(1)})^{-1}\|_\infty$ is treated similarly. □

From the proof of Lemma 2.1, we readily infer the following useful result.

Corollary 2.1. *If, under the assumptions of Lemma* 2.1, *the matrix A is irreducible, then the inequality*

$$\|(A_i^{(2)})^{-1}\|_\infty > \|(A_i^{(1)})^{-1}\|_\infty$$

is equivalent to the inequality

$$g_i < g_{i+1}, \quad \text{where } g = A^{-1}e.$$

Note also that in the second part of Lemma 2.1, the assumption that A is irreducible can be weakened as follows.

Corollary 2.2. *Let A be a nonsingular M-matrix and let $A_i^{(2)}$ be a nonsingular M-matrix. If $A^{-1}e_i$ is a positive vector, then*

$$\|A^{-1}\|_\infty = \|(A_i^{(2)})^{-1}\|_\infty$$

if and only if $g_i = g_{i+1}$.

Proof of Theorem 2.1. From definitions (1.7) and (2.3) it readily follows that for an arbitrary partitioning $\langle m\rangle = \bigcup_{i=1}^n M_i$, with $1 \leqslant n < m$, each of the aggregated matrices $A^{(i_1,\ldots,i_n)}$ from the collection (1.7) can be obtained from A as a result of successively aggregating pairs of consecutive columns. Thus, for every fixed partitioning $\langle m\rangle = \bigcup_{i=1}^n M_i$, we obtain a sequence (which is in general not uniquely determined) of partitionings of the index set, starting with the entrywise partitioning $\langle m\rangle = \bigcup_{i=1}^m \{i\}$ and terminating with the given one. Note that the order of the associated aggregated matrices successively decreases from m to n, and, by Corollary 1.1, all the intermediate aggregated matrices are nonsingular M-matrices. Thus, the bounds (2.1) stem from Lemma 2.1.

In order to prove the second assertion of Theorem 2.1, let A be irreducible. First assume that

$$\|A^{-1}\|_\infty = \max_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty. \tag{2.14}$$

We will show that (2.14) implies (2.2). To this end, it is obviously sufficient to demonstrate that if $|M_k| \geqslant 2$, where $1 \leqslant k \leqslant n$, then for all $i,j \in M_k, i \neq j$, we have $(A^{-1}e)_i = (A^{-1}e)_j$. Without loss of generality, we may assume that

$$M_k = \{j_k,\ldots,j_k + |M_k| - 1\}.$$

In this case, it is sufficient to show that from (2.14) it follows that for all $i, j_k \leqslant i < j_k + |M_k| - 1$,

$$(A^{-1}e)_i = (A^{-1}e)_{i+1}. \tag{2.15}$$

Indeed, by Lemma 2.1 and (2.1), we have

$$\|A^{-1}\|_\infty \leqslant \max_{l=1,2} \|(A_i^{(l)})^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty. \tag{2.16}$$

From (2.16) and (2.14) we immediately obtain that

$$\|A^{-1}\|_\infty = \max_{l=1,2} \|(A_i^{(l)})^{-1}\|_\infty$$

and, consequently, (2.15) holds by Lemma 2.1.

The fact that equality on the left-hand side of (2.1) implies (2.2) is established similarly.

Finally, assume that condition (2.2) is fulfilled. Then, by Lemma 2.1, aggregation of columns i and $i+1$, where $i, i+1$ belong to the same set M_k, does not change the infinity norm of the inverse. In addition, each of the inverse matrices $(A_i^{(1)})^{-1}$ and $(A_i^{(2)})^{-1}$ still satisfies (2.2), with M_k replaced by

$M_k \setminus \{i+1\}$ and $M_k \setminus \{i\}$, respectively. Thus, proceeding by induction, we conclude that both inequalities in (2.1) are equalities.

Theorem 2.1 is proved completely. □

Following [7], we say that a partitioning:

$$\langle m \rangle = \bigcup_{i=1}^{n} M_i \tag{2.17}$$

of the set $\langle m \rangle$ into disjoint nonempty subsets is *finer* than a partitioning

$$\langle m \rangle = \bigcup_{i=1}^{n'} M_i' \tag{2.18}$$

and (2.18) is *coarser* than (2.17) if $n > n'$ and each of the sets $M_i', i = 1, \ldots, n'$, is a union of some sets $M_i, i = 1, \ldots, n$.

In this terminology, from the proof of Theorem 2.1 we infer the following monotonicity result.

Theorem 2.2. *Let $A \in \mathbb{R}^{m\times m}, m \geqslant 1$, be a PM-matrix with respect to a partitioning (2.17). Then A is a PM-matrix with respect to every finer partitioning (2.18), and the following inequalities hold*:

$$\min_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty \leqslant \min_{i_1',\ldots,i_{n'}'} \|(A^{(i_1',\ldots,i_{n'}')})^{-1}\|_\infty \leqslant \|A^{-1}\|_\infty \tag{2.19}$$

and

$$\|A^{-1}\|_\infty \leqslant \max_{i_1',\ldots,i_{n'}'} \|(A^{(i_1',\ldots,i_{n'}')})^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty. \tag{2.20}$$

Here, the minima and maxima are taken over all $i_k' \in M_k', k = 1, \ldots, n'$, and all $i_k \in M_k, k = 1, \ldots, n$.

3. An upper bound for *PH*-matrices

In view of Theorems 1.1 and 1.2, the following upper bound for the infinity norm of the inverse of a *PH*-matrix is an immediate consequence of Theorem 2.1.

Theorem 3.1. *If $A \in \mathbb{C}^{m\times m}, m \geqslant 1$, is a PH-matrix with respect to a partitioning $\langle m \rangle = \bigcup_{i=1}^{n} M_i, 1 \leqslant n \leqslant m$, of the index set into disjoint nonempty subsets, then it is an H-matrix, and its inverse satisfies the upper bound*

$$\|A^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|(\mathscr{M}(A)^{(i_1,\ldots,i_n)})^{-1}\|_\infty. \tag{3.1}$$

The following monotonicity property of the upper bound (3.1) readily stems from Theorem 2.2.

Theorem 3.2. *Let $A \in \mathbb{C}^{m\times m}, m \geqslant 1$, be a PH-matrix with respect to a partitioning*

$$\langle m \rangle = \bigcup_{i=1}^{n} M_i, \quad 1 \leqslant n \leqslant m, \tag{3.2}$$

of the index set into disjoint nonempty subsets. Then A is a PH-matrix with respect to an arbitrary partitioning

$$\langle m \rangle = \bigcup_{i=1}^{n'} M_i', \quad 1 \leqslant n' \leqslant m, \tag{3.3}$$

that is finer than (3.2), and

$$\|A^{-1}\|_\infty \leqslant \max_{i_1',\ldots,i_{n'}'} \|(\mathscr{M}(A)^{(i_1',\ldots,i_{n'}')})^{-1}\|_\infty \leqslant \max_{i_1,\ldots,i_n} \|(\mathscr{M}(A)^{(i_1,\ldots,i_n)})^{-1}\|_\infty, \tag{3.4}$$

where the maxima are taken over all $i_k \in M_k, k = 1, \ldots, n$, *and all* $i'_k \in M'_k, k = 1, \ldots, n'$.

4. Comparison with known results

First we note that if $n = m$, i.e., no nontrivial block partitioning is imposed on A, then $A^{(i_1,\ldots,i_n)} = A$, and the upper bound of Theorem 3.1 reduces to the Ostrowski result (1.5).

If $n = 1$, then A is a PH-matrix if and only if

$$p_i(A) := |a_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}| > 0, \quad i = 1, \ldots, m, \tag{4.1}$$

i.e., A is a strictly diagonally dominant matrix, and the upper bound of Theorem 3.1 reduces to the classical Ahlberg–Nilson–Varah bound (see [1,13])

$$\|A^{-1}\|_\infty \leqslant \max_{i\in\langle m\rangle}\{1/p_i(A)\}. \tag{4.2}$$

In addition, for a PM-matrix A, Theorem 2.1 supplements the upper bound (4.2) with its lower counterpart

$$\|A^{-1}\|_\infty \geqslant \min_{i\in\langle m\rangle}\{1/p_i(A)\}, \tag{4.3}$$

which is almost trivial and was presented in [10]. Furthermore, if A is an irreducible PM-matrix, then, by Theorem 2.1, the bounds (4.2) and (4.3) simultaneously hold with equality if and only if

$$A^{-1}e = ce,$$

where c is a positive constant; otherwise both of them hold strictly.

Since the trivial partitioning $\langle m\rangle = M_1$ is coarser than any partitioning (3.2) with $n \geqslant 2$, by Theorem 3.2 we have

$$\|A^{-1}\|_\infty \leq \max_{i_1,\ldots,i_n} \|(\mathscr{M}(A)^{(i_1,\ldots,i_n)})^{-1}\|_\infty \leqslant \max_{i\in\langle m\rangle}\{1/p_i(A)\}. \tag{4.4}$$

Thus, Theorem 3.2 provides an improvement of the Ahlberg–Nilson–Varah upper bound (4.2), which is, in addition, applicable under milder assumptions on A (because if A is sdd, then all the matrices $A^{(i_1,\ldots,i_n)}$ are sdd as well). Furthermore, if A is a PM-matrix with respect to a partitioning with $n \geqslant 2$, then, by Theorem 2.2, we also have the lower counterpart of inequalities (4.4), namely

$$\|A^{-1}\|_\infty \geq \min_{i_1,\ldots,i_n} \|(A^{(i_1,\ldots,i_n)})^{-1}\|_\infty \geqslant \min_{i\in\langle m\rangle}\{1/p_i(A)\}. \tag{4.5}$$

In the case where $n = 2$ and $\langle m\rangle = M_1 \cup M_2$, a matrix $A \in \mathbb{C}^{m\times m}, m \geqslant 2$, represented as

$$A = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \quad \text{where } A_{ij} = A[M_i, M_j], \; i,j = 1,2,$$

is a PH-matrix if and only if all the matrices

$$A_{ij} := \begin{bmatrix} p_i(A_{11}) & -r_i(|A_{12}|)\\ -r_j(|A_{21}|) & p_j(A_{22})\end{bmatrix}, \quad 1 \leqslant i \leqslant |M_1|, \; 1 \leqslant j \leqslant |M_2|,$$

are nonsingular M-matrices, or, equivalently

$$p_i(A_{11}) > 0 \quad \text{for all } i, \; 1 \leqslant i \leqslant |M_1|$$

and

$$p_i(A_{11})p_j(A_{22}) > r_i(|A_{12}|)r_j(|A_{21}|) \quad \text{for all } i,j, \; 1 \leqslant i \leqslant |M_1|, \; 1 \leqslant j \leqslant |M_2|.$$

Such matrices were studied in a number of papers (e.g., see [9,6] and the references therein). In this case, the upper bound of Theorem 3.1 reduces to the bound

$$\|A^{-1}\|_\infty \leqslant \max_{i,j} \|A_{ij}^{-1}\|_\infty, \tag{4.6}$$

which was first proved in [6]. However, it should be mentioned that the bound (4.6) actually coincides with the bound established in [9], which is in terms of the entries of A and suggests no extension to the case $n \geqslant 2$. Note that in [5] the same upper bound as in [9] was proved for the narrower class consisting of matrices that are PH-matrices with respect to the specific partitioning of the index set into two subsets one of which corresponds to the strictly diagonally dominant rows, whereas the other corresponds to the rows that are not strictly diagonally dominant.

The lower counterpart of (4.6),

$$\|A^{-1}\|_\infty \geqslant \min_{i,j} \|A_{ij}^{-1}\|_\infty, \tag{4.7}$$

valid for a PM-matrix A by Theorem 2.1, and the analysis of the equality cases in (4.6) and (4.7) for a PM-matrix A are new.

In conclusion, we show that the upper bounds of Theorems 2.1 and 3.1 are is general incomparable with an old-known block bound, which is recalled below.

Given a block-partitioned matrix (1.3), define the matrix

$$\widetilde{N}(A) = \begin{bmatrix} \|A_{11}^{-1}\|_\infty^{-1} & -\|A_{12}\|_\infty & \cdots & -\|A_{1n}\|_\infty \\ -\|A_{21}\|_\infty & \|A_{22}^{-1}\|_\infty^{-1} & \cdots & -\|A_{2n}\|_\infty \\ \cdots & \cdots & \cdots & \cdots \\ -\|A_{n1}\|_\infty & -\|A_{n2}\|_\infty & \cdots & \|A_{nn}^{-1}\|_\infty^{-1} \end{bmatrix}. \tag{4.8}$$

As is known (see [3]), if $\widetilde{N}(A)$ is a nonsingular M-matrix, then A is nonsingular, and the nonnegative matrix

$$N(A^{-1}) = \begin{bmatrix} \|A'_{11}\|_\infty & \cdots & \|A'_{1n}\|_\infty \\ \cdots & \cdots & \cdots \\ \|A'_{n1}\|_\infty & \cdots & \|A'_{nn}\|_\infty \end{bmatrix}, \tag{4.9}$$

where we denote $A^{-1} = (A'_{ij})_{i,j=1}^n$, satisfies the inequality

$$N(A^{-1}) \leqslant \widetilde{N}(A)^{-1}. \tag{4.10}$$

On the other hand, we trivially have

$$\|A^{-1}\|_\infty \leqslant \|N(A^{-1})\|_\infty. \tag{4.11}$$

Thus, in view of (4.10) and (4.11), for $\|A^{-1}\|_\infty$ we have the block bound

$$\|A^{-1}\|_\infty \leqslant \|\widetilde{N}(A)^{-1}\|_\infty. \tag{4.12}$$

Note that (4.12) and the Ahlberg–Nilson–Varah bound (4.2) immediately imply the bound [13]

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{1\leqslant i\leqslant n}\left\{\|A_{ii}^{-1}\|_\infty^{-1} - \sum_{j\neq i}\|A_{ij}\|_\infty\right\}}, \tag{4.13}$$

which holds under the assumption that $\widetilde{N}(A)$ is strictly diagonally dominant, and generalizes the bound (4.2) to the block case.

Relation (4.12) is an upper bound for $\|A^{-1}\|_\infty$ in terms of the infinity norm of the inverse to the $n \times n$ matrix $\widetilde{N}(A)$, which is assumed to be a nonsingular M-matrix. Thus, it is natural to attempt to compare (4.12) with the upper bounds provided by Theorems 2.1 and 3.1, which are stated in terms of the aggregated $n \times n$ matrices. To this end, we consider two examples.

First let $n = 1$ and let A be an $m \times m$, $m \geqslant 2$, nonsingular M-matrix. In this case, $\widetilde{N}(A) = \|A^{-1}\|_\infty^{-1}$, so that (4.12) obviously holds with equality. On the other hand, if, in addition, A is sdd, then Theorem 2.1 yields

$$\min_{i\in\langle m\rangle}\{1/p_i(A)\} \leqslant \|A^{-1}\|_\infty \leqslant \max_{i\in\langle m\rangle}\{1/p_i(A)\}$$

and if A is irreducible and $p(A) = (p_i(A))$ is not a constant vector, then both inequalities are strict. Thus, in the case considered, the bound (4.12) is applicable under weaker assumptions and is, in general, better than the upper bound of Theorem 2.1.

However, if we assume that $n > 1$, that A is a PM-matrix, and that

$$A_{ii}e = c_i e, \quad c_i > 0, \ i = 1, \dots, n, \tag{4.14}$$

then, obviously,

$$\|A_{ii}^{-1}\|_\infty = \|A_{ii}^{-1}e\|_\infty = 1/c_i, \quad i = 1, \dots, n,$$

whence the diagonal entries of the matrix $\widetilde{N}(A)$, defined in (4.8), coincide with the respective diagonal entries of each of the matrices $A^{(i_1,\dots,i_n)}$. Since, in addition, we have

$$r_{i_k}(A_{kj}) = -r_{i_k}(|A_{kj}|) \geqslant -\|A_{kj}\|_\infty, \quad i_k \in M_k, \ k \neq j,$$

we conclude that

$$A^{(i_1,\dots,i_n)} \geqslant \widetilde{N}(A) \quad \text{for all } i_k \in M_k, \ k = 1, \dots, n. \tag{4.15}$$

Thus, it may happen that A is a PM-matrix, but the matrix $\widetilde{N}(A)$ is not a nonsingular M-matrix. Furthermore, under the assumption that $\widetilde{N}(A)$ is a nonsingular M-matrix, from (4.15) it follows (e.g., see [4, p. 131]) that

$$\widetilde{N}(A)^{-1} \geqslant (A^{(i_1,\dots,i_n)})^{-1} \quad \text{for all } i_k \in M_k, \ k = 1, \dots, n, \tag{4.16}$$

and, consequently,

$$\|\widetilde{N}(A)^{-1}\|_\infty \geqslant \max_{i_1,\dots,i_n} \|(A^{(i_1,\dots,i_n)})^{-1}\|_\infty. \tag{4.17}$$

Thus, in this case, the bound (4.12) is not necessarily applicable and is no better than the upper bound of Theorem 2.1. Furthermore, as is not difficult to realize, inequality (4.17) may hold strictly. For instance, it is strict for the matrix

$$A = \begin{bmatrix} 3 & 0 & -2 & -1 \\ 0 & 3 & -1 & -\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha > 1,$$

which is a PM-matrix with respect to the partitioning $\{1, 2, 3, 4\} = \{1, 2\} \cup \{3\} \cup \{4\}$. Indeed, for this partitioning we have

$$\widetilde{N}(A) = \begin{bmatrix} 3 & -2 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^{(1,3,4)} = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^{(2,3,4)} = \begin{bmatrix} 3 & -1 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

whence

$$\widetilde{N}(A)^{-1} = \begin{bmatrix} 1/3 & 2/3 & \alpha/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(A^{(1,3,4)})^{-1} = \begin{bmatrix} 1/3 & 2/3 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (A^{(2,3,4)})^{-1} = \begin{bmatrix} 1/3 & 1/3 & \alpha/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} \|\widetilde{N}(A)^{-1}\|_\infty &= 1 + \alpha/3 > \max\{1 + 1/3, 2/3 + \alpha/3\} \\ &= \max\{\|(A^{(1,3,4)})^{-1}\|_\infty, \|(A^{(2,3,4)})^{-1}\|_\infty\}. \end{aligned}$$

The above examples demonstrate that the bound (4.12) is in general incomparable with the upper bounds of Theorems 2.1 and 3.1.

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