Generalizing homotopy analysis method to solve Lotka–Volterra equation

L. Zou, Z. Zong, G.H. Dong

Abstract

In the paper, we generalized the homotopy analysis method to solve Lotka–Volterra equation. As a simple but typical example, it illustrates the validity and the great potential of the homotopy analysis method to solve differential difference equations. Comparisons are made between the results of the proposed method and exact solutions. The results reveal that the proposed method is valid for the Lotka–Volterra equation.

Keywords:
Homotopy analysis method
Differential difference equation
Homotopy–Padé technique
Explicit analytic solution
Lotka–Volterra equation

1. Introduction

Based on the homotopy in topology, Liao [1] proposed a kind of analytic technique, namely the homotopy analysis method (HAM) [2–7]. The validity of the HAM is independent of whether or not nonlinear problems under consideration contain “small parameters”. Thus, the HAM is an attractive method for nonlinear problems. T. Hayat and M. Sajid successfully solve the problem of a magneto-hydrodynamic boundary layer flow of an upper-convected Maxwell fluid by homotopy analysis method [8]. C. Wang solve the nonlinear mKdV equation with the homotopy analysis method [9], and so on. However, the applications of HAM are restricted to integral differential equations [10–13]. So far, HAM have not been applied to nonlinear differential difference equations (DDEs) directly. Here, we generalize the method to solve Lotka–Volterra equation.

In recent years, considerable interest in differential difference equations has been stimulated due to their numerous applications in the areas of mathematical ecology, physics and engineering. The DDEs play an important role in modeling complicated physical phenomena such as particle vibrations in lattices, current flow in electrical networks, and pulses in biological chains. The solutions of these DDEs can provide numerical simulations of nonlinear partial differential equations, queueing problems, and discretizations in solid state and quantum physics. Since the work of Fermi, Pasta and Ulam in the 1950s [14], there were quite a number of research works developed during the last decades on DDEs. For instance, Levi and his co-workers analyzed the condition for existence of higher symmetries for a class of DDEs [15,16], Yamilov and his co-workers [17,18] made outstanding contribution to the classification of DDEs, integrability tests and connections between integrable PDEs and DDEs [19,20] and a lot of works were developed to analyze the properties of solutions of DDEs [21–26].

The aim of this paper is to apply the HAM to consider the explicit analytic solution of the Lotka–Volterra equation.
Consider the Lotka–Volterra equation:

\[ u_n' = u_0(u_{n-1} - u_{n+1}) + u_n(u_{n-2} - u_{n+2}) \]  

(1)

with the initial condition

\[ u_n(0) = n \]  

(2)

whose exact solution is

\[ u_n(t) = \frac{n}{1 + 6t}. \]  

(3)

The paper has been organized as follows. In Section 2, we extended homotopy analysis method to solve nonlinear differential difference equations with initial conditions, a brief outline of differential difference equation-homology analysis method (DDE-HAM) is presented. In Section 3, we apply DDE-HAM to Lotka–Volterra equation and obtain an explicit analytic solution for the discussed problem. In Section 4, a brief analysis of the obtained results is given. A short summary and discussion are given in the final section.

2. Basic idea of differential difference equation-homotopy analysis method

The traditional homotopy analysis method is applied to this kind of nonlinear differential equations, like,

\[ \mathcal{N}(u(t)) = 0 \]  

(4)

where \(\mathcal{N}\) is a nonlinear operator, \(t\) denotes an independent variable, \(u(t)\) is an unknown function, respectively. While in this paper, we consider the following differential difference equation

\[ \mathcal{N}(u_n(t), u_{n+1}(t), u_{n-1}(t), u_{n+2}(t), u_{n-2}(t), \ldots) = 0 \]  

(5)

where \(\mathcal{N}\) is a nonlinear differential operator, \(n\) and \(t\) are independent variables, and \(u_n(t)\) is an unknown function. For simplicity, boundary or initial conditions are not considered here. Different from integral differential equation, differential difference equation is semi-discretized with the discrete variable \(n\). So we cannot apply the traditional homotopy analysis method to the DDEs directly. We make some changes on HAM, and generalize the HAM to solve DDEs. Based on the constructed zero-order deformation equation by Liao [5], we give the following zero-order deformation equation

\[ (1 - q)\mathcal{N}[\phi_n(t; q) - u_{n,0}(t)] = qhH_n(t)\mathcal{N}[\phi_0(t; q), \phi_{n+1}(t; q), \phi_{n-1}(t; q), \ldots] \]  

(6)

where \(q \in [0, 1]\) is the embedding parameter, \(h\) is a nonzero auxiliary parameter, \(H_n(t)\) is a nonzero auxiliary function, \(\mathcal{N}\) is an auxiliary linear operator, \(u_{n,0}(t)\) is an initial guess of \(u_n(t)\), \(\phi_0(t; q)\) is a unknown function on independent variables \(n, t, q\). It is important that one has great freedom to choose auxiliary parameter \(h\) in differential difference equation-homotopy analysis method (DDE-HAM). When \(q = 0\) and \(q = 1\), we have from the zero-order deformation equation (5) that \(\phi_0(t; 0) = u_{n,0}(t)\) and \(\phi_0(t; 1) = u_n(t)\).

Thus, as \(q\) increases from 0 to 1, the solution \(\phi_n(t; q)\) varies from the initial guess \(u_{n,0}(t)\) to the solution \(u_n(t)\). Defining

\[ u_{n,m}(t) = \frac{1}{m!} \frac{\partial^m \phi_n(t; q)}{\partial q^m} \bigg|_{q=0} \]  

(7)

and expanding \(\phi_n(t; q)\) in Taylor series with respect to the embedding parameter \(q\), we have

\[ \phi_n(t; q) = \phi_n(t; 0) + \sum_{m=1}^{+\infty} u_{n,m}(t)q^m. \]  

(8)

Namely,

\[ \phi_n(t; q) = u_{n,0}(t) + \sum_{m=1}^{+\infty} u_{n,m}(t)q^m \]  

(9a)

\[ \phi_{n+k}(t; q) = u_{n+k,0}(t) + \sum_{m=1}^{+\infty} u_{n+k,m}(t)q^m, \quad k \in N \]  

(9b)

\[ \phi_{n-k}(t; q) = u_{n-k,0}(t) + \sum_{m=1}^{+\infty} u_{n-k,m}(t)q^m, \quad k \in N. \]  

(9c)
If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$ and the auxiliary function $H_n(t)$ are properly chosen, the series equation (7) converges at $q = 1$, one has

$$u_n(t) = u_{n,0}(t) + \sum_{m=1}^{+\infty} a_{n,m} t^m$$  \hspace{1cm} (10a)

$$u_{n+k}(t) = u_{n+k,0}(t) + \sum_{m=1}^{+\infty} a_{n+k,m}(t), \quad k \in N$$  \hspace{1cm} (10b)

$$u_{n-k}(t) = u_{n-k,0}(t) + \sum_{m=1}^{+\infty} a_{n-k,m}(t), \quad k \in N.$$  \hspace{1cm} (10c)

Define the vector

$$\vec{u}_{n,m}(t) = \{u_{n,0}(t), u_{n,1}(t), \ldots, u_{n,m}(t)\} \quad \text{where} \quad n \in N, m \in N. \hspace{1cm} (11a)$$

$$\vec{u}_{n+k,m}(t) = \{u_{n+k,0}(t), u_{n+k,1}(t), \ldots, u_{n+k,m}(t)\} \quad \text{where} \quad n \in N, m \in N, k \in N. \hspace{1cm} (11b)$$

$$\vec{u}_{n-k,m}(t) = \{u_{n-k,0}(t), u_{n-k,1}(t), \ldots, u_{n-k,m}(t)\} \quad \text{where} \quad n \in N, m \in N, k \in N. \hspace{1cm} (11c)$$

Differentiating the zero-order deformation equation (5) $m$ times with respect to $q$, and finally dividing by $m!$, we have the $m$th-order deformation equation

$$\mathcal{L}[u_{n,m}(t) - \chi_{n,m} u_{n,m-1}(t)] = hH_n(t)\mathcal{D}_m(\vec{u}_{n,m-1}(t), \vec{u}_{n+1,m-1}(t), \vec{u}_{n-1,m-1}(t), \ldots)$$  \hspace{1cm} (12)

and

$$\chi_{n,m} = \begin{cases} 0 & \text{when} \quad m \leq 1, \\ 1 & \text{when} \quad m > 1. \end{cases} \hspace{1cm} (13)$$

The $m$th-order deformation equation (11) is linear and thus can be easily solved, especially by means of symbolic computation software Maple, Mathematica and so on.

3. Application to the Lotka–Volterra equation

To verify the validity and the potential of DDE-HAM in solving differential difference equations, we apply it to Eq. (1) with the initial condition (2).

It is straightforward to use the set of base functions

$$\{t^m | m = 0, 1, 2, 3, \ldots\},$$

to represent $u_n(t)$, i.e.

$$u_n(t) = \sum_{m=1}^{+\infty} a_{n,m} t^m.$$  \hspace{1cm} (14)

According to the rule of solution expression [5], proposed by Liao, we choose the initial approximation

$$u_{n,0}(t) = n + t$$  \hspace{1cm} (15)

when $t = 0$, the initial guess $u_{n,0}(0) = n$, which satisfies the initial condition (2).

Then we choose the auxiliary linear operator

$$\mathcal{L}[\phi_n(t; q)] = \frac{\partial \phi_n(t; q)}{\partial t}$$  \hspace{1cm} (16)

possessing the property

$$\mathcal{L}(C_1) = 0$$  \hspace{1cm} (17)

where $C_1$ is an integral constant to be determined by initial condition. Furthermore, Eq. (1) suggests to define the nonlinear operator

$$\mathcal{N}[\phi_n(t; q), \phi_{n+1}(t; q), \phi_{n-1}(t; q), \ldots]$$

$$= \frac{\partial \phi_n(t; q)}{\partial t} - \phi_n(t; q)\phi_{n-1}(t; q) + \phi_n(t; q)\phi_{n+1}(t; q) - \phi_n(t; q)\phi_{n-2}(t; q) + \phi_n(t; q)\phi_{n+2}(t; q). \hspace{1cm} (18)$$
Let \( q \in [0, 1] \) denote an embedding parameter, \( h \neq 0 \) an auxiliary parameter, and \( H_n(t) \neq 0 \) an auxiliary function. Using the above definitions, we construct the zero-order deformation equation in the similar way as (5), with the initial conditions \( \phi_0(0, q) = n \). Obviously, when \( q = 0 \) and \( q = 1 \),

\[
\phi_0(t, 0) = u_{n,0}(t), \quad \phi_0(t, 1) = u_n(t). \tag{19}
\]

According to (11) and (12), we get the \( m \)-th order deformation equation, especially,

\[
R_{n,m}(u_{n,m-1}(t), u_{n+1,m-1}(t), u_{n-1,m-1}(t), \ldots) = \frac{\partial u_{n,m-1}(t)}{\partial t} - \sum_{j=0}^{m-1} u_{n,j}(t)u_{n-1,m-1-j}(t) + \sum_{j=0}^{m-1} u_{n,j}(t)u_{n+1,m-1-j}(t) - \sum_{j=0}^{m-1} u_{n,j}(t)u_{n-2,m-1-j}(t) + \sum_{j=0}^{m-1} u_{n,j}(t)u_{n+2,m-1-j}(t). \tag{20}
\]

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [5], the corresponding auxiliary function can be determined uniquely

\[
H_n(t) = 1. \tag{21}
\]

It should be emphasized that \( u_{n,m}(t), \ (m \geq 1) \) is governed by the linear equation (21) with the linear initial conditions (22). We get all the solutions as follows:

\[
\begin{align*}
  u_{n,0}(t) &= n + t \\
  u_{n,1}(t) &= h(n + 1)t + 3ht^2 \\
  u_{n,2}(t) &= (1 + 6n)h(1 + h)t + (36h^2n + 6h^2 + 3h)t^2 + 18ht^3 \\
  \cdots \cdots \cdots 
\end{align*} \tag{22}
\]

Then the solution expression (14) can be written in an accurate form as follows

\[
  u_n(t) = \sum_{m=1}^{+\infty} a_{n,m}(h)t^m. \tag{23}
\]

Thus, DDE-HAM provides us with a family of solution expression in the auxiliary parameter \( h \). The convergence region of solution series depend upon the value of \( h \).

To increase the accuracy and convergence of the solution, Liao [5] has applied the homotopy analysis method to provide an analytic solution for the classical problem of nonlinear progressive waves in deep water and developed a new technique, namely the Homotopy–Padé technique.

4. Results and analysis

It is important to ensure that the solution series (22) are convergent. Note that the solution series (22) contain the auxiliary parameter \( h \), which we can choose properly by plotting the so-called \( h \)-curves to ensure solution series converge, as suggested by Liao [5]. In this way, we choose a valid value of \( h \). Here we choose the auxiliary parameter \( h = -1 \) by \( h \)-curve. Comparisons are made between the analytic 3rd-order, 5th-order approximation, 20th-order approximation and exact solution, when \( n = 10 \), as shown in Fig. 1. It can be seen from Fig. 1 that the present method could give good results if the order of approximation is high enough. A 20th-order approximation agrees well with the exact solution.

We applied Homotopy–Padé (H–P) approximation to our solution, we get [1, 1] H–P approximation:

\[
  U_n(t, -1) = \frac{n}{1 + 6t} \tag{24}
\]

when \( h = -1 \). Obviously, this is an exact solution of Eq. (1) with the initial condition \( u_n(0) = n \).

Homotopy–Padé approximation is an effective method to accelerate the convergence of the result and enlarge the convergence field. The application of the Homotopy–Padé approximation to the solution series achieves a high convergence rate over a considerably large convergence region.

5. Discussions and conclusions

In this paper, we successfully overcome the discrete variable \( n \) in nonlinear differential difference equation, and apply HAM to solve Lotka–Volterra equation. Comparison of the result obtained by the present method with the exact solution reveals that the present method is very effective and convenient for Lotka–Volterra equation.
References


Fig. 1. Comparison of the exact solution with the DDE-HAM solution when \(n = 10\) and \(h = -1\).