The Existence of Equilibria for Noncompact Generalized Games

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Abstract—The purpose of this paper is to establish general existence of equilibria for noncompact generalized games (respectively, noncompact abstract economics) under general setting of noncompact conditions and in which the L-majorized preference mappings may not have lower semicontinuity, and constraint correspondences are only lower or upper semicontinuous. In our model, strategic (respectively, commodity) spaces are not compact, the set of players (respectively, agents) are countable or uncountable, and underlying spaces are either finite- or infinite-dimensional locally topological vector spaces. Our results might be regarded as a unified theory for the corresponding results in the existing literatures in the study of generalized games (respectively, abstract economics) theory.

Keywords—Generalized game, Abstract economics, L-majorized mapping, (WC) condition.

1. INTRODUCTION

The aim of this paper is to establish the existence of equilibria for noncompact generalized games (respectively, noncompact abstract economy) under general setting of noncompact conditions and in which strategic (respectively, commodity) spaces are noncompact convex sets in locally topological vector spaces. By using so-called approximate technique due to Yuan [1], Yuan and Tan [2], Tulcea [3,4], and Chang [5], we establish general existence results of equilibria for the noncompact model below in which preference correspondences may not have lower semicontinuity and constraint correspondences are only lower or upper semicontinuous; in particular, the underlying setting of noncompact generalized games is general enough to cover corresponding noncompact generalized game models in the existing literatures as special cases. In order to exhibit clearly the idea and method we used in this paper, we only deal with so-called L-majorized mappings whose roots go back to Gale and Mas-Colell's pioneering work (e.g., see [6]) which has been carried forward by Shafer and Sonnenschein [7] and further developed by Borglin and Keiding [8].

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A generalized game (respectively, abstract economy) is a family of quadruples \( \Gamma = (X_i; A_i, B_i; P_i)_{i \in I} \) where \( I \) is a (possible uncountable) set of players (respectively, agents) such that for each \( i \in I \), \( X_i \) is a nonempty strategic (respectively, commodity) subset of a topological vector space and \( A_i, B_i : X = \Pi_{j \in I} X_j \to 2^{X_i} \) are constraint correspondences and \( P_i : X \to 2^{X_i} \) is a preference correspondence. When \( I = \{1, \ldots, N\} \), where \( N \) is a positive integer, \( \Gamma = (X_i; A_i, B_i; P_i)_{i \in I} \) is also called an \( N \)-person game. An equilibrium of \( \Gamma \) is a point \( \hat{x} \in X \) such that for each \( i \in I \), \( \hat{x}_i = \pi_i(\hat{x}) \in B_i(\hat{x}) \) and \( A_i(\hat{x}_i) \cap P_i(\hat{x}_i) = \emptyset \).

Since the existence of equilibria in an abstract economy with compact strategy sets in \( \mathbb{R}^n \) was proved in a seminal paper of Debreu [9] (see also [10] for more details) which is classical Arrow-Debreu-McKenzie model of exchange economies under the perfect information competition, there have been many generalizations of Debreu’s theorem. As a result, a number of existence theorems of equilibria for generalized games (respectively, abstract economics) have been established. However, we note that almost all of those results so far established in the literature need assumptions such as either preference mappings and/or constraint mappings are lower and/or upper semicontinuous, and the set of players (respectively, agents) are finite, or only countably infinite and strategic (respectively, commodity) spaces also require to be compact (or even noncompact, but the underlying setting of noncompact conditions are much stronger than the one we study in this paper).

Motivated by those shortages of results in the existing literatures for the study of generalized games and the question explained above, it is our aim in this paper to investigate the existence of equilibria for noncompact generalized games (respectively, noncompact abstract economics) and to solve above question involving \( L \)-majorized mappings which have been extensively used in the existing literatures, e.g., see [2,3,8,11–16]. In our models, strategic (respectively, commodity) spaces may not be compact, we allow that the set of players (respectively, agents) may be infinite and even uncountable and underlying spaces are infinite dimensional locally topological vector spaces. Our results might be regarded as a unified theory for corresponding results in the study of existence of equilibria for noncompact generalized games (respectively, abstract economies) in the literatures under very general noncompact setting.

Let \( X \) be a topological space and \( Y \) a nonempty subset of a vector space, and \( \phi : X \to 2^E \) a set-valued mapping. Then

1. \( \phi \) is said to be of class \( L \) if for each \( x \in X \), \( \co \phi(x) \subset Y \), \( x \notin \co \phi(x) \), and the set \( \phi^{-1}(y) := \{ x \in X : y \in \phi(x) \} \) is compactly open in \( X \) for each \( y \in Y \);
2. a mapping \( \phi_x : X \to 2^Y \) is said to be an \( L_x \)-majorant of \( \phi \) at \( x \in X \) if there exists an open neighborhood \( N_x \) of \( x \) such that
   a. for each \( z \in N_x \), \( \phi(z) \subset \phi_x(z) \), and \( z \notin \co \phi_x(z) \);
   b. for each \( z \in X \), \( \co \phi_x(z) \subset Y \); and
   c. for each \( y \in Y \), the set \( \phi_x^{-1}(y) \) is compactly open in \( X \); and
3. \( \phi \) is said to be \( L \)-majorized if for each \( x \in X \) with \( \phi(x) \neq \emptyset \), there exists \( \phi_x \)-majorant of \( \phi \) at \( x \in X \).

By the definition, it is clear that each class of \( L \) mapping is \( L \)-majorized, but the converse is not true in general.

Let \( X = \Pi_{i \in I} X_i \) be a product of nonempty convex sets \( X_i \) in a Hausdorff topological vector space \( V_i \) for each \( i \in I \), where \( I \) is any (possibly countable or uncountable) index set, and \( A_i : X \to 2^{X_i} \) is a set-valued mapping for each \( i \in I \). We shall denote by \( M_S(X, X_i)_{i \in I} \) (respectively, \( L_S(X, X_i)_{i \in I} \)) the set of all families of \( \{A_i\}_{i \in I} \) such that for each \( i \in I \), \( A_i : X \to X_i \) is \( L \)-majorized (respectively, is of class \( L \)). Moreover, if the index set \( I \) is a singleton, the mapping \( A \in M_S(X, X) \) (respectively, \( A \) in \( L(X, X) \)) is said to be \( L \)-majorized (respectively, of class \( L \)) from \( X \) to \( Y \).

Throughout this paper, all topological spaces are assumed to be Hausdorff and all notions are the same as those used in [2] unless otherwise specified.
2. THE EXISTENCE OF EQUILIBRIA FOR QUALITATIVE GAMES

We first recall the following result which is a special case of Lemma 2.1 of [2].

**Lemma 2.1.** Let $X$ be a regular topological space and $Y$ be a nonempty subset of a vector space $E$. Let $0 : X \to E$ and $P : X \to 2^Y$ be $L_{\theta,C}$-majorized. If the support set $B$ of the mapping $P$ is open and paracompact, then there exists an $L_{\theta,C}$-class mapping $\phi : X \to 2^Y$ such that $P(x) \subseteq \phi(x)$ for each $x \in X$.

In order to establish the existence of maximal elements for the family of $L$-majorized mappings in which domains are not compact, we now recall the following notions.

Let $X = \prod_{i \in I} X_i$ be a product space, where $X_i$ is a nonempty convex subset of a Hausdorff topological vector space for each $i \in I$ and $A_i : X \to 2^{X_i}$ is a set-valued mapping. Then $\{A_i\}_{i \in I}$ is said to be a KF family if the following assumptions are satisfied:

(i) for each $i \in I$, $A_i(x)$ is convex (may be empty) for each $x \in X$;
(ii) for each $x \in X$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$; and
(iii) for each $i \in I$, the set $A_i^{-1}(y_i)$ is compactly open in $X$ for each $y_i \in X_i$.

We shall denote by $\text{KF}(X, X_i)_{i \in I}$ the set of all KF families. For each $i \in I$, if $x_i \notin A_i(x)$ for all $x \in X$, then the family $\{A_i\}_{i \in I} \in M_S(X, X_i)_{i \in I}$. If the case $I$ is a singleton, the mapping $A \in \text{KF}(X, X)$ is also called a KF mapping.

**Theorem 2.2.** Let $X = \prod_{i \in I} X_i$ be a product space, where each $i \in I$, $X_i$ is a nonempty compact convex subset of a topological vector space. Suppose that the family $\{A_i\}_{i \in I} \in M_S(X, X_i)_{i \in I}$ is such that $\bigcup_{i \in I} \{x \in X : A_i(x) \neq \emptyset\} = \bigcup_{i \in I} \text{int}\{x \in X : A_i(x) \neq \emptyset\}$. Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) = \emptyset$ for each $i \in I$.

**Proof.** It is a special case of Theorem 3.3 of [2].

**Lemma 2.3.** Let $X$ be a compact Hausdorff topological space and $\{A_i\}_{i \in I} \in \text{KF}(X, X_i)_{i \in I}$. Then there is a subset $D = \prod_{i \in I} D_i$ of $X$ such that for each $x \in X$, there exists $i \in I$ such that $A_i(x) \cap D_i \neq \emptyset$. Moreover, for each $i \in I$, $D_i$ is a polytope and all of those polytope, except a finite number, consist of a single point.

**Proof.** By following Deguire and Lassonde [17], for each $x \in X$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$, so that $X = \bigcup_{i \in I} \bigcup_{y_i \in X_i} A_i^{-1}(y_i)$. Since $X$ is compact, there exists a finite subset $J$ of $I$ such that for each $j \in J$, there exists a finite subset $J_j = \{y_j^1, \ldots, y_j^{m_j}\}$ of $X_j$ with $X = \bigcup_{j \in J} \bigcup_{i = 1}^{m_j} A_j^{-1}(y_j^i)$. Take any fixed $y^0 = \{y_i^0\}_{i \in I} \in X$ and define

$$D_j = \begin{cases} \text{co}(J_j), & \text{if } j \in J, \\ \{y_j^0\}, & \text{if } j \notin J. \end{cases}$$

Then $D = \prod_{i \in I} D_i$ is the required set.

Now we have the following existence theorem of maximal elements for a family with noncompact domains.

**Theorem 2.4.** Let $X = \prod_{i \in I} X_i$ be a product space and $I$ any index set, where $X_i$ is a nonempty convex subset of a topological vector space for each $i \in I$ and $\{A_i\}_{i \in I} \in L_S(X, X_i)_{i \in I}$. Suppose there exist a nonempty compact subset $K$ (not necessarily convex) of $X$ and a nonempty compact convex subset $C_i$ of $X_i$ for each $i \in I$ with the property that for each $x \in X \setminus K$, there exists $i \in I$ such that $A_i(x) \cap C_i \neq \emptyset$. Then there exists $x \in K$ such that $A_i(x) = \emptyset$ for all $i \in I$.

**Proof.** By following Yuan [1], suppose it is not true. Then for each $x \in K$, there exists $i \in I$ such that $A_i(x) \neq \emptyset$. By Lemma 2.3, there exists a subset $D = \prod_{i \in I} D_i$ such that for each $x \in K$, there exists $i \in I$ such that $A_i(x) \cap D_i \neq \emptyset$, where $D_i$ is a nonempty polytope (thus, it is nonempty compact and convex) for each $i \in I$. 
It is clear that $D$ is a nonempty compact subset of $X$. For each $i \in I$, we set $H_i = \text{co}(C_i \cup D_i)$, then $H_i$ is a nonempty compact and convex subset of $X_i$. If $x \in X \setminus K$, then there exists $i \in I$ such that $\emptyset \neq A_i(x) \cap C_i \subset A_i(x) \cap H_i$ by our hypothesis above. If $x \in K$, by our argument above, there exists $i \in I$ such that $\emptyset \neq A_i(x) \cap D_i \subset A_i(x) \cap H_i$. Now for each $i \in I$, define a set-valued mapping $A'_i : X \to 2^{H_i}$ by $A'_i(x) = A_i(x) \cap H_i$ for each $x \in X$. Thus, in any case, for each $x \in X$, there exists $i \in I$ such that $A_i(x) \cap H_i \neq \emptyset$. (*)

Now set $H := \Pi_{i \in I} H_i$ and $X_i := H_i$ for each $i \in I$ in Theorem 2.2. Then the family $\{A'_i\}_{i \in I}$ satisfies all hypotheses of Theorem 2.2. By Theorem 2.2, there exists $x_0 \in H$ such that $A'_i(x_0) = \emptyset$ for all $i \in I$, which contradicts (*). Therefore, there must exist $x \in K$ such that $A_i(x) = \emptyset$ for all $i \in I$ and we complete the proof.

Now as applications of Theorem 2.1 and Theorem 2.4, we have the following existence theorem of maximal elements for $L$-majorized family in topological vector spaces under noncompact setting.

**Theorem 2.5.** Let $X = \Pi_{i \in I} X_i$ be a product space and $I$ any index set, where $X_i$ is a nonempty convex subset of a topological vector space for each $i \in I$. Suppose that the family $\{A_i\}_{i \in I} \in \mathcal{M}_S(X, X)$ is such that the set $\{x \in X : A_i(x) \neq \emptyset\}$ is open and paracompact for each $i \in I$. Moreover, there exists a nonempty compact subset $K$ (not necessarily convex) of $X$ and a nonempty compact convex subset $C_i$ of $X_i$ for each $i \in I$ with the property that for each $x \in X \setminus K$, there exists $j \in I$ (not necessarily for all $j \in I$) such that $A_j(x) \cap C_j \neq \emptyset$. Then there exists $x \in K$ such that $A_i(x) = \emptyset$ for all $i \in I$.

**Proof.** For each $i \in I$, by the assumption of Theorem 2.5, it follows that by Theorem 2.1, there exists a set-valued mapping $B_i : X \to 2^{H_i}$, which is of class $L$ such that $A_i(x) \subset B_i(x)$ for each $x \in X$. Hence, the family $\{B_i\}_{i \in I}$ satisfies all hypotheses of Theorem 2.4. By Theorem 2.4, there exists $x \in X$ such that $B_i(x) = \emptyset$ for each $i \in I$, so that $A_i(x) = \emptyset$ for each $i \in I$ as $A_i(x) \subset B_i(x)$.

### 3. THE EXISTENCE OF EQUILIBRIA

In this section, we shall study general existence of equilibria for noncompact generalized games with noncompact hypothesis which is weaker than those noncompact condition in existing literatures. In order to do so, we first state the following notion for so-called **Noncompact Condition (WC)**.

Let $X = \Pi_{i \in I} X_i$ be a product space and $I$ any possibly countable or uncountable set, where $X_i$ is a nonempty convex subset of a topological vector space for each $i \in I$. Suppose that $\{A_i\}_{i \in I}$ is a family of mappings such that for each $i \in I$, $A_i$ is a set-valued correspondence from $X$ to $X_i$. Then the family $\{A_i\}_{i \in I}$ is said to satisfy the condition (WC) if there exist a nonempty compact subset $K$ (not necessarily convex) of $X$ and a nonempty compact convex subset $C_i$ of $X_i$ for each $i \in I$ with the property that for each $x \in X \setminus K$, there exists $j \in I$ (not necessarily for all $j \in I$) such that $A_j(x) \cap C_j \neq \emptyset$. Then it is clear that above weaker noncompact (WC) condition is much weaker than corresponding noncompact conditions for generalized games used by Ding and Tan [13], Tan and Yuan [18], Tarafdar [19], Yuan and Tan [2], and related references therein. We also recall that a topological space $X$ is said to be a perfectly normal space if $X$ is a normal space and every closed subset of $X$ is a $G_\delta$-set (recall that a $G_\delta$-set is countable intersections of open sets) (e.g., see [20, p. 45]. Clearly, a normal space $X$ is perfectly normal if and only if every open subset of $X$ is an $F_{\sigma}$-set.

**Theorem 3.1.** Let $G = (X; A_i, B_i; P_i)_{i \in I}$ be a generalized game such that $X = \Pi_{i \in I} X_i$ is perfectly normal and paracompact. Suppose that the following conditions are satisfied for each $i \in I$:
(a) $X_i$ is a nonempty convex subset of a locally convex topological vector space $E_i$;
(b) $A_i : X \to 2^{X_i}$ is lower semicontinuous such that for each $x \in X$, $A_i(x)$ is nonempty and $\text{co } A_i(x) \subset B_i(x)$;
(c) the mapping $A_i \cap P_i$ is $L$-majorized;
(d) the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in $X$;
(e) the mapping family $\{A_i \cap P_i\}_{i \in I}$ satisfies the weaker noncompact condition (WC).

Then $G$ has an equilibrium point in $K$, i.e., there exists a point $\hat{x} = (\hat{x}_i)_{i \in I} \in K$ such that for each $i \in I$, $\hat{x}_i \in B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

**PROOF.** We first prove that for given any $V = \Pi_{i \in I} V_i$ where for each $i \in I$, $V_i$ is an open convex neighborhood of zero in $E_i$, then the generalized game $\Gamma_V = (X_i; A_i, B_i; P_i)_{i \in I}$ has an equilibrium point in $K$; i.e., there exists a point $x_V = (x_V)_i \in K$ such that $x_V \in B_V(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$, where $B_V(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$ and each $i \in I$.

Let $V = \Pi_{i \in I} V_i$ be given where for each $i \in I$, $V_i$ is an open convex neighborhood of zero in $E_i$. Fix any $i \in I$ and define $A_{V_i}, B_{V_i} : X \to 2^{X_i}$ by $A_{V_i}(x) = (\text{co } A_i(x) + V_i) \cap X_i$ and $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. By (b), $A_i$ is lower semicontinuous so that $\text{co } A_i$ is also lower semicontinuous. Hence $A_{V_i}$ is lower semicontinuous. It follows from Lemma 4.1 of [5, p. 244] or from [3, p. 7] that $A_{V_i}$ has an open graph in $X \times X$. Now let $F_{V_i} = \{x \in X : x_i \notin B_{V_i}(x)\}$.

Then $F_{V_i}$ is open in $X$. Define a set-valued mapping $Q_{V_i} : X \to 2^{X_i}$ by

$$Q_{V_i}(x) = \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_{V_i}, \\ A_i(x), & \text{if } x \in F_{V_i}. \end{cases}$$

By following exactly the same idea used in the proof of Theorem 3.3 of [2], we can show that the qualitative game $T = (X_i, Q_{V_i})_{i \in I}$ satisfies all conditions of Theorem 2.5. By Theorem 2.5, there exists a point $x_T = (x_T)_i \in K$ such that $Q_{V_i}(x_T) = \emptyset$ for all $i \in I$. Since for each $i \in I$, $A_i(x)$ is nonempty, we must have $x_T \in B_T(x_T)$ and $A_i(x_T) \cap P_i(x_T) = \emptyset$.

Second, for each $i \in I$, let $B_i$ be the collection of all open convex neighborhoods of zero in $E_i$ and $B = \Pi_{i \in I} B_i$. Given any $V \in B$, let $V = \Pi_{j \in J} V_j$, where $V_j \in B_j$ for each $j \in I$. By the result we just proved above, there exists a $\tilde{x}_V \in K$ such that $\tilde{x}_V \in B_V(\tilde{x}_V)$ and $A_i(\tilde{x}_V) \cap P_i(\tilde{x}_V) = \emptyset$ for each $i \in I$, where $B_V(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. It follows that the set $Q_V := \{x : x_j \notin B_{V_j}(x)\}$ is a nonempty closed subset of $K$ by (d). Now we want to prove that the family $\{Q_V : V \in B\}$ has the finite intersection property. Let $\{V_1, \ldots, V_n\}$ be any finite subset of $B$. For each $i = 1, \ldots, n$, let $V_i = \Pi_{j \in J} V_{ij}$, where $V_{ij} \in B_j$ for each $j \in I$; let $V = \Pi_{j \in J} (\bigcap_{i=1}^n V_{ij})$, then $Q_V \neq \emptyset$. Clearly, $Q_V \subset \bigcap_{i=1}^n Q_{V_i}$ so that $\bigcap_{i=1}^n Q_{V_i} \neq \emptyset$.

Therefore, the family $\{Q_V : V \in B\}$ has the finite intersection property. Since $K$ is compact, $\bigcap_{V \in B} Q_V \neq \emptyset$. Now take any $\tilde{x} \in \bigcap_{V \in B} Q_V$, then for each $i \in I$, $\tilde{x}_i \in B_{V_i}(\tilde{x})$ for each $V_i \in B_i$ and $A_i(\tilde{x}) \cap P_i(\tilde{x}) = \emptyset$. By Lemma 5.3 of [18], it follows that for each $i \in I$, $\tilde{x}_i \in B_i(\tilde{x})$.

As a special case of Theorem 3.1, we have the following corollary.

**COROLLARY 3.2.** Let $G = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized game, where the set $I$ of players is countable. Suppose that the following conditions are satisfied for each $i \in I$:

(a) the strategic set $X_i$ is a nonempty metrizable convex subset of a locally convex topological vector space;
(b) the mapping $A_i$ is lower semicontinuous and for each $x \in X$, $A_i(x)$ is nonempty and $\text{co } A_i(x) \subset B_i(x)$;
(c) the mapping $A_i \cap P_i$ is $L$-majorized;
(d) the set $E^i = \{x \in X : A_i \cap P_i(x) \neq \emptyset\}$ is open in $X$;
(e) the mapping family $\{A_i \cap P_i\}_{i \in I}$ satisfies the weaker noncompact condition (WC).

Then $G$ has an equilibrium point in $X$.

In what follows, we shall establish the existence of equilibria for noncompact generalized games in which preference mappings are not necessarily lower semicontinuous and constraint mappings
are only upper semicontinuous under the weaker noncompact condition (WC). Now we have the following second main result in this paper.

**Theorem 3.3.** Let \( \Gamma = (X_i; F_i; P_i)_{i \in I} \) be a generalized game such that \( X \) is perfectly normal and paracompact and each \( X_i \) has the property (K) (i.e., the convex hull of each subset in \( X_i \) is relatively compact in \( E_i \)). Suppose that the following conditions are satisfied for each \( i \in I \):

(i) the strategic set \( X_i \) is a nonempty closed convex set in a Hausdorff locally convex topological vector space;

(ii) the constraint mapping \( F_i \) is compact and upper semicontinuous with nonempty closed convex values;

(iii) the mapping \( F_i \cap P_i : X \to 2^{X_i} \) is \( L \)-majorized;

(iv) the set \( E_i = \{ x \in X : F_i(x) \cap P_i(x) \neq \emptyset \} \) is open in \( X \);

(v) the mapping family \( \{ F_i \cap P_i \}_{i \in I} \) satisfies the weaker noncompact condition (WC).

Then \( \Gamma \) has an equilibrium \( \hat{x} \in K \), i.e., there exists \( \hat{x} \in K \) such that \( \hat{x} \in F_i(\hat{x}) \) and \( F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset \) for all \( i \in I \).

**Proof.** By Lemma 2.4 of [2, p. 640] (see also [4]), there is a common filtering set \( J \) such that for every \( i \in I \), there exists a family \( (F_{ij})_{j \in J} \) of regular correspondences between \( X \) and \( X_i \), such that both \( (F_{ij})_{j \in J} \) and \( (\overline{F_{ij}})_{j \in J} \) are upper approximating families for \( F_i \). Note that \( E_i \) is paracompact and \( F_i \cap P_i \) is \( L \)-majored in \( E_i \) from \( E_i \) to \( E_i \). As \( X \) is perfectly normal and paracompact, it follows that the set \( E_i \) is also paracompact. By Lemma 2.1, it follows that there exists an \( L \)-class mapping \( \phi_i : E_i \to 2^{X_i} \) such that

(a) \( (F_i \cap P_i)(x) \subset \phi_i(x) \) and \( x_i \notin \phi_i(x) \) for all \( x \in X \); and

(b) \( \phi_i^{-1}(y) \) is compactly open in \( X \) for each \( y \in X_i \).

For each \( j \in J \), define a set-valued mapping \( \phi_{ij} : X \to 2^{X_i} \) by \( \phi_{ij}(x) = F_{ij}(x) \cap \phi_i(x) \) for each \( x \in X \). Note that \( F_{ij} \) is regular, it follows that \( F_{ij} \) has an open graph and thus, the mapping \( \phi_{ij} \) is of class \( L \). Second, for each \( i \in I \), the set \( \{ x \in X : F_{ij}(x) \cap \phi_{ij}(x) \neq \emptyset \} = \bigcup_{y \in X_i} F_{ij}^{-1}(y) \cap \phi_i^{-1}(y) \), which is open in \( X \). Therefore, the game \( \Gamma_j = (X_i; F_{ij}, \overline{F_{ij}}, \phi_{ij})_{i \in I} \) satisfies all hypotheses of Theorem 3.1 above. By Theorem 3.1, it follows that the \( \Gamma_j \) has an equilibrium \( \hat{x}^j \in K \) such that \( F_{ij}(\hat{x}^j) \cap \phi_i(\hat{x}^j) = \emptyset \), and \( \phi_i(\hat{x}^j) \in F_{ij}(\hat{x}^j) \) for all \( i \in I \). Since \( F_i(\hat{x}^j) \subset F_{ij}(\hat{x}^j) \), it follows that \( F_i(\hat{x}^j) \cap \phi_i(\hat{x}^j) = \emptyset \). Therefore, \( \{ \hat{x}_j \}_{j \in J} \subset E_i^c \), where the set \( E_i^c = \{ x \in X : x \notin E_i \} \) is closed in \( X \) by Condition (iv). On the other hand, note that \( \{ \hat{x}_j \}_{j \in J} \) is a net in the compact set \( K \), without loss of generality, we may assume that \( \{ \hat{x}_j \}_{j \in J} \) converges to \( x^* \in K \). Then for each \( i \in I \), \( x_i^* = \lim_{j \in J} \hat{x}_j \). As \( x^* \in E_i^c \) for all \( i \in I \), it follows that \( F_i(x^*) \cap P_i(x^*) = \emptyset \). Since \( \hat{x}^j \) is an equilibrium point of \( \Gamma_j \) and \( F_{ij} \) is regular, for each \( x \in X \), \( \text{cl} F_{ij}(x) = \overline{F_{ij}}(x) \), therefore, \( \hat{x}_j^i \) is an equilibrium point of \( \Gamma_j \) and \( F_{ij} \) is regular, for each \( x \in X \), \( \text{cl} F_{ij}(x) = \overline{F_{ij}}(x) \), therefore, \( \hat{x}_j^i \in \text{cl} B_{ij}(\hat{x}_j^i) = F_{ij}(\hat{x}_j^i) \). As \( \overline{F_{ij}} \) has a closed graph, it follows that \( \{ x^*, x_i^* \} \in \text{Graph} F_{ij} \) for every \( i \in I \). For each \( i \in I \), since \( \text{cl}(F_{ij})_{j \in J} \) is an upper approximation family for \( F_i \), it follows that \( \bigcap_{j \in J} F_{ij}(x) \subset F_i(x) \) for each \( x \in X \) so that \( \{ x^*, x_i^* \} \in \text{Graph} F_i \). Therefore, for each \( i \in I \), \( F_i(x^*) \cap P_i(x^*) = \emptyset \) and \( \pi_i(x^*) \in \overline{F_i}(x^*) \) and this completes the proof.

Note that each metrizable space is perfectly normal and paracompact by Theorem 4.1.13 of [20, p. 254], and as an application of Theorem 3.3, we have the following.

**Corollary 3.4.** Let \( \Gamma = (X_i; F_i; P_i)_{i \in I} \) be a generalized game such that \( X \) is metrizable and each \( X_i \) has the property (K), where \( I \) is countable. Suppose that the following conditions are satisfied for each \( i \in I \):

(i) the strategic set \( X_i \) is a nonempty closed convex set (indeed, it is metrizable) in a Hausdorff locally convex topological vector space;

(ii) the constraint mapping \( F_i \) is compact and upper semicontinuous with nonempty closed convex values;

(iii) the mapping \( F_i \cap P_i \) is \( L \)-majored;

(iv) the set \( E_i = \{ x \in X : F_i(x) \cap P_i(x) \neq \emptyset \} \) is open in \( X \);

(v) the mapping family \( \{ F_i \cap P_i \}_{i \in I} \) satisfies the weaker noncompact condition (WC).
Then $\Gamma$ has an equilibrium $\hat{x}$ in $K$ such that $\hat{x}_i \in F_i(\hat{x})$ and $F_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for all $i \in I$.

REMARK 3.5. The Example A of [2] shows that the conclusions of Theorem 3.3 and Corollary 3.4 do not hold if we withdraw the hypothesis ‘(iv) for each $i \in I$, the set $E_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in $X$’;

REMARK 3.6. We first note that Theorems 3.1 and 3.3 show that Theorems 3.4 and 3.8 of [2] hold without the hypothesis of lower semicontinuity for the preference mappings under our noncompact condition (WC) which is weaker than many those noncompact conditions used in the existing literature. Second, our Corollary 3.4 also shows Corollary 1 of [22] still holds when the set of players (respectively, agents) is uncountable instead of countable and the strategic (respectively, commodity) spaces may not be compact. Thus, our results in this paper include many corresponding existence of equilibria for (even noncompact) generalized games (respectively, abstract economics) in existing literatures as special cases. Finally, we should point out that in order to establish our general theory for the existence of equilibria, we need the hypothesis so-called ‘perfectly normal’ (of course this hypothesis is automatically satisfied when the space $X$ is metrizable), but we do not know whether the conclusions of both Theorem 3.1 and Theorem 3.3 still hold without the hypothesis of ‘perfectly normal paracompactness’ on the set $X$.

REFERENCES