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The Thomas Rotation Formalism Underlying a Nonassociative Group Structure for Relativistic Velocities

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Physics is a major external source of mathematical theories. It is shown here that the Thomas rotation effect of special theory of relativity (STR) gives rise to a "group" structure for the set of relativistically admissible velocities. This group structure turns out to be noncommutative and *nonassociative*. The term *nonassociative group* is justified by two illustrative examples demonstrating that the new concept of the nonassociative group is forced on us by the study of the laws of relativistic velocities.

Thomas rotation is studied in STR as an isolated notion; and the bizarre and counterintuitive noncommutativity and nonassociativity of the relativistic composition of nonparallel admissible velocities is sometimes interpreted as a peculiarity of STR. However, it turns out that the Thomas rotation plays a central role in STR, giving rise to an elegant formalism underlying the noncommutative, nonassociative "group" of relativistically admissible velocities.

To demonstrate the Thomas rotation formalism and the group structure to which it gives rise let $\mathbb{R}_c^3$,

$$\mathbb{R}_c^3 = \left\{ \mathbf{x}\in\mathbb{R}^3 :\ |\mathbf{x}| < c \right\}, \tag{1}$$

be a subset of the Euclidean 3-space $\mathbb{R}^3$ where c is a positive constant representing the speed of light. The elements $\mathbf{x}$ of $\mathbb{R}_c^3$ are known as relativistically *admissible velocities*. The relativistic velocity composition law is given by the equation

$$\mathbf{x}*\mathbf{y} = \frac{\mathbf{x}+\mathbf{y}}{1+\frac{\mathbf{x}\cdot\mathbf{y}}{c^2}} + \frac{1}{c^2}\frac{\gamma_x}{\gamma_x+1}\frac{\mathbf{x}\times(\mathbf{x}\times\mathbf{y})}{1+\frac{\mathbf{x}\cdot\mathbf{y}}{c^2}}, \qquad \mathbf{x},\mathbf{y}\in\mathbb{R}_c^3, \tag{2}$$

where γ_x is the *Lorentz factor,*

$$\gamma_x = \frac{1}{\sqrt{1-(\frac{x}{c})^2}}, \tag{3}$$

associated with the velocity $\mathbf{x}$ whose magnitude is $x = |\mathbf{x}|$, and where $\cdot$ and $\times$ signify the usual dot (scalar) and cross (vector) product between two vectors in $\mathbb{R}^3$. The magnitude of $\mathbf{x}*\mathbf{y}$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$,

$$(\mathbf{x}*\mathbf{y})^2 = \left[\frac{\mathbf{x}+\mathbf{y}}{1+\frac{\mathbf{x}\cdot\mathbf{y}}{c^2}}\right]^2 - \frac{1}{c^2}\left[\frac{\mathbf{x}\times\mathbf{y}}{1+\frac{\mathbf{x}\cdot\mathbf{y}}{c^2}}\right]^2. \tag{4}$$

Let $\mathbf{x},\mathbf{y}\in\mathbb{R}_c^3$ be nonparallel, $\mathbf{x}\times\mathbf{y}\neq 0$. The two vectors $\mathbf{x}*\mathbf{y}$ and $\mathbf{y}*\mathbf{x}$ lie in the plane normal to $\mathbf{x}\times\mathbf{y}$ and have equal nonzero magnitudes. Hence, there is a unique rotation operator that transforms $\mathbf{y}*\mathbf{x}$ into $\mathbf{x}*\mathbf{y}$ by a rotation about an axis parallel to $\mathbf{x}\times\mathbf{y}$. This rotation, denoted by tom[x; y], is the *Thomas rotation* generated by x and y [1].

The Thomas rotation has interesting properties, some of which are listed below. For all $\mathbf{x},\mathbf{y}\in\mathbb{R}_c^3$,

(a) $\mathrm{tom}^{-1}[\mathbf{x};\mathbf{y}] = \mathrm{tom}^t[\mathbf{x};\mathbf{y}] = \mathrm{tom}[\mathbf{y};\mathbf{x}]$

(b) $\det \mathrm{tom}[\mathbf{x};\mathbf{y}] = 1$

(c) $\mathrm{tom}[-\mathbf{x};-\mathbf{y}] = \mathrm{tom}[\mathbf{x};\mathbf{y}]$

(d) $\mathrm{tom}[\mathbf{x};-\mathbf{y}] = \mathrm{tom}[\mathbf{x}*\mathbf{x};-(\mathbf{x}*\mathbf{y})]\,\mathrm{tom}[\mathbf{x};\mathbf{y}]$

(e) $\mathrm{tom}[\mathbf{x};\mathbf{y}] = I$ if and only if $\mathbf{x}\times\mathbf{y}=0$

(f) $\mathrm{tom}[\mathbf{x};\mathbf{y}] = \mathrm{tom}[\mathbf{x}*\mathbf{y};\mathbf{y}] = \mathrm{tom}[\mathbf{x};\mathbf{y}*\mathbf{x}]$.

The matrices $\mathrm{tom}^{-1}[\mathbf{x};\mathbf{y}]$ and $\mathrm{tom}^t[\mathbf{x};\mathbf{y}]$ are respectively the inverse and the transpose of $\mathrm{tom}[\mathbf{x};\mathbf{y}]$, and $\det m$ is the determinant of a matrix m. Eq. (d) describes the Thomas rotation $\mathrm{tom}[\mathbf{x};-\mathbf{y}]$ as the Thomas rotation $\mathrm{tom}[\mathbf{x};\mathbf{y}]$ followed by another Thomas rotation. Eq. (e) asserts that Thomas rotation vanishes, $\mathrm{tom}[\mathbf{x};\mathbf{y}]=I$, if and only if either $\mathbf{x}\,\|\,\mathbf{y}$ or $\mathbf{x}=0$ or $\mathbf{y}=0$. Eq. (f) shows that the parametrization of the Thomas rotation, $\mathrm{tom}[\mathbf{x};\mathbf{y}]$, by two vector parameters, $\mathbf{x},\mathbf{y}\in\mathbb{R}_c^3$, is not unique.

The Thomas rotation was accidentally discovered by Thomas in 1926 as a means to reconcile a conflict in the spinning electron of the Goudsmit-Uhlenbeck model that gave twice the observed precession effect [2-5]. It is interesting to realize that the Thomas rotation, which plays an important role in the development of quantum mechanics and atomic spectra, is a source for a new mathematical concept, the *nonassociative group*. Let $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}_c^3$ be admissible velocities. Then

(i)	$\mathbf{x}*\mathbf{y}\in\mathbb{R}_c^3$	Closure
(ii)	$\mathbf{x}*\mathbf{y}=\mathrm{tom}[\mathbf{x};\mathbf{y}](\mathbf{y}*\mathbf{x})$	Weak commutative law
$(iiia)$	$\mathbf{x}*(\mathbf{y}*\mathbf{z})=(\mathbf{x}*\mathbf{y})*\mathrm{tom}[\mathbf{x};\mathbf{y}]\mathbf{z}$	Right weak associative law
$(iiib)$	$(\mathbf{x}*\mathbf{y})*\mathbf{z}=\mathbf{x}*(\mathbf{y}*\mathrm{tom}[\mathbf{y};\mathbf{x}]\mathbf{z})$	Left weak associative law
(iv)	$0*\mathbf{x} = \mathbf{x}*0 = \mathbf{x}$	Existence of identity
(v)	$(-\mathbf{x})*\mathbf{x} = \mathbf{x}*(-\mathbf{x}) = 0$	Existence of inverse

Eqs. (i)-(v) exhibit the basic properties of the *noncommutative, nonassociative* group, $(\mathbb{R}_c^3, *)$, of the set $\mathbb{R}_c^3$ of relativistically admissible velocities, with the group operation given by relativistic velocity composition. A use of this group structure is illustrated in the following two examples, where properties of the operators $*$ and *tom* are employed.

EXAMPLE 1. Let $\mathbf{a},\mathbf{b}\in\mathbb{R}_c^3$. Solve the equation

$$\mathbf{a}*\mathbf{x} = \mathbf{b} \tag{5}$$

for $\mathbf{x}\in\mathbb{R}_c^3$.

SOLUTION Employing the left weak associative law we have

$$\begin{aligned} \mathbf{x} &= 0*\mathbf{x} \\ &= ((-\mathbf{a})*\mathbf{a})*\mathbf{x} \\ &= (-\mathbf{a})*(\mathbf{a}*\mathrm{tom}[\mathbf{a};-\mathbf{a}]\mathbf{x}) \\ &= (-\mathbf{a})*(\mathbf{a}*\mathbf{x}) \\ &= (-\mathbf{a})*\mathbf{b}. \end{aligned} \tag{6}$$

The Thomas rotation involved in eq. (6) vanishes by eq. (e).

EXAMPLE 2. Let $\mathbf{a}, \mathbf{b} \in I\!R_c^3$. Solve the equation

$$\mathbf{x} * \mathbf{a} = \mathbf{b} \tag{7}$$

for $\mathbf{x} \in I\!R_c^3$.

SOLUTION Employing the right weak associative law and eq. (*f*) we have

$$\begin{aligned} \mathbf{x} &= \mathbf{x} * \mathbf{0} \\ &= \mathbf{x} * (\mathbf{a} * (-\mathbf{a})) \\ &= (\mathbf{x} * \mathbf{a}) * \mathrm{tom}[\mathbf{x}; \mathbf{a}]\,(-\mathbf{a}) \\ &= (\mathbf{x} * \mathbf{a}) * \mathrm{tom}[\mathbf{x} * \mathbf{a}; \mathbf{a}]\,(-\mathbf{a}) \\ &= \mathbf{b} * \mathrm{tom}[\mathbf{b}; \mathbf{a}]\,(-\mathbf{a})\,. \end{aligned} \tag{8}$$

In contrast with Example 1, the solution to the problem in Example 2 involves a non-vanishing Thomas rotation.

An expression for the effect of a Thomas rotation on an element $\mathbf{z} \in I\!R_c^3$ can be obtained from the first weak associative law (*iiia*),

$$\mathrm{tom}[\mathbf{x}; \mathbf{y}]\,\mathbf{z} = (-(\mathbf{x} * \mathbf{y})) * (\mathbf{x} * (\mathbf{y} * \mathbf{z})), \qquad \mathbf{x}, \mathbf{y}, \mathbf{z} \in I\!R_c^3. \tag{9}$$

The solution (8) of eq. (7) can therefore be expressed without a Thomas rotation by the equation

$$\begin{aligned} \mathbf{x} &= \mathbf{b} * ((-(\mathbf{b} * \mathbf{a})) * (\mathbf{b} * (\mathbf{a} * (-\mathbf{a})))) \\ &= \mathbf{b} * ((-(\mathbf{b} * \mathbf{a})) * \mathbf{b}). \end{aligned} \tag{10}$$

By letting c tend to infinity, $c \to \infty$, the composition operator, $*$, is continuously deformed into the ordinary vector addition operator, +, and the Thomas rotation is continuously deformed into the identity mapping, I. We thus see that for $c \to \infty$ the noncommutative, nonassociative group $(I\!R_c^3, *)$ reduces to the standard Euclidean group $(I\!R^3, +)$; eqs. (5) and (7) reduce to the equation $\mathbf{x} + \mathbf{a} = \mathbf{b}$; and their solutions in equations (6), (8) and (10) reduce to $\mathbf{x} = \mathbf{b} - \mathbf{a}$. Further details will be presented elsewhere.

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