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The spectral data for Hamiltonian stationary Lagrangian tori in $\mathbb{R}^4$

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A B S T R A C T

This article determines the spectral data, in the integrable systems sense, for all weakly conformally immersed Hamiltonian stationary Lagrangian in $\mathbb{R}^4$. This enables us to describe their moduli space and the locus of branch points of such an immersion. This is also an informative example in integrable systems geometry, since the group of ambient isometries acts non-trivially on the spectral data and the relevant energy functional (the area) need not be constant under deformations by higher flows.

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1. Introduction

A smooth immersion of a surface $f : M \to \mathbb{R}^4$ is Hamiltonian stationary Lagrangian[1] (HSL) if $f(M)$ is a Lagrangian submanifold whose area is stationary for all variations by (compactly supported) Hamiltonian vector fields. This is a natural variational problem for Lagrangian submanifolds and occurs in the study of volume minimisers in families of Lagrangian submanifolds [16,17]. The Euler–Lagrange equations were derived by Oh [14] and can be phrased in terms of the mean curvature 1-form $\sigma_H = f^*(H \lrcorner \omega)$, where H denotes the mean curvature vector and ω is the standard Kähler form on $\mathbb{R}^4 \simeq \mathbb{C}^2$. For any Lagrangian submanifold of $\mathbb{R}^{2n}$ it can be shown that $d\sigma_H = 0$, and f is Hamiltonian stationary when $d * \sigma_H = 0$, i.e., σ_H is harmonic. Consequently, for a compact HSL surface σ_H represents a cohomology class. It turns out that, after scaling, this is the Maslov class $\mu \in H^1(M, \mathbb{Z})$, which is an important Hamiltonian isotopy invariant of $f(M)$. Oh conjectured in [14] that the Clifford torus in $S^3 \subset \mathbb{R}^4$ minimises area in its Hamiltonian isotopy class and this proved to be a challenging question, which remains unanswered (although see [8,1,2] for a partial solution).

Rather surprisingly, all weakly conformal HSL immersions of a torus $\mathbb{C}/\Gamma$ into $\mathbb{R}^4$ can be explicitly described in terms of the Fourier components of the immersion: this was discovered by Hélein and Romon [7], who showed that the equations can be reduced to a system of linear equations. They arrived at this through an investigation of the HSL equations as the Maurer–Cartan equations for a loop of flat connexions, where the connexion form takes values in the Lie algebra of the group G of symplectic isometries of $\mathbb{R}^4$. This is an "integrable systems" approach, and although this proved to be somewhat superfluous to writing down the Fourier component solution, Hélein and Romon showed that the problem does admit the structure normally associated with an integrable system. The loop algebra valued Maurer–Cartan form is quadratic in the loop parameter,

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[1] The original terminology for these was H-minimal or Hamiltonian minimal.

$$\alpha_\zeta = \zeta^{-2}\alpha_{-2} + \zeta^{-1}\alpha_{-1} + \alpha_0 + \zeta\alpha_1 + \zeta^2\alpha_2, \tag{1}$$

and satisfies a Lax equation, $d\alpha_\zeta = [\xi_\zeta, \alpha_\zeta]$ where ξ_ζ is a Laurent polynomial in ζ (that is to say, HSL tori possess a "polynomial Killing field"). However, the Lax equations have a significant difference from those which are most often seen in the integrable systems study of, say, harmonic maps or minimal surfaces. The group G is not semi-simple (or even reductive): it is the semi-direct product $U(2) \ltimes \mathbb{C}^2$ and its complexification is most naturally realised inside a parabolic subgroup of $GL_5(\mathbb{C})$. Together with the fundamental linearity of this particular set of equations, this raises the question of what the solution to these Lax equations look like, and particularly whether there is any effective "spectral data" which reproduces the Fourier component solutions. Here we are referring to the expectation of a correspondence between a HSL torus and algebro-geometric data of the type which one typically sees in the solution of Lax equations.

Our aim here is to show that such a correspondence does exist and not only does it reproduce entirely the solutions from the Fourier component method, it provides extra information which that method does not yield so easily. In particular, given that the original formulation allows for branch points it would be nice to know to what extent branched immersions make up the vector space of solutions found in [7]. Our approach leads to the conclusion that unbranched immersions are generic, and generically amongst branch immersions the branch points are isolated (see Remark 5.5).

We first show that every weakly conformal HSL immersion $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ is determined by a triple of data $(X, \lambda, \mathcal{L})$ consisting of a complete algebraic curve X, a rational function λ on X and a line bundle $\mathcal{L}$ (or possibly a non-invertible rank 1 sheaf) over X. To achieve this we cannot use just the naive characteristic polynomial spectral curve of one polynomial Killing field ξ_ζ, since this does not yield enough information. Similarly, Leschke and Romon showed in [10] that the "multiplier spectral curve" defined in [4] is always the Riemann sphere in this case, so it too fails to provide enough information for classification. Instead we use a commutative subalgebra of polynomial Killing fields. The linear nature of the problem forces X to be a rational (and in this case, reducible) curve: its singularities correspond precisely to the non-trivial Fourier modes of f. More surprisingly, the parabolic nature of the group G means the spectral data is *not* invariant under symplectic isometries. This is a consequence of the fact not all commutative subalgebras of polynomial Killing fields are isomorphic, so that the choice of subalgebra becomes part of the spectral data.

When this is all put together the full picture has a simple geometry. Fix a conformal type Γ for the torus and fix a Maslov class $\beta_0 \in \Gamma^* \simeq H^1(\mathbb{C}/\Gamma, \mathbb{Z})$. This fixes the spectral curve (X, λ). Let $\mathcal{S}(\Gamma, \beta_0)$ be the set of triples $(X, \lambda, \mathcal{L})$ for weakly conformal HSL immersions $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ of Maslov class β_0 satisfying the base point condition $f(0) = 0$ (and a similar condition on the Lagrangian angle function). This condition is preserved by dilations, which always preserve the HSL condition, and a subgroup G_0 of symplectic isometries isomorphic to $SU(2)$. Let N denote the maximum number of non-trivial Fourier modes such a map can possess. We prove:

Theorem 1.1. $\mathcal{S}(\Gamma, \beta_0) \simeq \mathbb{CP}^{N-1}$ *and corresponds to the space of based weakly conformal HSL immersions f modulo dilations and the action of a maximal torus $S^1 \subset G_0$. Varying the base point of f induces an action of the real Lie group $\mathbb{C}/\Gamma$ on $\mathcal{S}(\Gamma, \beta_0)$ through a map $p \mapsto (X, \lambda, \mathcal{L}_p)$. The map f has a branch point at $p \in \mathbb{C}/\Gamma$ precisely when $\mathcal{L}_p$ lies on the intersection of two hyperplanes Θ_∞ and Θ_0 in $\mathcal{S}(\Gamma, \beta_0)$.*

The quotient of $\mathcal{S}(\Gamma, \beta_0)$ by the action of G_0 is a S^2-bundle over the moduli space $\mathcal{M}(\Gamma, \beta_0)$ of these HSL tori. The natural map $\mathcal{S}(\Gamma, \beta_0) \to \mathcal{M}(\Gamma, \beta_0)$ can be realised as the fibration $\mathbb{CP}^{N-1} \to \mathbb{HP}^{N/2-1}$, and the fibres are the G_0-orbits of spectral data. We provide an explicit expression for the hyperplanes Θ_∞, Θ_0 in the homogeneous coordinates on $\mathcal{S}(\Gamma, \beta_0)$. Each can be interpreted as a translate of the θ-divisor for the θ-function of a rational curve: this is explained in Appendix A. It follows that the parameterisation by spectral data turns the question of locating branch points into a problem in projective geometry.

As one expects, the Jacobi variety of X, or more precisely a real subgroup J_R of it, acts on the spectral data to produce new HSL tori. The Maslov class alone determines (X, λ) completely, so that $\mathcal{S}(\Gamma, \beta_0)$ is a union of J_R-orbits of different dimensions. The sheaf $\mathcal{L}$ is only a line bundle on the largest orbit. The base point translation action of $\mathbb{C}/\Gamma$ factors through a homomorphism of $\mathbb{C}/\Gamma$ into J_R. In integrable systems language the action of J_R generates the "higher flows". We show in Section 6 that these do not all correspond to Hamiltonian variations. This too is a surprising departure from the study of minimal tori, where all higher flows preserve the area. It reflects the fact that the area functional is a non-constant function on the moduli space $\mathcal{S}(\Gamma, \beta_0)$.

We are hopeful this study will provide a useful testing ground for deeper investigations in the theory of spectral data in surface theory. In particular, the biggest challenge at this time is to understand how the appropriate energy functional depends on the spectral genus. For HSL tori in $\mathbb{R}^4$ the spectral data is elementary enough to see (cf. Remarks 6.2 and 6.3 below) how the area functional depends upon the conformal class and the spectral data. We hope this relationship which may provide insight into how this works more generally in integrable surface theory.

2. Lagrangian surfaces in $\mathbb{R}^4$

Let $\mathbb{R}^4$ be equipped with its Euclidean metric, its standard complex structure J, and Kähler form ω. We will represent its group of symplectic isometries in the form

$$G = \left\{ (g, u) \in SO(4) \ltimes \mathbb{R}^4 : gJg^{-1} = J \right\} \simeq U(2) \ltimes \mathbb{C}^2.$$

Now suppose we have a conformally immersed orientable surface $f : M \to \mathbb{R}^4$. If it is Lagrangian (i.e., $f^*\omega = 0$) then its Gauss map $\gamma : M \to \mathrm{Lag}(\mathbb{R}^4)$ takes values in the Grassmannian $\mathrm{Lag}(\mathbb{R}^4)$ of oriented Lagrangian 2-planes in $\mathbb{R}^4$. Let $H : M \to TM^\perp$ be the mean curvature field for f, then one knows that the mean curvature form $\sigma_H = f^*(H \lrcorner \omega)$ is closed and f is Hamiltonian stationary precisely when σ_H is also co-closed. Now, since $\mathrm{Lag}(\mathbb{R}^4) \simeq U(2)/SO(2)$ we can post-compose γ with the well-defined map $\det : U(2)/SO(2) \to S^1$ induced by the determinant on $U(2)$. Let $s : M \to S^1$ be defined by $s = \det \circ \gamma$, then by definition the Maslov form $\mu \in \Omega^1_M$ of f is

$$\mu = \frac{1}{\pi i} s^{-1} \, ds.$$

By a theorem of Morvan [12] this is related to the mean curvature form by $\mu = \frac{2}{\pi}\sigma_H$ and therefore

$$s^{-1} ds = 2i\sigma_H. \tag{2}$$

Thus f is Hamiltonian stationary if and only if s is a harmonic map.

Now we restrict our attention to the case where M is a torus, represented in the form $M = \mathbb{C}/\Gamma$ where Γ is a lattice. We can write $s = \exp(i\beta)$ for a function $\beta : \mathbb{C} \to \mathbb{R}$, which is called the Lagrangian angle, and f is Hamiltonian stationary when β is a harmonic function. Note that the action of the centre of $U(2)$ on f is by $f \mapsto e^{\theta J} f$ (for some $\theta \in [0, 2\pi]$), under which the Lagrangian angle changes by $\beta \mapsto \beta + 2\theta$. Therefore we may (and will) assume that $\beta(0) = 0$. Following [7], and using the inner product $\langle z, w \rangle = \mathrm{Re}(z\bar{w})$ on $\mathbb{C}$, we can write β as

$$\beta(z) = 2\pi \langle \beta_0, z \rangle \tag{3}$$

for a constant $\beta_0 \in \Gamma^* \subset \mathbb{C}$, i.e., $\beta(z) \in 2\pi\mathbb{Z}$ for every $z \in \Gamma$.

Remark 2.1. The Maslov class of f is the cohomology class of $[\mu] \in H^1(M, \mathbb{Z})$. Under the natural identification $H^1(M, \mathbb{Z}) \simeq \Gamma^*$ we can think of β_0 as the Maslov class. One knows that HSL surfaces are constrained Willmore surfaces [3] (i.e., the Willmore energy $W(f) = \int_M |H|^2$ is critical for variations through conformal immersions). Because of the relation $H = -\frac{1}{2}J\nabla\beta$ the Willmore energy for a conformally immersed HSL torus $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ is "quantized" by the Maslov class β_0:

$$W(f) = \pi^2 |\beta_0|^2 A(\mathbb{C}/\Gamma), \tag{4}$$

where $A(\mathbb{C}/\Gamma)$ is the area of the flat torus $\mathbb{C}/\Gamma$ with metric $|dz|^2$.

2.1. Twistor lift and frames

Let E denote the pullback $f^{-1}T\mathbb{R}^4$ of the tangent bundle of $\mathbb{R}^4$. Then $E = TM \oplus TM^\perp$ and since f is Lagrangian $TM^\perp = JTM$. Let J_M denote the intrinsic complex structure carried by M. Then since f is conformal it induces another complex structure $S = J_M \oplus J J_M J$ on E, with the property that $JS = -SJ$. We may think of S as a twistor lift [5] of f, i.e., $S : M \to Z$ where Z is the twistor bundle of complex structures on $T\mathbb{R}^4$. Inside Z lies the S^1-subbundle of all complex structures which anti-commute with J, which is where S takes values. In fact this S^1-bundle is an embedding of a 4-symmetric space G/G_0 into Z, where G_0 is the fixed point subgroup of an order 4 outer automorphism τ of G. To see this, first let $\varepsilon_1, \ldots, \varepsilon_4$ be the standard oriented orthonormal basis of $\mathbb{R}^4$ for which $\varepsilon_2 = J\varepsilon_1$, $\varepsilon_4 = J\varepsilon_3$, and define $L \in SO(4)$ to be the complex structure on $\mathbb{R}^4$ characterised

$$L\varepsilon_1 = \varepsilon_3, \qquad L\varepsilon_2 = -\varepsilon_4.$$

We observe that $LJ = -JL$ and that every complex structure anti-commuting with J is of the form gLg^{-1} for some $g \in U(2) \subset SO(4)$. Using L we define an order 4 outer automorphism

$$\tau : G \to G; \qquad \tau(g, u) = (-LgL, -Lu).$$

The fixed point subgroup is $G_0 = \{(g, 0) \in G : gL = Lg\} \simeq SU(2)$ and the 4-symmetric space G/G_0 is an S^1-bundle over $\mathbb{R}^4$. On the other hand the twistor bundle has description

$$Z = \left\{ (gJg^{-1}, u) \subset SO(4) \times \mathbb{R}^4 : g \in SO(4) \right\},$$

which is isomorphic to the homogeneous space $(SO(4) \ltimes \mathbb{R}^4)/U(2)$. Now we have the embedding

$$G/G_0 \to Z; \qquad (g, u)G_0 \mapsto (gLg^{-1}, u),$$

whose image is the bundle of complex structures anti-commuting with J.

Let us now give a description of $S : M \to Z$ in terms of natural frames, and show that S is essentially the Lagrangian angle function. Any conformal Lagrangian torus possesses a natural frame on the universal cover $\mathbb{C}$, called the fundamental frame, $\tilde{f} : \mathbb{C} \to G$, given by $\tilde{f} = (F, f)$, where $F : \mathbb{C} \to SO(4)$ is chosen so that $F\varepsilon_j = f_j$, where

$$f_1 = e^{-\rho} f_x, \qquad f_2 = e^{-\rho} J f_x, \qquad f_3 = e^{-\rho} f_y, \qquad f_4 = e^{-\rho} J f_y,$$

for $|df|^2 = e^{2\rho}|dz|^2$, where $z = x + iy$. Setting $\epsilon = (\varepsilon_1 - i\varepsilon_3)/2$ we see

$$df = e^{\rho} F(\epsilon \, dz + \bar{\epsilon} \, d\bar{z}).$$

It follows that $s = \det(F)$.[2]

From the definition of S in terms of the complex structures J_M and J it follows that $Sf_1 = f_3$ and $Sf_2 = -f_4$. Therefore $S = (FLF^{-1}, f)$. But, as observed in [7], we can also take one of two spinor frames for f:

$$U_\pm = \big(\pm \exp(J\beta/2), f\big).$$

Notice that $\tilde{f} = U_\pm K$ where $K = (\pm \det(F)^{-1/2} F, 0)$. Since K takes values in G_0 we have

$$FLF^{-1} = e^{J\beta/2} L e^{-J\beta/2} = e^{J\beta} L,$$

and therefore S is essentially the Lagrangian angle function.

Remark 2.2. This gives us another perspective on the Hamiltonian stationary condition, namely, a map $f : M \to \mathbb{R}^4$ is conformal Lagrangian if and only if it is S-holomorphic for some $S : M \to G/G_0$ (i.e., $Sf_z = if_z$). In that case we necessarily have $S = e^{\beta J} L$ for some function β. Then f is Hamiltonian stationary if further $\Delta\beta = 0$.

Throughout the remainder of this article we choose to work with the spinor lift U_+ since it has a particularly nice Maurer–Cartan form: we will define

$$\alpha = U_+^{-1} dU_+ = \left(\frac{1}{2} J \, d\beta, \, e^{-J\beta/2} df\right).$$

We also assume, without any loss of generality, that $f(0) = 0$ and therefore there is a unique spinor lift U_+ determined by α satisfying the initial condition $U_+(0) = (I, 0)$. The subgroup of symplectic isometries preserving the two conditions $f(0) = 0$ and $\beta(0) = 0$ is G_0.

2.2. Extended Maurer–Cartan form

The automorphism τ induces an order 4 automorphism (which we shall also call τ) on $\mathfrak{g}^{\mathbb{C}}$, the complexification of the Lie algebra $\mathfrak{g}$ of G. We will represent $\mathfrak{g}^{\mathbb{C}}$, as vector space, by

$$\mathfrak{g}^{\mathbb{C}} = \big\{(X, x) \in \mathfrak{so}_4(\mathbb{C}) \times \mathbb{C}^4 : [X, J] = 0\big\}.$$

The automorphism τ takes the form

$$\tau(X, x) = (-LXL, -Lx).$$

Let $\mathfrak{g}_j \subset \mathfrak{g}^{\mathbb{C}}$ be the i^j-eigenspace for τ, then one computes

$$\mathfrak{g}_{-1} = \mathbb{C}\epsilon \oplus \mathbb{C}J\bar{\epsilon}, \qquad \mathfrak{g}_0 = \big\{(X, 0) \in \mathfrak{g}^{\mathbb{C}} : [X, L] = 0\big\},$$

$$\mathfrak{g}_1 = \bar{\mathfrak{g}}_{-1}, \qquad \mathfrak{g}_2 = \big\{(rJ, 0) : r \in \mathbb{C}\big\}.$$

Notice that $\mathfrak{g}_{-1}$ is the i-eigenspace for $L \in \mathrm{End}(\mathbb{C}^4)$. We notice that the components of α in this decomposition are

$$\alpha_{-1} = \left(0, e^{-J\beta/2} \frac{\partial f}{\partial z} dz\right), \qquad \alpha_1 = \left(0, e^{-J\beta/2} \frac{\partial f}{\partial \bar{z}} d\bar{z}\right), \qquad \alpha_0 = (0, 0), \qquad \alpha_2 = \left(\frac{1}{2} J d\beta, 0\right).$$

We define the extended Maurer–Cartan form to be the loop of 1-forms

$$\alpha_\zeta = \zeta^{-2}\alpha_2' + \zeta^{-1}\alpha_{-1} + \alpha_0 + \zeta\alpha_1 + \zeta^2\alpha_2''. \tag{5}$$

It is the principal observation of Hélein and Romon [7] that the Maurer–Cartan equations for α_ζ are satisfied if and only if f is Hamiltonian stationary. As usual, we think of α_ζ as a 1-form with values in a loop algebra. It possesses two symmetries, namely, a real symmetry and τ-equivariance:

$$\overline{\alpha_\zeta} = \alpha_{\bar{\zeta}^{-1}}, \qquad \tau(\alpha_\zeta) = \alpha_{i\zeta}, \tag{6}$$

where $\overline{(X, x)} = (\bar{X}, \bar{x})$ is simply complex conjugation. Therefore we may think of α_ζ as taking values in the twisted loop algebra $\Lambda^\tau \mathfrak{g}$ of τ-equivariant real analytic maps $\xi_\zeta : S^1 \to \mathfrak{g}^{\mathbb{C}}$ possessing the real symmetry. This is a real subalgebra of the complex algebra $\Lambda^\tau \mathfrak{g}^{\mathbb{C}}$ of τ-equivariant real analytic maps $\xi_\zeta : S^1 \to \mathfrak{g}^{\mathbb{C}}$.

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[2] Beware here that $\det(F)$ is the determinant taken in $U(2)$ not in its representation as a subgroup of $O(4)$.

3. Polynomial Killing fields

Hélein and Romon [7] have shown that every Hamiltonian stationary Lagrangian torus in $\mathbb{R}^4$ has an adapted polynomial Killing field, i.e., a map $\xi_\zeta : \mathbb{C}/\Gamma \to \Lambda^\tau \mathfrak{g}$ satisfying

(a) $d\xi_\zeta = [\xi_\zeta, \alpha_\zeta]$,
(b) $\xi_\zeta = \zeta^{-4d-2}\alpha_{-2} + \zeta^{-4d-1}\alpha_{-1} + \cdots$.

However, there are infinitely many linearly independent adapted polynomial Killing fields. Following the principle in [11] we would like to say that, by dropping condition (b) and allowing ξ_ζ to take values in $\Lambda^\tau \mathfrak{g}^\mathbb{C}$, polynomial Killing fields come in complex algebras, and that the solution of the Lax equation (and the geometry of the original map) should be able to be reconstructed from spectral data determined by this algebra. However, matrix multiplication does not preserve the loop algebra $\Lambda^\tau \mathfrak{g}^\mathbb{C}$. The reason is that $\mathfrak{g}$ itself is not closed under matrix multiplication. We rectify this by working in the larger matrix algebra

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & a \\ 0 & b \end{pmatrix} \in \mathfrak{gl}_5(\mathbb{C}) \colon A \in \mathfrak{gl}_4(\mathbb{C}), \ [A, J] = 0, \ a \in \mathbb{C}^4, \ b \in \mathbb{C} \right\}.$$

This contains $\mathfrak{g}^\mathbb{C}$ as the subalgebra for which $A \in \mathfrak{so}_4(\mathbb{C})$ and $b = 0$. Rather than use the block matrix notation, it will be more convenient to use a hybrid notation,

$$\begin{pmatrix} B & a \\ 0 & b \end{pmatrix} = (A, a) + bI, \tag{7}$$

where I stands for the identity matrix in $\mathfrak{gl}_5$ and we have defined $A = B - bI_4$. In this notation the Lie bracket is given by

$$\left[(X, x) + yI, (A, a) + bI \right] = \left([X, A], Xa - Ax \right).$$

Notice that τ extends to $\mathfrak{p}$ as the Lie algebra automorphism for which

$$\tau : (X, x) + yI \mapsto (-LXL, -Lx) + yI.$$

To properly understand the algebra of polynomial Killing fields and their spectral data, we will work in a different realisation of the twisted loop algebra, in which the twisting is partially removed. First we embed $\Lambda^\tau \mathfrak{g}^\mathbb{C}$ in $\Lambda^\mu \mathfrak{p}$, where $\mu = \tau^2$. The involution μ is inner,

$$\mu = \mathrm{Ad} \begin{pmatrix} -I_4 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore $\Lambda^\mu \mathfrak{p}$ is an algebra with unit under matrix multiplication. It is easy to check that

$$(X_\zeta, x_\zeta) + y_\zeta I \in \Lambda^\mu \mathfrak{p} \quad \Longleftrightarrow \quad X_{-\zeta} = X_\zeta, \ x_{-\zeta} = -x_\zeta, \ y_{-\zeta} = y_\zeta. \tag{8}$$

To remove this twisting define $\kappa_\zeta : S^1 \to K$ to be the homomorphism

$$\kappa_\zeta = \begin{pmatrix} \zeta I_4 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from (8) that

$$\mathrm{Ad}\kappa_\zeta \cdot (X_\zeta, x_\zeta) + y_\zeta I = (X_\zeta, \zeta x_\zeta) + y_\zeta I$$

is an untwisted loop when considered as a function of $\lambda = \zeta^2$. This gives a matrix algebra automorphism from $\Lambda^\mu \mathfrak{p}$ to the algebra $\Lambda \mathfrak{p}$ of untwisted loops in $\mathfrak{p}$. From now on we shall assume this has been applied to all the objects under study: the notation will imply that the untwisting has been applied by writing all loops as a function of $\lambda = \zeta^2$.

In $\Lambda\mathfrak{p}$ the extended Maurer–Cartan form has the shape

$$\alpha_\lambda = \left(\frac{\pi}{2} \left(\lambda^{-1} \bar\beta_0 \, dz + \lambda \beta_0 \, d\bar z \right) J, \, e^{-J\beta/2} (f_z \, dz + \lambda f_{\bar z} \, d\bar z) \right). \tag{9}$$

Since α_ζ is both τ-equivariant and satisfies the reality condition (6), this form α_λ has the induced symmetries $\bar\rho^*(\alpha_\lambda) = \alpha_\lambda$, $\tau^*(\alpha_\lambda) = \alpha_\lambda$, where $\bar\rho^*, \tau^*$ are the commuting, respectively $\mathbb{R}$-linear and $\mathbb{C}$-linear, algebra involutions of $\Lambda\mathfrak{p}$ defined by

$$\bar\rho^*(\xi_\lambda) = \mathrm{Ad} R_\lambda \cdot \overline{\xi_{\bar\lambda^{-1}}}, \qquad \tau^*(\xi_\lambda) = \mathrm{Ad} T \cdot \tau(\xi_{-\lambda}), \tag{10}$$

where

$$R_\lambda = \begin{pmatrix} I_4 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad T = \begin{pmatrix} iI_4 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\xi_\lambda = (X_\lambda, x_\lambda) + y_\lambda I$ these look like

$$\bar\rho^*(\xi_\lambda) = (\bar X_{\bar\lambda^{-1}}, \lambda \bar x_{\bar\lambda^{-1}}) + \bar y_{\bar\lambda^{-1}} I, \qquad \tau^*(\xi_\lambda) = (-LX_{-\lambda}L, -iLx_{-\lambda}) + y_{-\lambda} I. \tag{11}$$

Both $\bar\rho^*$ and τ^* are loop algebra automorphisms. As a result of this symmetry of α_λ, if ξ_λ is a polynomial Killing field then so are both $\bar\rho^*(\xi_\lambda)$ and $\tau^*(\xi_\lambda)$.

From now on we extend the definition of a polynomial Killing field to include any solution of (a) with values in $\Lambda\mathfrak{p}$. Our aim is to understand the algebra $\mathcal{K}$ of these polynomial Killing fields and, more precisely, to describe certain maximal abelian subalgebras. For a polynomial Killing field of the form $(X_\lambda, x_\lambda) + y_\lambda I$ the polynomial Killing field equations become

$$dX_\lambda = 0,$$
$$dy_\lambda = 0,$$
$$dx_\lambda + \frac{\pi}{2} \left(\lambda^{-1} \bar\beta_0 \, dz + \lambda \beta_0 \, d\bar z \right) J x_\lambda = X_\lambda e^{-J\beta/2} (f_z \, dz + \lambda f_{\bar z} \, d\bar z). \tag{12}$$

Therefore X_λ and y_λ depend on λ alone. Since these equations are linear over $\mathbb{C}[\lambda, \lambda^{-1}]$ we may assume that X_λ is a polynomial of degree N, $X_\lambda = \sum_{j=0}^N X_j \lambda^j$. Expanding x_λ similarly we obtain the equations

$$\partial x_j / \partial z + \frac{\pi}{2} \bar\beta_0 J x_{j+1} = X_j e^{-J\beta/2} f_z, \tag{13}$$
$$\partial x_j / \partial \bar z + \frac{\pi}{2} \beta_0 J x_{j-1} = X_{j-1} e^{-J\beta/2} f_{\bar z}. \tag{14}$$

It follows that $x_j = 0$ for $j \leqslant 0$ and $j > N$. It is also clear that if X_λ is identically zero then so is x_λ, i.e., there are no non-trivial polynomial Killing fields of the form $(0, x_\lambda)$. Now consider polynomial Killing fields for which $X_\lambda = q_\lambda R$, where $q_\lambda \in \mathbb{C}[\lambda]$ and R is a constant projection matrix (i.e., $R^2 = R$), which we will assume to be orthogonal projection onto some subspace V of $\mathbb{C}^4$: since $[R, J] = 0$ this subspace V is J-invariant.[3]

Lemma 3.1. *For each orthogonal projection matrix R commuting with J there exists a unique monic polynomial $p(\lambda)$, of minimal degree N with $p(0) \neq 0$, such that there is a polynomial Killing field of the form $(p(\lambda)R, x_\lambda)$. Further, for any polynomial Killing field of the form $(q(\lambda)R, y_\lambda)$, with $q(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}]$, there exists $r(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}]$ for which $q(\lambda) = r(\lambda)p(\lambda)$ and $y_\lambda = r(\lambda)x_\lambda$. Finally, x_λ takes values in the image of R.*

Proof. Suppose $(p(\lambda)R, x_\lambda)$, $(p'(\lambda)R, x'_\lambda)$ are both non-trivial polynomial Killing fields for which $p(\lambda), p'(\lambda)$ are monic of the same degree, minimal for polynomial Killing fields of this type. Then $((p(\lambda) - p'(\lambda))R, x_\lambda - x'_\lambda)$ is again a polynomial Killing field and so $p(\lambda) = p'(\lambda)$, otherwise the degree of their difference is less than the minimal degree. It follows that $x_\lambda = x'_\lambda$.

Now suppose $(q(\lambda)R, y_\lambda)$ is a polynomial Killing field, then there exists $k \in \mathbb{N}$ for which $\lambda^k q(\lambda)$ is a polynomial. Clearly we can always find a polynomial $s(\lambda)$ which $\lambda^k q(\lambda) - s(\lambda)p(\lambda)$ has degree less than $p(\lambda)$. But the polynomial Killing field equations are linear over $\mathbb{C}[\lambda, \lambda^{-1}]$, therefore must have

$$\lambda^k q(\lambda) - s(\lambda)p(\lambda) = 0, \qquad \lambda^k y_\lambda - s(\lambda)x_\lambda = 0.$$

Finally, $x_\lambda = Rx_\lambda + R^\perp x_\lambda$, where $R^\perp$ is the complementary orthogonal projection onto $U^\perp$. Since $[R^\perp, J] = 0$, $R^\perp x_\lambda$ satisfies (12) with $X_\lambda = 0$, hence it is identically zero. $\quad\square$

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[3] We know such polynomial Killing fields must exist since in [7] it is shown that there are polynomial Killing fields of the form $(p(\lambda)J, y_\lambda)$ and multiplication on the left by $-RJ$ produces solutions of (12) by linearity.

For convenience, we will refer to the polynomial Killing field with this unique, minimal degree, monic polynomial multiplier of R as the minimal polynomial Killing field for the projector R. Note that if $\dim(V) = 1$ then $x_\lambda = s_\lambda v$ for some non-zero $v \in V$ and some function $s_\lambda(z)$ which is polynomial in λ.

Everything we need to know about the spectral curve is encoded in the minimal polynomial Killing field of the form $(p(\lambda)I_4, x_\lambda)$. From now on we will use $p(\lambda), x_\lambda$ exclusively for this polynomial Killing field, and denote the degree of $p(\lambda)$ by N. To understand $p(\lambda)$ we note first that x_λ is completely determined by $p(\lambda)$, given u, using the recursion implicit in Eqs. (13) we have

$$x_{j+1} = -\left(\frac{2}{\pi \bar{\beta}_0} J\right)^{j+1} \sum_{k=0}^{j} \left(-\frac{\pi \bar{\beta}_0}{2} J\right)^k p_k \frac{\partial^{j-k} u}{\partial z^{j-k}}, \quad 0 \leqslant j \leqslant N, \tag{15}$$

where for simplicity we set $u := \alpha_{-1}(\partial/\partial z) = e^{-J\beta/2} f_z$ (following the notations in [7, §3.2]). Recall also that the Maurer–Cartan equation for α_ζ implies in particular

$$u_{\bar{z}} = \frac{\pi}{2} \bar{\beta}_0 J \bar{u}. \tag{16}$$

This recursion has an important consequence for the roots of $p(\lambda)$. Indeed, considering (15) for $j = N$, we see that, since $x_{N+1} = 0$,

$$\sum_{k=0}^{N} \left(-\frac{\pi \bar{\beta}_0}{2} J\right)^k p_k \frac{\partial^{N-k} u}{\partial z^{N-k}} = 0. \tag{17}$$

Now use the Fourier expansions of f and u,

$$f = \sum_{\gamma \in \Gamma^*} f_\gamma e_\gamma, \qquad u = \sum_{\gamma \in \frac{1}{2}\Gamma^*} u_\gamma e_\gamma,$$

where $e_\gamma(z) = \exp(2\pi i\langle \gamma, z\rangle)$ and we recall from [7] that u is *a priori* only 2Γ-periodic. Let Δ_f denote the set of frequencies $\gamma \in \frac{1}{2}\Gamma^*$ such that $u_\gamma \neq 0$. It is a subset of

$$\Gamma_{\beta_0}^* = \left\{\gamma \in \Gamma^* + \frac{\beta_0}{2}: |\gamma| = \frac{|\beta_0|}{2}, \; \gamma \neq \pm\frac{\beta_0}{2}\right\}$$

which is invariant under $\gamma \mapsto -\gamma$ (see [7]). Define also $u^+ := \frac{1}{2}(1 - Ji)u$, the projection on i-eigenspace of J (and similarly define u^- for the $(-i)$-eigenspace). Then Eq. (17) projects down to the i-eigenspace as

$$\forall \gamma, \quad 0 = (i\pi \bar{\gamma})^N \sum_{k=0}^{N} \left(-\frac{\bar{\beta}_0}{2\bar{\gamma}}\right)^k p_k u_\gamma^+$$

and similarly for the $(-i)$-eigenspace. One easily shows that u_γ^+ and u_γ^- are both non-zero for $\gamma \in \Delta_f$ so that

$$p\left(\frac{\bar{\beta}_0}{2\bar{\gamma}}\right) = 0 \quad \text{whenever } \gamma \in \Delta_f. \tag{18}$$

Since $|\gamma| = |\beta_0/2|$ these roots of $p(\lambda)$ lie on the unit circle. On top of this, the $\bar{\rho}^*$ and τ^* symmetries of the polynomial Killing field equations, together with the uniqueness result in Lemma 3.1, tell us that

$$\left(\overline{p(\bar{\lambda}^{-1})}I_4, \lambda \overline{x(\bar{\lambda}^{-1})}\right) = \overline{p(0)}\lambda^{-N}\left(p(\lambda)I_4, x(\lambda)\right), \qquad \left(p(-\lambda)I_4, -iLx(-\lambda)\right) = (-1)^N\left(p(\lambda)I_4, x(\lambda)\right). \tag{19}$$

We note that $\overline{p(0)} = 1/p(0)$ and N must be even for p to have minimal degree. Taken altogether this information allows us to completely determine $p(\lambda)$.

Lemma 3.2. *The minimal polynomial Killing field $(p(\lambda)I_4, x_\lambda)$ has*

$$p(\lambda) = \prod_{\gamma \in \Delta_f} (\lambda - 2\gamma/\beta_0). \tag{20}$$

In particular, $p(\lambda)$ is even, its roots are simple and all lie on the unit circle.

We will label these roots s_j, $j = 1, \ldots, N$ and note that

$$s_j = 2\gamma_j/\beta_0 = \bar{\beta}_0/2\bar{\gamma}_j, \tag{21}$$

where $\{\gamma_j : j = 1, \ldots, N\} = \Delta_f$. It will be convenient for us later to label these so that, for $s_{j+N/2} = -s_j$ (i.e., $\gamma_{j+N/2} = -\gamma_j$).

Proof. Assume that $p(\lambda)$ is given by (20) and define x_λ according to (15). This necessarily satisfies (13): we must show that it also satisfies (14) (equivalently, that it has the real symmetry in (19)). But that follows from (16) by a straightforward computation. $\square$

Remark 3.3. A particular consequence of (15) and (19) is that $x_N = -\frac{2}{\pi\beta_0} J\bar{u} = p_0\bar{x}_1$. Under the assumption that f is conformal this means x_1, x_N never vanish. Later we will consider the possibility that f is only weakly conformal: the branch point of f will occur precisely when the degree of x_λ drops.

Finally, consider the effect of the group of symplectic isometries on the minimal polynomial Killing field. Since we wish to retain the conditions $f(0) = 0$ and $\beta(0) = 0$ we are only interested in the action of $G_0 \subset G$.

Lemma 3.4. *If* $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ *has minimal polynomial Killing field* $(p(\lambda)I_4, x_\lambda)$ *then, for any* $g \in G_0$, *gf has minimal polynomial Killing field* $(p(\lambda)I_4, gx_\lambda)$.

The proof follows at once from the construction above.

4. The spectral data

As with other surface geometries which arise from integrable systems, the spectral data for a HSL torus consists of a complete algebraic curve, a rational function on that curve and a line bundle over that curve. Two features which distinguish this geometry are that: (a) the spectral curve is rational (indeed reducible), (b) the spectral data is not invariant under the action of ambient symmetries (the symplectic isometries of $\mathbb{R}^4$). Both of these features oblige us to take extra care when formulating the correspondence between spectral data and HSL tori.

4.1. The spectral curve

As a general principle (see e.g. [11]) the spectral data should be a geometric realisation of the algebra $\mathcal{K}$ of all polynomial Killing fields. As a straightforward consequence of Lemma 3.1 we obtain a complete description of $\mathcal{K}$, by thinking of it as a module over the algebra $\mathcal{B} = \mathbb{C}[\lambda^{-1}, \lambda]$ in the obvious way.

Lemma 4.1. *Let* $(p(\lambda)I_4, x_\lambda)$ *be the minimal polynomial Killing field of this type for f and let $R \in \mathrm{End}(\mathbb{C}^4)$ be an orthogonal projection commuting with J. Then the minimal polynomial Killing field for R is $(\frac{p}{r}R, \frac{1}{r}Rx_\lambda)$ where $r(\lambda)$ is the monic polynomial whose zeroes are exactly the common zeroes of $p(\lambda)$ and Rx_λ. Thus $\mathcal{K}$ is generated over $\mathcal{B}$ by the set of all these minimal polynomial Killing fields together with the constant polynomial Killing field I.*

In particular, since $p(\lambda)$ and x_λ do not have common zeroes, $(p(\lambda)R, Rx_\lambda)$ is the minimal polynomial Killing field for R unless x_λ lies in $\ker(R)$ at some zeroes of $p(\lambda)$.

Since $\mathcal{K}$ is non-commutative it is necessary to construct the spectral curve by taking a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{K}$, as the completion by smooth points of the affine curve $\mathrm{Spec}(\mathcal{A})$. Given our symmetries $\mathcal{A}$ should correspond to a choice of real maximal torus $\mathfrak{t}$ in the commutator $\mathfrak{z}^J \subset \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4)$ of J for which $\mathrm{Ad}L \cdot \mathfrak{t} = \mathfrak{t}$. Let us first consider the maximal torus $\mathfrak{t}_0 \subset \mathfrak{z}^J$ generated by

$$\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix},$$

where I_2 is the identity, and J_2 the standard complex structure, on $\mathbb{R}^2$. This maximal torus is characterised by its decomposition of $\mathbb{C}^4$ into (Hermitian orthogonal) invariant lines given by

$$V_1 = \mathbb{C}.(\varepsilon_1 - i\varepsilon_2), \qquad V_2 = \mathbb{C}.(\varepsilon_3 - i\varepsilon_4), \qquad V_3 = \bar{V}_1 = LV_2, \qquad V_4 = \bar{V}_2 = LV_1. \tag{22}$$

Notice that $\mathfrak{t}_0$ is completely determined by V_1, which is a complex line in the i-eigenspace $V \subset \mathbb{C}^4$ of J. Every other maximal torus of $\mathfrak{z}^J$ for which $\mathrm{Ad}L \cdot \mathfrak{t} = \mathfrak{t}$ is given by $\mathrm{Ad}g \cdot \mathfrak{t}_0$ for some $g \in G_0$.

For a fixed f we can define

$$\mathcal{A}_f = \left\{ (q(\lambda)R, y_\lambda) + u(\lambda)I \in \mathcal{K} : R \in \mathfrak{t}_0^{\mathbb{C}} \right\}.$$

By Lemma 4.1 $\mathcal{A}_f$ is abelian. Further, it is maximal abelian. For if $\xi \in \mathcal{K}$ commutes with every element of $\mathcal{A}_f$ then $\xi = (q(\lambda)R, y_\lambda)$ where $R \in \mathfrak{t}_0^{\mathbb{C}}$ since $\mathfrak{t}_0^{\mathbb{C}}$ is maximal.

However, quite surprisingly, it is *not* true that $\mathcal{A}_{gf} \simeq \mathcal{A}_f$ for every $g \in G_0$ (recall that G_0 is the group of symplectic isometries preserving the base point conditions $f(0) = 0$ and $\beta(0) = 0$). This isomorphism is only true generically in G_0, as we will show below. The generic algebra is isomorphic to an algebra $\mathcal{A}$ which we will now describe. For each $k = 1, \ldots, 4$ let $R_k \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$ denote the unitary projection matrix corresponding to orthogonal projection onto V_k. These span $\mathfrak{t}_0^{\mathbb{C}}$ and satisfy $R_j R_k = \delta_{jk} R_k$ (where δ_{jk} is the Kronecker delta) and

$$R_3 = -LR_2 L, \qquad R_4 = -LR_1 L, \qquad R_3 = \bar{R}_1, \qquad R_4 = \bar{R}_2. \tag{23}$$

Now define $\xi_k = (p(\lambda)R_k, R_k x_\lambda)$. Then

$$\xi_j \xi_k = \delta_{jk} p(\lambda) \xi_k. \tag{24}$$

Finally, define

$$\mathcal{A} = \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4, \lambda I, \lambda^{-1} I].$$

This is clearly an abelian subalgebra of $\mathcal{K}$. Notice that any $\eta \in \mathcal{A}$ can be uniquely written in the form

$$\eta = \sum_{j=1}^{4} q_j \xi_j + q_5 I, \quad q_j \in \mathbb{C}[\lambda, \lambda^{-1}].$$

It is easy to see from this that $\eta = 0$ if and only if each $q_j = 0$ and therefore $\mathcal{A}$ has no non-trivial relations other than those in (24). We deduce the following.

Lemma 4.2. $\mathcal{A} \simeq \mathbb{C}[Z_1, Z_2, Z_3, Z_4, Z_5, Z_5^{-1}]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $Z_k^2 - p(Z_5)Z_k$, $Z_j Z_k$ for $j, k = 1, \ldots, 4$ and $j \neq k$.

The next lemma describes the situation under which $\mathcal{A}$ is maximal abelian.

Lemma 4.3. $\mathcal{A} \subseteq \mathcal{A}_f$, and $\mathcal{A} = \mathcal{A}_f$ if and only if $R_k(x(s_j, 0)) \neq 0$ for $k = 1, \ldots, 4$, $j = 1, \ldots, N$.

Proof. By definition $\mathcal{A} \neq \mathcal{A}_f$ if and only if there exists a minimal polynomial Killing field for $R \in \mathfrak{t}_0^{\mathbb{C}}$, which we write as (qR, y), not in $\mathcal{A}$. Since it is minimal there exists $r \in \mathbb{C}[\lambda]$ such that $(rqR, ry) = (pR, Rx)$. This is equivalent to saying that p and Rx have at least one common zero, i.e., there exists at least one s_j for which $R_k(x(s_j, z)) = 0$, for some $k = 1, \ldots, 4$ and for all $z \in \mathbb{C}$. Further, the common zeroes are independent of z, for at the zeroes of $p(\lambda)$ (12) reduces to

$$d\left(\exp\left(\frac{1}{2}\beta_\lambda J\right)x_\lambda\right) = 0, \qquad \beta_\lambda = \pi\left(\lambda^{-1}\bar{\beta}_0 z + \lambda\beta_0\bar{z}\right),$$

hence

$$x(s_j, z) = \exp\left(-\frac{1}{2}\beta(s_j, z)J\right)x(s_j, 0), \quad j = 1, \ldots, N. \tag{25}$$

It follows that

$$R_k\left(x(s_j, z)\right) = e^{\pm i\beta(s_j, z)/2} R_k\left(x(s_j, 0)\right),$$

and therefore $\mathcal{A} \neq \mathcal{A}_f$ if and only if $R_k(x(s_j, 0)) = 0$ for some j, k. $\quad\square$

Now we can show that generically $\mathcal{A} = \mathcal{A}_f$. To be precise, consider the set

$$\mathcal{U}_f = \left\{g \in G_0 \colon R_k\left(gx(s_j, 0)\right) \neq 0 \; \forall k = 1, \ldots, 4, \; j = 1, \ldots, N \; z \in \mathbb{C}\right\}.$$

Lemma 4.4. The non-empty subset $\mathcal{U}_f \subset G_0$ is proper, open and $\mathcal{A}_{gf} \simeq \mathcal{A}$ if and only if $g \in \mathcal{U}_f$.

Proof. Recall that $G_0 \subset U(2) \subset SO(4)$ is the commutator of J and L. Therefore G_0 preserves both $V = V_1 \oplus V_2$ and $\bar{V}$, the $\pm i$-eigenspaces of J. Further, G_0 acts transitively on $\mathbb{P}V$ and the subset of $g \in G_0$ for which $gV_1 \cap V_1 = \{0\}$ is an open subset of G_0 (hence, the same statements are true for the action of G_0 on $\bar{V}$ and the orbit of $\bar{V}_1$). Finally, whenever $|\lambda| = 1$ the real symmetry in (19) implies that the line $\ell = \mathbb{C}.x(\lambda, 0)$ has $\bar{\ell} = \ell$ and therefore has non-trivial components in both V and $\bar{V}$. It follows that for each root s_j of $p(\lambda)$ there is a proper non-empty open subset $\mathcal{U}_{kj} \subset G_0$ for which each $R_k(gx(s_j, 0))$ is

non-zero: $\mathcal{U}_f$ is the intersection of the finitely many open subsets $\mathcal{U}_{kj}$. It follows from Lemmas 4.2 and 4.3 that $\mathcal{A}_{gf} \simeq \mathcal{A}$ if and only if $g \in \mathcal{U}_f$. $\quad\square$

Given f we define its affine spectral curve to be the affine scheme $X_0 = \mathrm{Spec}(\mathcal{A})$. The natural inclusion of algebras $\mathcal{B} \to \mathcal{A}$ is dual to a finite morphism $\lambda: X_0 \to \mathbb{C}^\times$. It follows from (24) that $\mathcal{A}$ is a free rank 5 module over $\mathbb{C}[\lambda, \lambda^{-1}]$, hence this morphism has degree 5. We define the spectral curve X to be the completion of X_0 by smooth points. By Lemma 4.2 X_0 is biregular to the curve in $\mathbb{C}^5 \setminus \{Z_5 = 0\}$ determined by the equations

$$Z_j\big(Z_j - p(Z_5)\big) = 0, \qquad Z_j Z_k = 0, \quad j, k = 1, \dots, 4, \ k \neq j. \tag{26}$$

Its normalisation $\varphi: \tilde{X}_0 \to X_0$ is dual to the algebra monomorphism

$$\varphi^*: \mathcal{A} \to \tilde{\mathcal{A}}; \qquad \sum_{j=1}^4 q_j \xi_j + q_5 I \to (q_1 p + q_5, \dots, q_4 p + q_5, q_5),$$

where $\tilde{\mathcal{A}}$ denotes $\mathcal{B}^5$ with the direct product structure. We obtain the following structure for the spectral curve X.

Proposition 4.5. *X is a reducible rational curve with five irreducible components $C_1, \dots, C_5$ each of which is a smooth rational curve. Any two intersect along the 0-dimensional subscheme $\mathfrak{S} \subset X_0$ given by $\mathfrak{S} = \mathrm{Spec}(\mathcal{A}/\mathcal{J})$ where $\mathcal{J}$ is the ideal in $\mathcal{A}$ generated by $p, \xi_1, \dots, \xi_4$. As a divisor, $\mathfrak{S}$ is just the union of the N singular points corresponding to the zeroes of $p(\lambda)$. In particular, X has arithmetic genus $g = 4(N-1)$.*

For simplicity, we will abuse notation by also using $\mathfrak{S}$ to denote the set $\lambda(\mathfrak{S}) = \{s_1, \dots, s_N\}$.

It will be convenient for us to identify the irreducible components as follows. The component C_5 has affine part given by $Z_j = 0$ for $j = 1, \dots, 4$, while the affine part of C_j, for $j \leqslant 4$, is the curve with equations

$$Z_j = p(Z_5), \qquad Z_k = 0, \quad k \neq j, \ k \leqslant 4.$$

The holomorphic function $\lambda = Z_5$ on X_0 extends to a rational function $\lambda: X \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. It has 5 points over $\lambda = 0$, which we will denote by $P_1, \dots, P_5$, with P_j lying on C_j. The corresponding points over $\lambda = \infty$ will be denoted $Q_1, \dots, Q_5$.

Since α_λ possesses the symmetries ρ^* and τ^* these act as involutions on $\mathcal{A}$. Therefore they induce involutions on X, which we will call ρ and τ, for which $\overline{h \circ \rho} = \bar{\rho}^* h$ and $h \circ \tau = \tau^* h$ for any $h \in \mathbb{C}[X_0]$. It is straightforward to establish the following characterisation of these involutions.

Proposition 4.6. *The involutions ρ and τ on X are, respectively, anti-holomorphic and holomorphic, and act as follows.*

$$\big(\overline{Z_1 \circ \rho}, \overline{Z_2 \circ \rho}, \overline{Z_3 \circ \rho}, \overline{Z_4 \circ \rho}, \overline{Z_5 \circ \rho}\big) = \left(\frac{Z_3}{Z_5^N p(0)}, \frac{Z_4}{Z_5^N p(0)}, \frac{Z_1}{Z_5^N p(0)}, \frac{Z_2}{Z_5^N p(0)}, \frac{1}{Z_5}\right),$$

$$\tau(Z_1, Z_2, Z_3, Z_4, Z_5) = (Z_4, Z_3, Z_2, Z_1, -Z_5).$$

Finally, let us explain the difference between $\mathrm{Spec}(\mathcal{A}_f)$ and $\mathrm{Spec}(\mathcal{A})$ when $\mathcal{A} \neq \mathcal{A}_f$. For each $k = 1, \dots, 4$ define

$$\mathfrak{S}_k = \big\{s_j: R_k\big(x(s_j, 0)\big) \neq 0\big\}. \tag{27}$$

Note that every s_j belongs to at least one $\mathfrak{S}_k$ since $p(\lambda)$ and $x(\lambda, 0)$ have no common zeroes. Now define

$$p_k(\lambda) = \prod_{s_j \in \mathfrak{S}_k} (\lambda - s_j).$$

As a consequence of the symmetries (19) and (23) we see that $\mathfrak{S}_1 = \mathfrak{S}_3$, $\mathfrak{S}_2 = \mathfrak{S}_4$ and $\mathfrak{S}_2 = -\mathfrak{S}_1$. Therefore $p_3 = p_1$, $p_4 = p_2$ and $p_2(\lambda) = \pm p_1(-\lambda)$. It follows that $\mathcal{A}_f = \mathbb{C}[\eta_1, \dots, \eta_4, \lambda I, \lambda^{-1} I]$ where

$$\eta_k = \frac{p_k}{p} \xi_k = \left(p_k R_k, \frac{p_k}{p} R_k x_\lambda\right)$$

and therefore $\mathrm{Spec}(\mathcal{A}_f)$ is biregular to the affine curve in $\mathbb{C}^5 \setminus \{Z_5 = 0\}$ with equations

$$Z_j\big(Z_j - p_j(Z_5)\big) = 0, \qquad Z_j Z_k = 0, \quad j, k = 1, \dots, 4, \ k \neq j.$$

We conclude from this the following structure.

Lemma 4.7. *Let X_f denote the completion by smooth points of* $\mathrm{Spec}(\mathcal{A}_f)$. *It has five irreducible components* C_j^f, $j = 1, \ldots, 5$, *each with* $C_j \simeq \hat{\mathbb{C}}$. *All five components intersect along* $\mathfrak{S}_1 \cap \mathfrak{S}_2$, *with further intersection relations*

$$C_1^f \cap C_3^f = \mathfrak{S}_1, \qquad C_2^f \cap C_4^f = \mathfrak{S}_2, \qquad C_k^f \cap C_5^f = \mathfrak{S}_k, \quad k = 1, \ldots, 4.$$

In particular, X_f has arithmetic genus $4(N_1 - 1)$, where $N_1 = \#\mathfrak{S}_1$.

The picture one should have in mind is that the natural inclusion $\mathcal{A} \hookrightarrow \mathcal{A}_f$ is dual to a finite morphism $\pi_f : X_f \to X$ which realises X_f as the desingularisation of X obtained by pulling apart the components $C_1, \ldots, C_5$ so that only the above intersection relations remain and the symmetries ρ and τ persist. Notice that, because of the symmetries ρ and τ, the structure of X_f is completely determined by knowing how C_1^f and C_5^f intersect.

Remark 4.8. In case this seems an overly elaborate way of obtaining the spectral curve, consider the alternatives. Taking the characteristic polynomial of $\xi_\lambda = (p(\lambda)I_4, x_\lambda)$ we obtain an unreduced planar curve with equation $\mu(\mu - p(\lambda))^4 = 0$. A slightly more sophisticated approach is to consider the curve of eigenlines of ξ_ζ, but its eigenlines are generated by $\varepsilon_1, \ldots, \varepsilon_4$, which are constant, and the vector

$$\bigl(x_1(\lambda, z), \ldots, x_4(\lambda, z), -p(\lambda)\bigr), \qquad x = \sum_{j=1}^{4} x_j \varepsilon_j.$$

This leads to a disconnected union of Riemann spheres. Neither of these approaches permits the vector $x(\lambda, z)$ to be encoded in the spectral data using a line bundle, or sheaf, over the spectral curve. The next section shows how our spectral data achieves this.

4.2. Line bundles over the spectral curve

In actuality $\mathcal{A}$ is a family of algebras. By evaluating every polynomial Killing field at a point z we construct an algebra $\mathcal{A}(z)$: these are all isomorphic under the map

$$\mathcal{A}(0) \to \mathcal{A}(z); \qquad \xi_\lambda(0) \mapsto \mathrm{Ad}U_\lambda(z)^{-1}\xi_\lambda(0), \tag{28}$$

where U_λ is the extended frame, i.e., $U_\lambda^{-1} dU_\lambda = \alpha_\lambda$ and $U_\lambda(0) = I$. The vector space $\mathcal{M} = \mathcal{B} \otimes \mathbb{C}^5$ is an $\mathcal{A}(z)$-module for each z. We will show that it determines a line bundle over each irreducible component and then explain how these fit together over X. Before we do this, we need to review the moduli space $\mathrm{Pic}(X)$ of line bundles over X.

Let $\tilde{X}$ be the normalisation of X: it is the disjoint union of the smooth rational curves $C_1, \ldots, C_5$. A line bundle over X can be thought of as a divisor equivalence class $[D]$ for a divisor D of smooth points on X. Recall (from e.g., [15]) that the equivalence is characterised by the property of being trivial if and only if D is the divisor of a rational function on X. Thus $\mathrm{Pic}(X)$ is isomorphic to the group of these divisor equivalence classes. Since X is reducible such a class cannot be assigned a single integer for its degree, but rather

$$\deg : \mathrm{Pic}(X) \to \mathbb{Z}^5; \qquad \deg([D]) = \bigl(\deg([D \cap C_1]), \ldots, \deg([D \cap C_5])\bigr).$$

In particular, the Jacobi variety $\mathrm{Jac}(X)$ is the subgroup of line bundles whose degree on each component is zero. The structure of $\mathrm{Jac}(X)$ is given by the next lemma, whose proof is straightforward.

Lemma 4.9. *Let $\mathrm{Div}_0(X)$ denote the set of all divisors with $\deg(D) = (0, \ldots, 0)$. For each $D \in \mathrm{Div}_0(X)$ there is a rational function f on $\tilde{X}$, unique up to scale, with divisor D. The map $\mathrm{Jac}(X) \to (\mathbb{C}^\times)^{4(N-1)}$ which assigns to $[D]$ the coordinates*

$$t_{kj} = \frac{f_k(s_j)}{f_k(s_N)} \frac{f_5(s_N)}{f_5(s_j)}, \quad 1 \leqslant k \leqslant 4, \ 1 \leqslant j \leqslant N-1, \tag{29}$$

in which $f_k = f|_{C_k}$, is an isomorphism of linear algebraic groups.

Lemma 4.10. *The $\mathcal{A}(z)$-module $\mathcal{M}$ determines a rank 1 sheaf $\mathcal{L}_f(z)$ over X. Either*:

(a) $\mathcal{A}(z) = \mathcal{A}_f(z)$ *and $\mathcal{L}_f(z)$ is a line bundle of degree $(N, N, N, N, 0)$, or,*
(b) $\mathcal{A}(z) \neq \mathcal{A}_f(z)$ *and $\mathcal{L}_f(z)$ is the direct image of a line bundle over X_f of degree $(N_1, N_1, N_1, N_1, 0)$.*

In either case $\mathcal{L}_f$ possesses the symmetries

$$\overline{\rho^*\mathcal{L}_f} \simeq \mathcal{L}_f(P_5 - Q_5), \qquad \tau^*\mathcal{L}_f \simeq \mathcal{L}_f. \tag{30}$$

For the purpose of the proof, and for subsequent use, it will be convenient to work with the Hermitian orthonormal basis $v_1, \ldots, v_4$ for $\mathbb{C}^4$, thought of as the first four dimensions in $\mathbb{C}^5$, with $v_j \in V_j$ defined by

$$v_1 = \frac{1}{\sqrt{2}}(\varepsilon_1 - i\varepsilon_2), \qquad v_2 = \frac{1}{\sqrt{2}}(\varepsilon_3 - i\varepsilon_4), \qquad v_3 = \bar{v}_1, \qquad v_4 = \bar{v}_2. \tag{31}$$

We note that $v_2 = L\bar{v}_1$, $v_4 = Lv_1$. In such a basis we write $x_\lambda(z) = \sum_{j=1}^4 \chi_j(\lambda, z)v_j$, so that $R_k x_\lambda = \chi_k(\lambda)v_k$. We also define $v_5 = (0,0,0,0,1)$.

Proof. First assume $\mathcal{A} = \mathcal{A}_f$ (for simplicity we drop the explicit dependence of the notation on z). Let $\mathcal{I}_j \subset \mathcal{A}$ denote the prime ideal corresponding to the component $C_j \cap X_0$. In terms of generators we have

$$\mathcal{I}_j = \langle \xi_j - pI, \xi_k : k \neq j \rangle, \quad j \leqslant 4,$$
$$\mathcal{I}_5 = \langle \xi_1, \ldots, \xi_4 \rangle.$$

Now define $\mathcal{A}_j = \mathcal{A}/\mathcal{I}_j$ (this is the coordinate ring for $C_j \cap X_0$) and $\mathcal{M}_j = \mathcal{M}/\mathcal{I}_j\mathcal{M}$. It is easy to see that $\mathcal{A}_j \simeq \mathcal{B}$ for each $j = 1, \ldots, 5$ and the points over $\lambda = 0, \infty$ on C_j correspond to the two gradings carried by $\mathcal{A}_j$, namely, the degree in λ^{-1} and λ, respectively. For $j = 1, \ldots, 4$ we have

$$\mathcal{I}_j\mathcal{M} = \mathcal{B}\langle pv_k, \chi_k v_k, \chi_j v_j - pv_5 : k \neq j \rangle.$$

But $p(\lambda)$ and $\chi_k(\lambda)$ have no common zeroes so the ideal in $\mathcal{B}$ generated by p and χ_k is $\mathcal{B}$ itself. Therefore

$$\mathcal{I}_j\mathcal{M} = \mathcal{B}\langle v_k, \chi_j v_j - pv_5 : k \neq j \rangle.$$

It follows that

$$\mathcal{M}_j \simeq \mathcal{B}\langle v_j, v_5 \rangle / \langle \chi_j v_j - pv_5 \rangle.$$

This is clearly a rank one module over $\mathcal{A}_j$ and torsion free since p and χ_j have no common zeroes. This determines a line bundle $\mathcal{L}_j$ over C_j with sections σ_j, σ_5, corresponding to v_j, v_5, satisfying $\chi_j\sigma_j - p\sigma_5 = 0$. Now

$$\mathcal{I}_5\mathcal{M} = \mathcal{B}\langle v_1, \ldots, v_4 \rangle$$

and therefore $\mathcal{M}_5 \simeq \mathcal{B}\langle v_5 \rangle$, which is clearly a rank one torsion free module over $\mathcal{A}_5$. It determines a trivial line bundle $\mathcal{L}_5$ with nowhere vanishing global section σ_5.

Therefore as an $\mathcal{A}$-module $\mathcal{M}$ determines a sheaf $\mathcal{L}_f$ over X for which $\mathcal{L}_f|C_j = \mathcal{L}_j$. It has a globally holomorphic section σ_5 which vanishes exactly at the zeroes of $\chi_j(\lambda, z)$: this gives a degree N divisor $D_j(z)$ on C_j, so $\deg(\mathcal{L}_j) = N$. Since $\mathcal{L}_5$ is trivial it has degree 0. Hence σ_5 has divisor

$$D(z) = D_1(z) + \cdots + D_4(z). \tag{32}$$

Since this divisor includes no singular points $\mathcal{L}(z)$ is invertible (i.e., a line bundle) with $\mathcal{L}(z) \simeq \mathcal{O}_X(D(z))$.

Now consider the case $\mathcal{A} \neq \mathcal{A}_f$. A simple adaptation of the arguments above shows that the $\mathcal{A}_f$-module $\mathcal{M}$ determines a line bundle over X_f of degree $(N_1, N_1, N_1, N_1, 0)$. Therefore its direct image $\mathcal{L}_f$ corresponds to $\mathcal{M}$ as an $\mathcal{A}$-module.

Finally, from the symmetries (19) and (31) we obtain

$$\chi_3(\lambda) = \lambda^{N+1}p(0)\overline{\chi_1(\bar{\lambda}^{-1})}, \qquad \chi_4(\lambda) = \lambda^{N+1}p(0)\overline{\chi_2(\bar{\lambda}^{-1})}, \qquad \chi_4(\lambda) = -i\chi_1(-\lambda), \qquad \chi_3(\lambda) = i\chi_2(-\lambda). \tag{33}$$

The first expression gives the equation of divisors

$$D_3 - NQ_3 = (N+1)(P_3 - Q_3) + \rho^*(D_1 - NQ_1).$$

It follows that $D_3 + Q_3 - P_3 = \rho^*D_1$. We obtain a similar equation relating D_4 and ρ^*D_2, and deduce

$$\rho^*D = D + \sum_{j=1}^4 (Q_j - P_j) \sim D + P_5 - Q_5,$$

using the fact that $\sum_{j=1}^5 (P_j - Q_j)$ is the divisor of λ. Similarly $\tau^*D = D$. $\square$

Remark 4.11. In this lemma we are implicitly assuming that the degree of $\chi_j(\lambda, z)$ is exactly N for each $z \in \mathbb{C}$. This means that $D_j(z)$ actually lies on $C_j \setminus \{Q_j\}$. But it is possible for the degree of χ_j to drop (e.g., at branch points of f: see below). We shall keep $\deg(D_j(z)) = N$ by allowing it to include the points at ∞.

It is an inevitable consequence of Lemma 4.4 that the group G_0 acts non-trivially on the spectral data. Indeed, the G_0-orbit of f takes $\mathcal{L}_f$ outside the Jacobi variety Jac(X) into a compactification which includes non-invertible sheaves, since there is at least one point in this orbit at which χ_k has a zero at some s_j. We will explain precisely what happens when we describe the moduli space of HSL tori in the next section. Meanwhile let us observe that the previous proof provides a convenient characterisation of the sheaf $\mathcal{L}_f$, since it shows that $\mathcal{L}_f$ is entirely determined by the divisor of zeroes of χ_1, even when these are allowed to include singularities or points at infinity.

Corollary 4.12. *Up to isomorphism $\mathcal{L}_f$ is determined by the positive divisor $E = D_1 - P_1$ of degree $N - 1$ on C_1, where D_1 is the divisor of zeroes of χ_1. The map f has a branch point at $z = 0$ precisely when E has the form $E = P_1 + Q_1 + E'$ for some positive divisor E' of degree $N - 3$.*

Proof. From (13) and (14) we know x_λ, and hence χ_1, has a zero at $\lambda = 0$. From Remark 3.3 we know f has a branch point at $z = 0$ precisely when the degree of x_λ drops. Using the symmetries (33) we see that this occurs precisely when χ_1 has an additional zero at P_1 and degree less then $N - 1$, i.e., a zero at Q_1. $\quad\square$

We finish this section by describing how the sheaf $\mathcal{L}_f(z)$ moves with base point translation. Since we started with a Lax equation we expect this motion to be linear, and that is indeed the case.

Proposition 4.13. *The map $\ell : \mathbb{C}/\Gamma \to \mathrm{Jac}(X)$ given by $\ell = \mathcal{L}(z) \otimes \mathcal{L}(0)^{-1}$ has image in the real analytic subgroup $J_R \subset \mathrm{Jac}(X)$ consisting of all line bundles possessing the symmetries*

$$\overline{\rho^* \ell} \simeq \ell, \qquad \tau^* \ell \simeq \ell. \tag{34}$$

Further, ℓ is a homomorphism of real groups characterised by the equation

$$\left.\frac{\partial \ell}{\partial z}\right|_{z=0} = \frac{i\pi \bar{\beta}_0}{2} \left.\frac{\partial \mathcal{A}}{\partial \lambda}\right|_{\lambda=0} \tag{35}$$

where $\mathcal{A} : \mathbb{C} \to \mathrm{Jac}(X)$ is defined by

$$\mathcal{A}(\lambda) = \mathcal{O}\left(\sum_{j=1}^{4}\left(P_j(\lambda) - P_j\right)\right), \tag{36}$$

with $P_j(\lambda)$ the point on C_j given by $Z_j = \lambda$ for $j = 1, 2$ but $Z_j = -\lambda$ for $j = 3, 4$.

Proof. The isomorphisms in (34) are a direct consequence of (30). To prove (35) we use the coordinates (29) on Jac(X), in which $L(z)$ has coordinates,

$$t_{kj}(z) = \frac{\chi_k(s_j, z)}{\chi_k(s_j, 0)} \frac{\chi_k(s_N, 0)}{\chi_k(s_N, z)}, \quad 1 \leqslant k \leqslant 4, \ 1 \leqslant j \leqslant N - 1.$$

Here we are using the fact that $D(z) - D(0)$ is the divisor for the rational function χ on $\tilde{X}$ whose restriction to $\tilde{C}_k$ is χ_k for $1 \leqslant k \leqslant 4$. Now using (25) we compute

$$\left.\frac{\partial t_{kj}}{\partial z}\right|_{z=0} = \epsilon_k \frac{-i\pi \bar{\beta}_0}{2}\left(s_j^{-1} - s_N^{-1}\right), \tag{37}$$

where ϵ_k equals $+1$ for $k = 1, 2$ and -1 for $k = 3, 4$. On the other hand we compute, in analogy with the computation in Appendix A, the coordinates of $\mathcal{A}(\lambda)$ to be

$$a_{kj}(\lambda) = \frac{\epsilon_k \lambda - s_j}{\epsilon_k \lambda - s_N} \frac{s_N}{s_j}, \quad 1 \leqslant k \leqslant 4, \ 1 \leqslant j \leqslant N - 1.$$

Therefore

$$\left.\frac{\partial a_{kj}}{\partial \lambda}\right|_{\lambda=0} = \epsilon_k\left(s_N^{-1} - s_j^{-1}\right).$$

Eq. (35) follows. $\quad\square$

The real subgroup J_R acts on the set of spectral data by tensor product $\mathcal{L} \mapsto \mathcal{L} \otimes L$, for $L \in J_R$. This action gives what are usually called the "higher flows" in integrable systems language.

Lemma 4.14. *The real subgroup $J_R \subset \mathrm{Jac}(X)$ is isomorphic, as a real group, to $(\mathbb{C}^\times)^{N-1}$. In the coordinates (29) on $\mathrm{Jac}(X)$ it corresponds to the subgroup*

$$\left\{ t_{kj} \in \mathbb{C}^\times : t_{1j} = \frac{t_j}{t_N},\ t_{2j} = \frac{\bar{t}_{j+n}}{\bar{t}_{N/2}},\ t_{3j} = \frac{\bar{t}_j}{\bar{t}_N},\ t_{4j} = \frac{t_{j+N}}{t_{N/2}},\ \exists t_j \in \mathbb{C}^\times,\ j \in \mathbb{Z}_N \right\}.$$

This is straightforward to prove: the action of $\mathrm{Jac}(X)$ equates to the action of $(\mathbb{C}^\times)^{4(N-1)}$ as the group of diagonal matrices in $(SL_N)^4$ on the $4N$ vector $\chi_k(s_j)$. The subgroup J_R is that which preserves the symmetries (33).

5. The moduli spaces of spectral data and of HSL tori

5.1. Reconstruction from the spectral data

We begin by showing that the spectral data reconstructs the HSL torus f, up to dilations and base point preserving isometries which preserve the maximal torus $\mathfrak{t}_0$. These isometries comprise

$$T_0 = \{ g \in G_0 \colon \mathrm{Ad}\, g \cdot \mathfrak{t}_0 = \mathfrak{t}_0 \} \simeq S^1.$$

Fix a Maslov class $\beta_0 \in \Gamma^*$ for which $\Gamma^*_{\beta_0}$ is non-empty, and choose a non-empty subset $\Delta \subseteq \Gamma^*_{\beta_0}$ which is closed under $\gamma \mapsto -\gamma$. Now define

$$p_\Delta(\lambda) = \prod_{\gamma \in \Delta} (\lambda - 2\gamma/\beta_0) \tag{38}$$

and let $\mathfrak{S}_\Delta = \{ 2\gamma/\beta_0 \colon \gamma \in \Delta \}$ (whose elements we label $s_1, \dots, s_N$ as above). Let X_Δ denote the completion by smooth points of the curve given by using p_Δ in Eqs. (26), and let λ be the rational function on it given by Z_5. Let $\mathcal{E}_\Delta$ denote the set of all positive divisors E of degree $N-1$ on the smooth component C_1 of X_Δ and with the property that if s_j lies on E then $-s_j$ does not. Let $\overline{\mathrm{Pic}(X)}$ be the set of rank 1 coherent sheaves over X. We can define a map

$$\mathcal{L} : \mathcal{E}_\Delta \to \overline{\mathrm{Pic}(X_\Delta)}, \qquad E \mapsto \mathcal{O}\left(E' + \rho^* E' + \tau^* E' + (\rho\tau)^* E' + \sum_{k=1}^{4} P_k \right), \tag{39}$$

where

$$E' = E - \sum_{s_j \in \mathrm{Supp}(E)} s_j$$

and whenever $E' \neq E$ we mean $\mathcal{L}(E)$ is the direct image of that line bundle over the partial desingularisation of X_Δ obtained in the manner of Lemma 4.7.

Lemma 5.1. *The map (39) is injective.*

Proof. Suppose $\mathcal{L}(E) = \mathcal{L}(F)$ for $E, F \in \mathcal{E}_\Delta$. Then there is a rational function h on X for which $h.\mathcal{L}(E)_P = \mathcal{L}(F)_P$ at every stalk. So both must be invertible on X if either is, in which case h has divisor $E - F$. Therefore h is constant on the component C_5 and hence constant globally since it has degree N on the other components. When $\mathcal{L}(E)$ is non-invertible, it and $\mathcal{L}(F)$ must both come by direct image of a line bundle over the same curve, and these are therefore equivalent line bundles upstairs. Again h is constant on C_5 and on the other components has degree equal to the number of intersection points, hence h is constant globally. $\square$

Proposition 5.2. *To each triple $(X_\Delta, \lambda, \mathcal{L}(E))$, with $E \in \mathcal{E}_\Delta$, there corresponds a based map $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$, determined up to dilations and the action of T_0, which is a weakly conformal HSL immersion with Maslov class β_0 for which the triple is its spectral data.*

Proof. Given E there is a unique, up to complex scaling, function $\chi_1(\lambda, 0)$ whose divisor is $E + P_1 - N.\infty$. Using the relations (33) this gives $x(\lambda, 0) = \sum \chi_j v_j$. Now define $x(s_j, z)$ using (25). Then $x(\lambda, z)$ is uniquely determined by all $x(s_j, z)$ by linear algebra since $p(\lambda)$ has N zeroes and $x(\lambda, z)$ has at most N non-trivial coefficients $x_1, \dots, x_N$ as a polynomial in λ. Finally, set

$$f(z) = \frac{1}{p(1)} \left(\exp\left(\frac{1}{2} \beta J \right) x(1, z) - x(1, 0) \right). \tag{40}$$

This is a based map into $\mathbb{R}^4$ and it is easy to check that effect of the scaling is to dilate and rotate f, with rotations from T_0.

It remains to show that this map is a HSL immersion. Since by definition we have $\beta_{z\bar{z}} = 0$ it suffices to show that $Sf_z = if_z$, where $S = e^{J\beta/2}Le^{-J\beta/2}$ (cf. Remark 2.2). Differentiating (40) gives

$$d\left(\exp\left(\frac{1}{2}\beta J\right)x(1,z)\right) = p(1)\,df,$$

which is (12) at $\lambda = 1$. Since $x(\lambda, z)$ also satisfies (25), Eq. (12) must hold (with $X_\lambda = p_\Delta(\lambda)I$) at the $N+1$ values $\lambda = 1, s_1, \ldots, s_N$, and therefore holds for all λ since both sides are polynomials in λ with at most $N+1$ non-trivial terms. Consequently (13) holds. Since $p_\Delta(\lambda)$ is an even polynomial, its coefficient $p_j \neq 0$ only if j is even. When p_j is non-zero we have

$$e^{-J\beta/2}f_z = \frac{1}{p_j}\left(\partial x_j/\partial z + \frac{\pi}{2}\bar{\beta}_0 Jx_{j+1}\right). \tag{41}$$

Now we use the symmetry $Lx(-\lambda, z) = ix(\lambda, z)$ to deduce that for j even $Lx_j = ix_j$ and $Lx_{j+1} = -ix_{j+1}$. So when L is applied to both sides of (41) we obtain

$$Le^{-J\beta/2}f_z = ie^{-J\beta/2}f_z,$$

using $LJ = -JL$. The real symmetry in (33) ensures that f takes values in $\mathbb{R}^4$ and the Γ-periodicity is ensured by the fact that the Fourier frequencies of f come from Γ^*.

Finally, the construction above is clearly the reconstruction from spectral data since it works via the polynomial Killing field $(p_\Delta(\lambda)I, x_\lambda)$. □

5.2. The moduli space of spectral data

From now on we fix a torus $\mathbb{C}/\Gamma$ and a Maslov class $\beta_0 \in \Gamma^*$. We fix $N = \#\Gamma^*_{\beta_0}$ and set

$$p(\lambda) = \prod_{j=1}^{N}(\lambda - s_j),$$

so that X is now the most singular curve possible to obtain a HSL torus with Maslov class β_0. From this point of view, (X, λ) is completely determined by (Γ, β_0). As before, we identify the singular set $\mathfrak{S}$ with $\{s_j: j = 1, \ldots, N\}$. For every non-empty $\Delta \subset \Gamma^*_{\beta_0}$ of the type used in the previous section X_Δ is a partial desingularisation of X, obtained by pulling apart the intersections at $\mathfrak{S} \setminus \mathfrak{S}_\Delta$. By Proposition 5.2 the moduli space of spectral data for HSL tori with Maslov class β_0 is $\mathcal{S}(\Gamma, \beta_0) = \bigcup_\Delta \mathcal{E}_\Delta$. Also by that proposition, we can think of $\mathcal{S}(\Gamma, \beta_0)$ as the moduli space of based, weakly conformal HSL immersions $f: \mathbb{C}/\Gamma \to \mathbb{R}^4$, with Maslov class β_0, up to dilations and isometries from T_0. We will show that this is naturally isomorphic to $\mathbb{CP}^{N-1}$ and provide coordinates in which the action of $J_\mathbb{R} \subset \text{Jac}(X)$ corresponds to the natural action on Fourier components.

First, let $\text{Div}_{N-1}(C_1)$ denote the set of all positive divisors of degree $N-1$ on C_1. For each Δ the map

$$\mathcal{E}_\Delta \to \text{Div}_{N-1}(C_1); \qquad E \mapsto E + \sum_{s_j \notin \mathfrak{S}_\Delta} s_j, \tag{42}$$

is clearly injective. In fact it gives a bijection from $\mathcal{S}(\Gamma, \beta_0)$ to $\text{Div}_{N-1}(C_1)$, since $E \in \mathcal{E}_\Delta$ is not permitted to contain both s_j and $-s_j$ for any s_j. On the other hand, $\text{Div}_{N-1}(C_1)$ is clearly isomorphic to the projective space

$$\mathbb{P}F_{N-1} = \mathbb{P}\{h(\lambda) \in \mathbb{C}[\lambda]: \deg(h) \leqslant N-1\}$$

since every polynomial is determined up to scale by its divisor of zeroes. Now we require a result (which must surely be classical) whose proof is given in Appendix A.

Proposition 5.3. *The map*

$$\iota: \mathbb{P}F_{N-1} \to \mathbb{CP}^{N-1}; \qquad [h] \mapsto [h(s_1), \ldots, h(s_N)], \tag{43}$$

is an isomorphism whose inverse is given by $[\mathbf{t}] \mapsto [\varphi(\lambda, \mathbf{t})]$ *where*

$$\varphi(\lambda, \mathbf{t}) = \left(\prod_{j=1}^{N}(\lambda - s_j)\right)\det\begin{pmatrix} 1 & s_1 & \ldots & s_1^{N-2} & t_1/(\lambda - s_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_N & \ldots & s_N^{N-2} & t_N/(\lambda - s_N) \end{pmatrix}, \tag{44}$$

whenever $[t_1, \ldots, t_N] = [\mathbf{t}]$. *The polynomial* $\varphi(\lambda, \mathbf{t})$ *has degree less than* $N-1$ *if and only if* $[\mathbf{t}]$ *lies on the hyperplane*

$$\Theta_\infty = \left\{ [\mathbf{t}] \in \mathbb{CP}^{N-1} \colon \sum_{j=1}^{N} \left(\prod_{m \neq j} \frac{1}{s_m - s_j} \right) t_j = 0 \right\}, \tag{45}$$

and has a zero at $\lambda = 0$ precisely when $[\mathbf{t}]$ lies on the hyperplane

$$\Theta_0 = \left\{ [\mathbf{t}] \in \mathbb{CP}^{N-1} \colon \sum_{j=1}^{N} \left(\prod_{m \neq j} \frac{1}{s_m - s_j} \right) \frac{t_j}{s_j} = 0 \right\}. \tag{46}$$

Consequently we have fixed an isomorphism $\mathcal{S}(\Gamma, \beta_0) \simeq \mathbb{CP}^{N-1}$, which we think of as assigning (homogeneous) coordinates to each based HSL immersion $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ with Maslov class β_0, and the proposition shows how to invert this. Before we describe the inversion, note that as a corollary to Lemma 4.14 the action of the real subgroup $J_R \subset \mathrm{Jac}(X)$ on $\mathcal{S}(\Gamma, \beta_0)$ corresponds in these coordinates to the standard action of $(\mathbb{C}^\times)^N / \mathbb{C}^\times$ on $\mathbb{CP}^{N-1}$ by diagonal action on homogeneous coordinates precomposed with the isomorphism

$$\left(\mathbb{C}^\times \right)^{N-1} \to \left(\mathbb{C}^\times \right)^N / \mathbb{C}^\times; \qquad (a_1, \ldots, a_{N-1}) \mapsto [a_1, \ldots, a_{N-1}, 1]. \tag{47}$$

In particular, it follows from (37) that the action of base point translation on $S(\Gamma, \beta_0)$ corresponds to the action of $\mathbb{C}/\Gamma$ via the real homomorphism

$$\ell : \mathbb{C}/\Gamma \to \left(\mathbb{C}^\times \right)^N / \mathbb{C}^\times; \qquad z + \Gamma \mapsto \left[e_{-\gamma_1}(z), \ldots, e_{-\gamma_N}(z) \right], \tag{48}$$

where we recall that $e_\gamma(z) = \exp(2\pi i \langle \gamma, z \rangle)$.

Now we describe the explicit inversion of the isomorphism from $\mathcal{S}(\Gamma, \beta_0)$ to $\mathbb{CP}^{N-1}$ which follows from Proposition 5.3. First, by combining (40) with (31) and (33) we have the formula

$$f(z) = 2 \operatorname{Re} \left[\frac{1}{p(1)} \left(\chi_1(1, z) e_{\beta_0/2} - \chi_1(1, 0) \right) v_1 \right] + 2 \operatorname{Im} \left[\frac{1}{p(1)} \left(\chi_1(-1, z) e_{-\beta_0/2} - \chi_1(-1, 0) \right) L v_1 \right]. \tag{49}$$

Thus, as a consequence of Corollary 4.12 and Proposition 5.3, we achieve the coordinate inversion and characterise the locus of branch points of f.

Proposition 5.4. *The isomorphism $\mathcal{S}(\Gamma, \beta_0) \to \mathbb{CP}^{N-1}$ given above is inverted by assigning to $[\mathbf{t}] = [t_j] \in \mathbb{CP}^{N-1}$ the map f given by* (49), *in which*

$$\chi_1(\lambda, z) = \lambda \varphi \left(\lambda, \mathbf{t}(z) \right), \qquad t_j(z) = t_j e_{-\gamma_j}(z).$$

Further, f has a branch point at $f(z)$ precisely when $[\mathbf{t}(z)]$ lies on the intersection of hyperplanes $\Theta_0 \cap \Theta_\infty$.

Later we will require the explicit expression for f, which follows from

$$\chi_1(1, z) = \sum_{j=1}^{N} (-1)^{j-1} \Delta_j t_j e_{-\gamma_j}, \tag{50}$$

$$\chi_1(-1, z) = \sum_{j=1}^{N} (-1)^{j-1} \tilde{\Delta}_j t_j e_{-\gamma_j}, \tag{51}$$

where

$$\Delta_j = \prod_{\substack{0 \leqslant k < \ell \leqslant N \\ k, \ell \neq j}} (s_\ell - s_k), \qquad \tilde{\Delta}_j = \prod_{\substack{1 \leqslant k < \ell \leqslant N+1 \\ k, \ell \neq j}} (s_\ell - s_k)$$

with $s_0 = 1$ and $s_{N+1} = -1$.

Remark 5.5. Proposition 5.4 above allows to make some general conclusions about the existence of branch points, by exploiting the projective geometry. For simplicity consider the case where $\beta_0/2 \in \Gamma^*$, so that $\Gamma^*_{\beta_0} \subset \Gamma^*$. This is the case referred to in [7] as "truly periodic", in that the immersion does not factor through a covered torus. In that case the real homomorphism ℓ in (48) has a complexification obtained in the following way. Fix generators τ_1, τ_2 for Γ and identify $\mathbb{C}/\Gamma$ with $S^1 \times S^1$ via

$$\mathbb{C}/\Gamma \to S^1 \times S^1 \subset \mathbb{C}^\times \times \mathbb{C}^\times; \qquad u\tau_1 + v\tau_2 + \Gamma \mapsto \left(e^{2\pi i u}, e^{2\pi i v} \right), \quad u, v \in \mathbb{R}.$$

The complexification of ℓ is

$$\ell^{\mathbb{C}} : \mathbb{C}^{\times} \times \mathbb{C}^{\times} \to \left(\mathbb{C}^{\times}\right)^N / \mathbb{C}^{\times}; \qquad (a, b) \mapsto \left[a^{l_1} b^{m_1}, \ldots, a^{l_N} b^{m_N}\right], \tag{52}$$

where $l_j = -\langle \gamma_j, \tau_1 \rangle$ and $m_j = -\langle \gamma_j, \tau_2 \rangle$ are integers. This gives a homomorphism of algebraic groups. Through every point $[\mathbf{t}] \in \mathbb{CP}^{N-1}$ this puts a $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$-orbit $\mathcal{O}_{\mathbf{t}}$. This extends to provide an algebraic map

$$\ell_{\mathbf{t}}^{\mathbb{C}} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^{N-1},$$

$$([a_0, a_1], [b_0, b_1]) \mapsto \left[\left(\frac{a_1}{a_0}\right)^{l_1} \left(\frac{b_1}{b_0}\right)^{m_1} t_1, \ldots, \left(\frac{a_1}{a_0}\right)^{l_N} \left(\frac{b_1}{b_0}\right)^{m_N} t_N \right]$$

whose image is the Zariski closure $\bar{\mathcal{O}}_{\mathbf{t}}$ of the orbit. We can think of $\bar{\mathcal{O}}_{\mathbf{t}}$ as foliated by $\mathbb{C}/\Gamma$-orbits. This gives a $(\mathbb{R}^+)^2$ family, although some will be degenerate orbits.

For generic $[\mathbf{t}]$ the pull-back along $\ell_{\mathbf{t}}^{\mathbb{C}}$ of the hyperplanes $\Theta_{\infty}, \Theta_0$ determines curves $\Sigma_{\infty}, \Sigma_0$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since Θ_{∞} and Θ_0 are clearly transverse so are these curves, therefore for generic $[\mathbf{t}]$ the $\mathbb{C}/\Gamma$-orbit of $[\mathbf{t}]$ will miss these points, and the corresponding HSL map will be an unbranched immersion.

For any branched immersion lying in $\bar{\mathcal{O}}_{\mathbf{t}}$ we can give an upper bound to the number of isolated branch points it can possess, using the intersection number $\Sigma_0 \cdot \Sigma_{\infty}$. Since $\Gamma_{\beta_0}^*$ is invariant under $\gamma \mapsto -\gamma$ the set of indices $\{l_j, m_j : j = 1, \ldots, N\}$ in (52) is also invariant under change of sign. The degree of the map $\ell^{\mathbb{C}}$ restricted to the first factor $\mathbb{CP}^1 \times \{\mathrm{pt}\}$ is, generically, $\max_{j,m} |l_j - l_m|$. Since the γ_j lie on a circle this equals $2l$, where $l = \max_j\{l_j\}$. Similarly $\ell^{\mathbb{C}}$ has degree $2m$ where $m = \max_j\{m_j\}$ when restricted to the second factor. Hence Σ_0 and Σ_{∞} are curves of type $(2l, 2m)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and have intersection number $8lm$.

5.3. The moduli space of HSL tori

Here we will give a simple description of the moduli space $\mathcal{M}(\Gamma, \beta_0)$ of all weakly conformal HSL tori $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$, based at $f(0) = 0$ and with Maslov class β_0, modulo base point preserving symplectic isometries and dilations of $\mathbb{R}^4$. As in the previous section, we still assume $N = \#\Gamma_{\beta_0}^*$ for each $j = 1, \ldots, N/2$. We have seen that $x(\lambda, 0)$ is uniquely determined by its values at the roots of $p(\lambda)$, and by the symmetries (19) it suffices to specify the $N/2$ vectors

$$w_j = \frac{1}{\sqrt{p(0)s_j^{N+1}}} x(s_j, 0) \in \mathbb{R}^4, \quad j = 1, \ldots, N/2. \tag{53}$$

It is convenient now to identify $\mathbb{R}^4$ with $\mathbb{H}$ so that J represents left multiplication by $i \in \mathbb{H}$ and L represents left multiplication by $j \in \mathbb{H}$. In that case the action of G_0 corresponds to the action by right multiplication of the group $\mathrm{Spin}(3)$ of unit quaternions. It follows from Lemma 3.4 that the assignment

$$(w_1, \ldots, w_{N/2}) \mapsto x(\lambda, 0) \mapsto f \tag{54}$$

given by (40) intertwines the action of $\mathbb{H}^*$ (by right multiplication) with the action of $\mathbb{R}^+ \times G_0$ (by dilations and symplectic isometries).

We may allow w_j to take any value in $\mathbb{H}$ provided they are not all zero: the subset on which they are non-zero will correspond to Δ_f. So (54) descends to a bijection. In fact if we give the latter the manifold structure it inherits from the Banach space $C^2(S^1 \times S^1, \mathbb{R}^4)$, equipped with norm of uniform convergence on derivatives up to second order, we see this must be a diffeomorphism.

Proposition 5.6. Set $n = \frac{N}{2} - 1$. Whenever $N > 0$, the map $\mathbb{HP}^n \to \mathcal{M}(\Gamma, \beta_0)$ defined by (54) is a diffeomorphism.

Of course, the two moduli spaces $\mathcal{S}(\Gamma, \beta_0)$ and $\mathcal{M}(\Gamma, \beta_0)$ are not isomorphic, since the first labels T_0-orbits and the second labels G_0-orbits. Hence the former will be a $G_0/T_0 \simeq S^2$ bundle over the latter. The next result explains exactly what this is.

Proposition 5.7. The action of G_0 on $\mathcal{S}(\Gamma, \beta_0)$ via base point preserving isometries is equivalent, in appropriate homogeneous coordinates, to the action of $SU(2)$ on $\mathbb{CP}^{N-1}$ whose orbits are the fibres of $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$.

Proof. If we take the coordinates on $\mathcal{M}(\Gamma, \beta_0)$ from (54) then we must make a change coordinates on the real manifold $\mathcal{S}(\Gamma, \beta_0)$: we identify the spectral data for f with the point

$$\left[\chi_1(s_1, 0), \chi_2(s_1, 0), \ldots, \chi_1(s_{n+1}, 0), \chi_2(s_{n+1}, 0)\right] \in \mathbb{CP}^{2n+1}. \tag{55}$$

Here we are using the fact, from (33), that $\chi_2(s_j, 0) = i s_j^{N+1} p(0) \overline{\chi_1(s_{j+N/2}, 0)}$ for $1 \leqslant j \leqslant n+1$. On the other hand the point in $\mathcal{M}(\Gamma, \beta_0)$ corresponding to f is identified by the quaternionic homogeneous coordinates

$$[w_1, \ldots, w_{n+1}]_{\mathbb{H}} \in \mathbb{HP}^n, \qquad w_j = \frac{1}{\sqrt{p(0) s_j^{N+1}}} x(s_j, 0).$$

Here the identification of $\mathbb{R}^4$ with $\mathbb{H}$ is via

$$w_j = \sum_{k=1}^{4} w_{jk} \varepsilon_k = (w_{j1} + w_{j2} J + w_{j3} L + w_{j4} J L) \varepsilon_1.$$

Now we observe that

$$w_j = \frac{1}{\sqrt{2}} \big((w_{j1} + i w_{j2}) v_1 + (w_{j3} + w_{j4}) v_2 + (w_{j1} - i w_{j2}) v_3 + (w_{j3} - w_{j4}) v_4 \big),$$

and therefore

$$\chi_1(s_j, 0) = \sqrt{\frac{s_j^{N+1} p(0)}{2}} (w_{j1} + i w_{j2}), \qquad \chi_2(s_j, 0) = \sqrt{\frac{s_j^{N+1} p(0)}{2}} (w_{j3} + i w_{j4}).$$

Therefore the natural fibration $\mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ obtained by the identification $\mathbb{H} = \mathbb{C} + \mathbb{C} j$ matches the map $\mathcal{S}(\Gamma, \beta_0) \to \mathcal{M}(\Gamma, \beta_0)$ in the chosen coordinates. $\quad\square$

6. Higher flows and Hamiltonian variations

The action of the real subgroup $J_R \subset \mathrm{Jac}(X)$ generates the so-called higher flows. These give Lagrangian variations for our immersed Lagrangian tori, and we will show that a codimension 1 subspace of these are actually Hamiltonian variations. First we need to explain what we mean by Lagrangian and Hamiltonian vector fields along a Lagrangian immersion.

For a Lagrangian immersion $f : M \to N$ into a Kähler manifold N a section $T \in \Gamma(f^{-1} T N)$ is called a Lagrangian vector field *along* f if $\sigma_T = f^*(T \lrcorner \omega)$ is a closed 1-form, and T is Hamiltonian when σ_T is exact. We denote the vector space of Lagrangian vector fields along f by $\mathcal{X}_{\mathrm{Lag}}(f)$ and use $\mathcal{X}_{\mathrm{Ham}}(f)$ to denote the subspace of Hamiltonian vector fields. As one expects, families of Lagrangian immersions give rise to Lagrangian vector fields along an immersion [6].

Now we consider the Lagrangian variations corresponding to the higher flows. For simplicity set $\mathcal{S} = \mathcal{S}(\Gamma, \beta_0)$. Let $\mathcal{H} \simeq \mathbb{C}^N \setminus \{0\}$ be the space of HSL tori in $\mathbb{R}^4$ given by (49), (50) and (51) parameterised by $\mathbf{t} \in \mathbb{C}^N \setminus \{0\}$. The map $\mathbb{C}^N \to \mathbb{CP}^{N-1}$ which sends $\mathbf{t}$ to $[\mathbf{t}]$ is then the parameterisation of the map $\mathcal{H} \to \mathcal{S}$ which assigns to each HSL torus its spectral data. Suppose $f \in \mathcal{H}$ corresponds to $\mathbf{t} = (t_1, \ldots, t_N)$. To each $\mathbf{a} = (a_j) \in \mathbb{C}^N$ we can assign the Lagrangian vector field $T = (\partial f / \partial t)_{t=0}$ along f corresponding to the Lagrangian deformation $f(t)$ determined by the real curve

$$t \mapsto \big(t_1 e^{t a_1}, \ldots, t_N e^{t a_N} \big), \quad t \in \mathbb{R}, \tag{56}$$

in $\mathcal{S}$. This gives a linear map

$$\mathcal{T}_f : \mathbb{C}^N \to \mathcal{X}_{\mathrm{Lag}}(f), \qquad \mathcal{T}_f(\mathbf{a}) = T. \tag{57}$$

It induces a linear map $\mathbb{C}^N / \mathbb{C} \to T_{[\mathbf{t}]} \mathcal{S}$ which is tangent to the action of $(\mathbb{C}^\times)^N / \mathbb{C}^\times$ which generates the higher flows for f (47). Notice that what we lose in the quotient are all dilations and the rotations from T_0.

Now recall that $\mathcal{Z}^1(\mathbb{C}/\Gamma)$ carries the Hodge inner product

$$(\sigma_1, \sigma_2) = \frac{1}{A(\mathbb{C}/\Gamma)} \int_{\mathbb{C}/\Gamma} \sigma_1 \wedge *\sigma_2,$$

from which we obtain the orthogonal decomposition $\mathcal{Z}^1(\mathbb{C}/\Gamma) = \mathcal{H}^1(\mathbb{C}/\Gamma) \oplus \mathcal{B}^1(\mathbb{C}/\Gamma)$, where the first summand is the space of harmonic 1-forms. Since $d\beta$ and $*d\beta$ span $\mathcal{H}^1(\mathbb{C}/\Gamma)$ we know that $\sigma_T \in \mathcal{Z}^1(\mathbb{C}/\Gamma)$ is exact, and thus T is Hamiltonian, precisely when

$$(\sigma_T, d\beta) = 0 = (\sigma_T, *d\beta).$$

Notice that we can write

$$\sigma_T = J T \cdot df,$$

using dot product notation for the metric on $\mathbb{R}^4$ and its complex bilinear extension.

Theorem 6.1. *Let $T = \mathcal{T}_f(\mathbf{a}) \in \mathcal{X}_{\text{Lag}}(f)$ be given by* (57). *Then T is Hamiltonian along f precisely when both real equations*

$$\sum_{j=1}^{N} \frac{\text{Re}(a_j)|t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2} = 0, \qquad \sum_{j=1}^{N} \frac{\text{Re}(s_j)}{\text{Im}(s_j)} \frac{\text{Re}(a_j)|t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2} = 0 \tag{58}$$

are satisfied.

The first equation in (58) is the statement that $\mathcal{T}_f(\mathbf{a})$ is a stationary variation for area (cf. the area formula (63) below). Together, Eqs. (58) can also be derived as the condition that the Liouville form, which when pulled back along f is the 1-form $\sigma_f = Jf \cdot df$ on $\mathbb{C}/\Gamma$, has stationary Γ-periods for the variation $\mathcal{T}_f(\mathbf{a})$. These periods are well-known to be Hamiltonian isotopy invariants.

Proof. Let us begin by writing $\sigma_T = c\,dz + \bar{c}\,d\bar{z}$, $c = JT \cdot f_z$. Since we are working over a torus, we can simplify computations significantly by observing that the orthogonal projection of σ_T onto $\mathcal{H}^1(\mathbb{C}/\Gamma)$ is just its Fourier zero mode, i.e., the coefficient of e_0 in the Fourier decomposition. Hence, if c_0 denotes the Fourier zero mode of c then

$$(\sigma_T, d\beta) = \pi (c_0\,dz + \bar{c}_0\,d\bar{z}, \bar{\beta}_0\,dz + \beta_0\,d\bar{z}) = 4\pi \,\text{Re}(c_0\beta_0), \tag{59}$$

$$(\sigma_T, *d\beta) = \pi (c_0\,dz + \bar{c}_0\,d\bar{z}, -i\bar{\beta}_0\,dz + i\beta_0\,d\bar{z}) = -4\pi \,\text{Im}(c_0\beta_0). \tag{60}$$

To compute c_0 we first simplify the formula (49) by writing it as

$$f = Av_1 + \bar{A}\bar{v}_1 + Bv_2 + B\bar{v}_2,$$

using

$$A = \frac{1}{p(1)}\big(\chi_1(1,z)e_{\beta_0/2} - \chi_1(1,0)\big), \qquad B = \frac{1}{ip(1)}\big(\chi_1(-1,z)e_{-\beta_0/2} - \chi_1(-1,0)\big).$$

Since $f(0) = 0$ we may write each of the functions $A, B : \mathbb{C}/\Gamma \to \mathbb{C}$ as

$$A = \sum_{j=1}^{N} A_j(e_{-\gamma_j+\beta_0/2} - 1), \qquad B = \sum_{j=1}^{N} B_j(e_{-\gamma_j-\beta_0/2} - 1),$$

and, using Eqs. (50) and (51),

$$A_j = \frac{(-1)^{j-1}}{p(1)} \Delta_j t_j, \qquad B_j = \frac{(-1)^{j-1}}{ip(1)} \tilde{\Delta}_j t_j.$$

The deformation $f(t)$ is given by

$$A(t) = \sum_{j=1}^{N} e^{ta_j} A_j(e_{-\gamma_j+\beta_0/2} - 1), \qquad B(t) = \sum_{j=1}^{N} e^{ta_j} B_j(e_{-\gamma_j-\beta_0/2} - 1).$$

Since $v_1, \bar{v}_1, v_2, \bar{v}_2$ is a Hermitian orthonormal frame for $\mathbb{C}^4$ we find that

$$Jf_t \cdot f_z = i\left[\frac{\partial A}{\partial t}\frac{\partial \bar{A}}{\partial z} - \frac{\partial \bar{A}}{\partial t}\frac{\partial A}{\partial z} - \frac{\partial B}{\partial t}\frac{\partial \bar{B}}{\partial z} + \frac{\partial \bar{B}}{\partial t}\frac{\partial B}{\partial z} \right].$$

We compute at $t = 0$

$$\partial A/\partial t = \sum_{j=1}^{N} a_j A_j(e_{\beta_0/2-\gamma_j} - 1), \qquad \partial \bar{A}/\partial t = \sum_{j=1}^{N} \bar{a}_j \bar{A}_j(e_{\gamma_j-\beta_0/2} - 1),$$

$$\partial A/\partial z = i\pi \sum_{j=1}^{N} \left(\frac{\bar{\beta}_0}{2} - \bar{\gamma}_j\right) A_j e_{\beta_0/2-\gamma_j}, \qquad \partial \bar{A}/\partial z = -i\pi \sum_{j=1}^{N} \left(\frac{\bar{\beta}_0}{2} - \bar{\gamma}_j\right) \bar{A}_j e_{\gamma_j-\beta_0/2}$$

and similar expressions for B. The Fourier zero mode c_0 of $JT \cdot f_z$ is given by the following calculation. To simplify it, we let V_N stand for the Vandermonde determinant $\prod_{1 \leqslant k < \ell \leqslant N}(s_\ell - s_k)$ (cf. Appendix A below).

$$c_0 = \pi \left(\sum_{j=1}^{N} \left(\frac{\bar{\beta}_0}{2} - \bar{\gamma}_j \right) (a_j + \bar{a}_j) |A_j|^2 - \sum_{j=1}^{N} \left(-\frac{\bar{\beta}_0}{2} - \bar{\gamma}_j \right) (a_j + \bar{a}_j) |B_j|^2 \right)$$

$$= 2\pi \sum_{j=1}^{N} \left(\left(\frac{\bar{\beta}_0}{2} - \bar{\gamma}_j \right) |A_j|^2 + \left(\frac{\bar{\beta}_0}{2} + \bar{\gamma}_j \right) |B_j|^2 \right) \mathrm{Re}(a_j)$$

$$= \frac{\pi \bar{\beta}_0}{|p(1)|^2} \sum_{j=1}^{N} \left((1 - \bar{s}_j) |\Delta_j|^2 + (1 + \bar{s}_j) |\tilde{\Delta}_j|^2 \right) \mathrm{Re}(a_j) |t_j|^2$$

$$= \pi \bar{\beta}_0 |V_N|^2 \sum_{j=1}^{N} \left(\frac{1}{1 - s_j} + \frac{1}{1 + s_j} \right) \frac{\mathrm{Re}(a_j) |t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2}$$

where we have used the property $p(1) = p(-1)$ to write

$$|\Delta_j|^2 = \frac{|p(1)|^2 |V_N|^2}{|1 - s_j|^2 \prod_{k \neq j} |s_k - s_j|^2} = \frac{|1 + s_j|^2}{|1 - s_j|^2} |\tilde{\Delta}_j|^2.$$

It follows that

$$(\sigma_T, d\beta) = 4\pi^2 |\beta_0|^2 |V_N|^2 \sum_{j=1}^{N} \frac{\mathrm{Re}(a_j) |t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2}, \tag{61}$$

$$(\sigma_T, *d\beta) = -4\pi^2 |\beta_0|^2 |V_N|^2 \sum_{j=1}^{N} \frac{\mathrm{Re}(s_j) \mathrm{Re}(a_j) |t_j|^2}{\mathrm{Im}(s_j) \prod_{k \neq j} |s_k - s_j|^2}. \qquad \square \tag{62}$$

Remark 6.2. It is interesting to note that, given the isomorphism $J_R \simeq (S^1)^{N-1} \times (\mathbb{R}^+)^{N-1}$, all the higher flows tangent to the compact factor of J_R are Hamiltonian, but not *all* higher flows are Hamiltonian, in the sense that there are vector fields $\mathcal{T}_f(\mathbf{a})$ which are not Hamiltonian and whose projections onto $\mathcal{S}$ give non-trivial higher flows. This is a reflection of the fact that the area functional is non-constant on $\mathcal{S}$. Using the expressions above it is straightforward to show that the area of $f(\mathbb{C}/\Gamma)$, when f corresponds to $[\mathbf{t}] \in \mathbb{CP}^{N-1}$, is given by

$$A(f) = \int_{\mathbb{C}/\Gamma} |f_z|^2 \, dz \wedge d\bar{z} = A(\mathbb{C}/\Gamma) \frac{\pi^2 |\beta_0|^2 |V_N|^2}{2} \sum_{j=1}^{N} \frac{|t_j|^2}{\prod_{k \neq j} |s_k - s_j|^2}. \tag{63}$$

Here, of course, we run into the problem that the dilations act on the area but have been factored out of the space $\mathcal{S}$. Therefore we restrict $A(f)$ to the unit sphere $S^{2N-1} \subset \mathbb{C}^N$: it then descends to $\mathbb{CP}^{N-1}$. It is tantalising to observe that, up to a constant independent of $\mathbf{t}$, $A(f)$ equals

$$\int_{\mathbb{C}/\Gamma} \left| \theta_\infty \left(\mathbf{t}(z) \right) \right|^2 dz \wedge d\bar{z} \tag{64}$$

where θ_∞ is defined in (66).

Remark 6.3. Our final remark concerns the dependence of $A(f)$ on the spectral genus of f. Here we interpret the spectral genus of f to be the arithmetic genus g_f of X_f. From the earlier discussion $g_f = 4(N_1 - 1)$ where N_1 is the complex dimension of the J_R-orbit of the spectral data for f in $\mathcal{S}(\Gamma, \beta_0^*)$. The closure of these J_R-orbits provides a stratification of $\mathcal{S}(\Gamma, \beta_0^*)$, and as we have observed this agrees with the stratification of $\mathbb{CP}^{N-1}$ by $(\mathbb{C}^\times)^N / \mathbb{C}^\times$-orbits. In particular, one can decrease the spectral genus by going to the boundary of a J_R-orbit. In our parameterisation this corresponds to taking the limit as some $t_j \to 0$.

When $A(f)$ is restricted to the unit sphere $S^{2N-1} \subset \mathbb{C}^N$, as in the previous remark, it descends to $\mathbb{CP}^{N-1}$. Clearly "going to the boundary of spectral genus" decreases $A(f)$. Moreover, it is easy to show, using Lagrange multipliers, that $A(f)$ has critical points only at the homogeneous tori (those which arise as orbits of a homomorphism from $\mathbb{C}/\Gamma$ into G). These are all minima of $A(f)$. These are also the HSL tori with lowest (non-trivial) spectral genus. While both of these observations are elementary for this geometry, they are an echo of what one hopes to find in other, more complicated, integrable surface theory (cf. the work of Kilian and Schmidt [9] on deformations of CMC tori into cylinders, which uses a form of "going to the boundary of spectral data" to decrease spectral genus).

Appendix A. The θ-function

Here we will prove Proposition 5.3 and also explain its geometric meaning: it is essentially the Riemann vanishing theorem for a singular curve related to our spectral curve X.

At an algebraic level the proof is straightforward. For any $h \in F_{N-1}$ write $h(\lambda) = \sum_{k=0}^{N-1} h_k \lambda^k$. Given $[\mathbf{t}] = [t_1, \ldots, t_N] \in \mathbb{CP}^{N-1}$ we have $\iota([h]) = [\mathbf{t}]$ precisely when we can solve the linear system

$$\begin{pmatrix} 1 & s_1 & \ldots & s_1^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_{N-1} & \ldots & s_{N-1}^{N-1} \\ 1 & s_N & \ldots & s_N^{N-1} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_{N-2} \\ h_{N-1} \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_{N-1} \\ t_N \end{pmatrix}. \tag{65}$$

The matrix has determinant $V_N = \prod_{m>j}(s_m - s_j)$ (it is a Vandermonde matrix), hence it is invertible whenever $s_1, \ldots, s_N$ are distinct. Hence ι is an isomorphism. Further, $\iota([\varphi(\lambda, \mathbf{t})]) = [\mathbf{t}]$ since

$$\varphi(s_j, \mathbf{t}) = \left(\prod_{m \neq j} (s_m - s_j) \right) \det \begin{pmatrix} 1 & s_1 & \ldots & s_1^{N-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_j & \ldots & s_j^{N-2} & t_j \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_N & \ldots & s_N^{N-2} & 0 \end{pmatrix} = t_j V_N.$$

Finally, by applying Cramer's rule to (65) it follows that $h_{N-1} = 0$ precisely when

$$\theta_\infty(\mathbf{t}) = \det \begin{pmatrix} 1 & s_1 & \ldots & s_1^{N-2} & t_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_N & \ldots & s_N^{N-2} & t_N \end{pmatrix} = V_N \sum_{j=1}^{N} \left(\prod_{m \neq j} \frac{1}{s_m - s_j} \right) t_j = 0. \tag{66}$$

Similarly, $h_0 = 0$ when

$$\theta_0(\mathbf{t}) = \begin{pmatrix} t_1 & s_1 & \ldots & s_1^{N-1} \\ \vdots & \vdots & & \vdots \\ t_N & s_N & \ldots & s_N^{N-1} \end{pmatrix} = \left(\prod_{j=1}^{N} s_j \right) V_N \sum_{j=1}^{N} \left(\prod_{m \neq j} \frac{1}{s_m - s_j} \right) \frac{t_j}{s_j} = 0. \tag{67}$$

This completes the proof.

The geometric meaning is as follows (cf. the discussion of rational nodal curves in [13, 3.251]). Let Y be the rational singular curve obtained from $\hat{\mathbb{C}}$ by identifying the points $s_1, \ldots, s_N$ into one singularity s. This curve has arithmetic genus $g = N - 1$ and one can show that $\mathrm{Jac}(Y) \simeq (\mathbb{C}^\times)^{N-1}$. It has an Abel map

$$\mathcal{A}_\infty : Y \setminus \{s\} \to \mathrm{Jac}(Y), \qquad Q \mapsto \mathcal{O}_Y(Q - \infty).$$

We can identify $Y \setminus \{s\}$ with $\hat{\mathbb{C}} \setminus \mathfrak{S}$ and $\mathrm{Jac}(Y)$ with

$$\{[t_1, \ldots, t_{N-1}, t_N] \in \mathbb{CP}^{N-1} : t_j \neq 0 \forall j\}.$$

Since $\mathcal{O}_Y(Q - \infty)$ has unique, up to scale, global section $\lambda - Q$ the Abel map extends holomorphically to the map

$$\mathcal{A}_\infty : \hat{\mathbb{C}} \to \mathbb{CP}^{N-1}, \qquad \mathcal{A}_\infty(Q) = [s_1 - Q, \ldots, s_N - Q].$$

By analogy with the case of compact Riemann surfaces, we say the θ-divisor for Y (given base point ∞) is the image of $(Y \setminus \{s\})^{(N-2)}$ in $\mathrm{Jac}(Y)$ under the Abel map on divisors (note that $N - 2 = g - 1$). This image is the set of all divisor classes of the form $[E - (N - 2).\infty]$, hence corresponds to polynomials of degree strictly less than $N - 1$. Therefore this θ-divisor is $\mathrm{Jac}(Y) \cap \Theta_\infty$. Similarly, Θ_0 is a translate of this θ-divisor. The function φ plays the role of a translate of the θ-function pulled back to Y along the Abel map. In this context Proposition 5.3 is the analogue of Riemann's vanishing theorem.

The link with our spectral curve X is that there is a isomorphism of real groups $\mathrm{Jac}(Y) \simeq J_R$ given by identifying $Y \setminus \{s\}$ with $C_1 \setminus \mathfrak{S}$ and mapping

$$\mathcal{O}_Y(D) \to \mathcal{O}_X(D + \rho^* D + \tau^* D + (\rho\tau)^* D), \quad D \in \mathrm{Div}_0(Y \setminus \{s\}).$$

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