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Co-recursive associated Jacobi polynomials

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Abstract

In this note we present some results on co-recursive associated Jacobi polynomials, with a special attention to some simple limiting cases. Explicit representations, orthogonality measures and fourth-order differential equations satisfied by the polynomials are presented.

Keywords: Orthogonal polynomials; Hypergeometric functions

1. Introduction

Starting from a sequence of orthogonal polynomials $\{P_n\}_{n \geqslant 0}$ defined by the recurrence relation

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geqslant 0 \tag{1}$$

and the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0, \tag{2}$$

with $\beta_n, \gamma_n \in \mathbb{C}$ and $\gamma_n \neq 0$, we obtain the co-recursive associated polynomials by replacing n by $n + c$ in the coefficients β_n and γ_n and β_0 by $\beta_0 + \nu$ (keeping $\gamma_n \neq 0$) [2].

The purpose of this note is to present preliminary results on co-recursive associated Jacobi (CAJ) polynomials. The methods used are the same as in the study of the co-recursive associated Laguerre (CAL) polynomials presented at the Granada Symposium [8]. In Section 2 we give an explicit expression for the CAJ polynomials and in Section 3 the absolutely continuous part of the orthogonality measure. In Section 4 we obtain, using Orr's method, the fourth-order differential equation satisfied by the CAJ polynomials in some limiting cases, among which are some new simple cases of associated Jacobi polynomials.

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2. Explicit representation

The recurrence relation of the associated Jacobi polynomials $P_n^{\alpha,\beta}(x;c)$ is [13]

$$(2n+2c+\alpha+\beta+1)\,[(2n+2c+\alpha+\beta+2)(2n+2c+\alpha+\beta)x+\alpha^2-\beta^2]\,p_n$$
$$=2(n+c+1)(n+c+\alpha+\beta+1)(2n+2c+\alpha+\beta)p_{n+1}$$
$$+\,2(n+c+\alpha)(n+c+\beta)(2n+2c+\alpha+\beta+2)p_{n-1}. \tag{3}$$

This recurrence relation (3) is invariant under the transformation $\mathscr{T}$ defined by

$$\mathscr{T}(c,\alpha,\beta)=(c+\alpha+\beta,\ -\alpha,\ -\beta). \tag{4}$$

As in [13] we use the more convenient shifted polynomials defined as

$$R_n^{\alpha,\beta}(x;c)=P_n^{\alpha,\beta}(2x-1;c). \tag{5}$$

A solution of the recurrence relation satisfied by the $R_n^{\alpha,\beta}(x;c)$ in terms of the hypergeometric function is [9, p. 280]

$$u_n=\frac{(c+\alpha+1)_n}{(c+1)_n}\,{}_2F_1\left(\begin{matrix}-n-c,n+c+\alpha+\beta+1\\1+\alpha\end{matrix};1-x\right) \tag{6}$$

and an other linearly independent solution is given by

$$v_n=\mathscr{T}u_n=\frac{(c+\beta+1)_n}{(c+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix}-n-c-\alpha-\beta,n+c+1\\1-\alpha\end{matrix};1-x\right). \tag{7}$$

The functions u_n and $x^{-\beta}(1-x)^{-\alpha}v_n$ are two independent solutions of the second-order differential equation

$$x(1-x)y''(x)+[1+\beta-(\alpha+\beta+2)x]y'(x)+(n+c)(n+c+\alpha+\beta+1)y(x)=0. \tag{8}$$

The associated Jacobi polynomials are defined by (3) and the initial condition

$$P_{-1}^{\alpha,\beta}(x;c)=0,\qquad P_0^{\alpha,\beta}(x;c)=1. \tag{9}$$

This gives for $P_1^{\alpha,\beta}(x;c)$

$$P_1^{\alpha,\beta}(x;c)=\frac{(2c+\alpha+\beta+1)(2c+\alpha+\beta+2)}{2(c+1)(c+\alpha+\beta+1)}\times\left[x+\frac{\alpha^2-\beta^2}{(2c+\alpha+\beta)(2c+\alpha+\beta+2)}\right]. \tag{10}$$

The CAJ polynomials $P_n^{\alpha,\beta}(x;c,\mu)$ satisfy the recurrence relation (3) with a shift μ on the monic polynomial of first degree. This corresponds to the initial condition on the shifted CAJ polynomials $R_n^{\alpha,\beta}(x;c,\mu)$

$$R_{-1}^{\alpha,\beta}(x;c,\mu)=D=-\frac{(2c+\alpha+\beta)(2c+\alpha+\beta+1)}{2(c+\alpha)(c+\beta)}\mu,\quad R_0^{\alpha,\beta}(x;c,\mu)=1, \tag{11}$$

with

$$R_n^{\alpha,\beta}(x; c, \mu) = P_n^{\alpha,\beta}(2x - 1; c, 2\mu). \tag{12}$$

If $c + \alpha \to 0$ or $c + \beta \to 0$, D is not defined; nevertheless this initial condition in (3) leads to a shift μ on the value of x in $P_1^{\alpha,\beta}(x; c)$.

As in [8] writing

$$R_n^{\alpha,\beta}(x; c, \mu) = Au_n + Bv_n \tag{13}$$

and using the initial condition (11) we obtain

$$A = \frac{1}{\Delta}[Dv_0 - v_{-1}] \quad \text{and} \quad B = -\frac{1}{\Delta}[Du_0 - u_{-1}], \tag{14}$$

where Δ is easily calculated using the fact that u_n and $x^{-\beta}(1 - x)^{-\alpha}v_n$ are two independent solutions of (8),

$$\Delta = u_{-1}v_0 - u_0v_{-1} = -\frac{\alpha(2c + \alpha + \beta)}{(c + \alpha)(c + \beta)}. \tag{15}$$

The condition $\Delta \neq 0$ leads to $\alpha \neq 0$ and $2c + \alpha + \beta \neq 0$. Noting the invariance of D and Δ under $\mathscr{T}$ one sees that $B = \mathscr{T}A$.

Grouping the two ${}_2F_1$ involved in expression (14) of A, we may write the CAJ polynomials

$$\begin{aligned} R_n^{\alpha,\beta}(x; c, \mu) = {} & (1 + \mathscr{T})\frac{c + \alpha + \beta - D(c + \beta)}{\alpha(2c + \alpha + \beta)}(c + \alpha)\frac{(c + \alpha + 1)_n}{(c + 1)_n} \\ & \times {}_3F_2\left(\begin{matrix} -c - \alpha - \beta, c, F + 1 \\ 1 - \alpha, F \end{matrix}; 1 - x\right) \\ & \times {}_2F_1\left(\begin{matrix} -n - c, n + c + \alpha + \beta + 1 \\ 1 + \alpha \end{matrix}; 1 - x\right) \end{aligned} \tag{16}$$

with

$$F = \frac{c[D(c + \beta) - c - \alpha - \beta]}{D(c + \beta) + c}. \tag{17}$$

Transforming the ${}_2F_1(1 - x)$ in (13) by [3, Eq. (1), p. 108] one obtains with a little algebra

$$\begin{aligned} R_n^{\alpha,\beta}(x; c, \mu) = {} & (1 + \mathscr{T})\frac{(-1)^n c(c + \alpha)}{\beta(2c + \alpha + \beta)} \\ & \times \frac{(c + \alpha + 1)_n}{(c + \alpha + \beta + 1)_n}\, {}_2F_1\left(\begin{matrix} -n - c - \alpha - \beta, n + c + 1 \\ 1 - \beta \end{matrix}; x\right) \\ & \times \left[\frac{c + \beta}{c} D\, {}_2F_1\left(\begin{matrix} -c, c + \alpha + \beta + 1 \\ 1 + \beta \end{matrix}; x\right) - {}_2F_1\left(\begin{matrix} 1 - c, c + \alpha + \beta \\ 1 + \beta \end{matrix}; x\right)\right]. \end{aligned} \tag{18}$$

This formula generalizes the one of [13, Eq. (28)] to the case of the CAJ polynomials. The representations (16) and (18) are valid only for $\alpha \neq 0, \pm 1, \pm 2 \ldots$ and $\beta \neq 0, \pm 1, \pm 2 \ldots$ but can be extended by limiting processes.

We obtain an explicit formula following the same way as in [13]. We first use [3, Eq. (14), p. 87] for each product of ${}_2F_1$ in (18) to obtain four series involving gamma functions and a ${}_4F_3$. For two of them we use [1, Eq. (1), p. 56]. The next step is to use for each ${}_4F_3$ twice [1, Eq. (3), p. 62]. After numerous cancellations only two series of ${}_4F_3$ remain, which we can group to obtain the following explicit form

$$R_n^{\alpha,\beta}(x; c, \mu) = (-1)^n \frac{(2c+\alpha+\beta+1)_n(\beta+c+1)_n}{n!\,(c+\alpha+\beta+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k} \times {}_5F_4\left(\begin{matrix} k-n, n+k+2c+\alpha+\beta+1, c, c+\beta, G+1 \\ c+k+1, c+\beta+k+1, 2c+\alpha+\beta+1, G \end{matrix}; 1\right) x^k, \tag{19}$$

where

$$G = \frac{2c(c+\beta)(2c+\alpha+\beta)}{2c(c+\beta)+\mu(2c+\alpha+\beta)(2c+\alpha+\beta+1)}. \tag{20}$$

3. Spectral measure

The Stieltjes transform of the measure of the shifted associated Jacobi polynomials $R_n^{\alpha,\beta}(x; c)$ is [13, Eqs. (63) and (64)]

$$s(p) = \frac{1}{p}\, {}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+2 \end{matrix}; \frac{1}{p}\right) {}_2F_1\left(\begin{matrix} c, c+\beta \\ 2c+\alpha+\beta \end{matrix}; \frac{1}{p}\right)^{-1}. \tag{21}$$

The CAJ polynomials $R_n^{\alpha,\beta}(x; c, \mu)$ satisfy the same recurrence relations as the $R_n^{\alpha,\beta}(x; c)$ with

$$R_1^{\alpha,\beta}(x; c, \mu) - R_1^{\alpha,\beta}(x; c) = \frac{(2c+\alpha+\beta+1)(2c+\alpha+\beta+2)}{2(c+1)(c+\alpha+\beta+1)}\,\mu. \tag{22}$$

Using continued J-fractions [4, 6, 12] whose denominators are $R_n^{\alpha,\beta}(x; c, \mu)$ and $R_n^{\alpha,\beta}(x; c)$ we can derive for the Stieltjes transform of the measure

$$\begin{aligned} s(p; \mu) &= s(p)\left(1 + \frac{\mu}{2} s(p)\right)^{-1} \\ &= \frac{{}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+2 \end{matrix}; \frac{1}{p}\right)}{p\, {}_2F_1\left(\begin{matrix} c, c+\beta \\ 2c+\alpha+\beta \end{matrix}; \frac{1}{p}\right) + \frac{\mu}{2}\, {}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+2 \end{matrix}; \frac{1}{p}\right)}, \end{aligned} \tag{23}$$

which, using contiguous relations, we can also write

$$s(p; \mu) = {}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+2 \end{matrix}; \frac{1}{p}\right)\left[{}_3F_2\left(\begin{matrix} c, c+\beta, G+1 \\ 2c+\alpha+\beta+1, G \end{matrix}; \frac{1}{p}\right)\right]^{-1}, \tag{24}$$

where G is given by (20). A sufficient condition for the positivity of the denominator in (24) on $(1, \infty)$ is

$$c \geqslant 0, \quad c > -\beta, \quad \alpha > -1, \quad \mu \geqslant -\frac{2c(c+\beta)}{(2c+\alpha+\beta)(2c+\alpha+\beta+1)}, \tag{25}$$

but other conditions are possible.

To obtain the absolutely continuous part of the spectral measure we need to evaluate $s^+(p;\mu) - s^-(p;\mu)$ where $s^\pm$ are the values of s above and below the cut $[0, 1]$. Using the analytic continuation [3, Eq. (2), p. 108] for each ${}_2F_1$ in (23) we find

$$\begin{aligned}\phi'(x) &= (1-x)^\alpha x^{\beta+2c} \left| {}_2F_1\left(\begin{matrix} c, c+\beta \\ 2c+\alpha+\beta \end{matrix}; \frac{\mathrm{e}^{\mathrm{i}\pi}}{x}\right) + \frac{\mu}{2x}\, {}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+2 \end{matrix}; \frac{\mathrm{e}^{\mathrm{i}\pi}}{x}\right)\right|^{-2} \\ &= (1-x)^\alpha x^{\beta+2c} \left| {}_3F_2\left(\begin{matrix} c, c+\beta, G+1 \\ 2c+\alpha+\beta+1, G \end{matrix}; \frac{\mathrm{e}^{\mathrm{i}\pi}}{x}\right)\right|^{-2}, \end{aligned} \tag{26}$$

which is certainly valid under conditions (25).

4. Fourth-order differential equation

To obtain the differential equation satisfied by the $R_n^{\alpha,\beta}(x; c, \mu)$ we use the Orr method [10]. In (16) the hypergeometric function ${}_2F_1$ is a solution of Eq. (8) and the ${}_3F_2$ is of the form

$${}_3F_2\left(\begin{matrix} a, b, e+1 \\ d, e \end{matrix}; x\right)$$

which we can prove to be a solution of the second-order differential equation

$$\begin{aligned} &x(x-1)[(a-e)(b-e)x + e(d-e-1)]\,y''(x) \\ &\quad + \{(a-e)(b-e)(a+b+1)x^2 + [e(a+b+1)(2d-e-2) - d(ab+e^2) + ab]x \\ &\qquad + de(e-d+1)\}\,y'(x) + ab[(a-e)(b-e)x + (e+1)(d-e-1)]\,y(x) = 0. \end{aligned} \tag{27}$$

The general form of the fourth-order differential equation is

$$c_4 y^{(4)}(x) + c_3 y^{(3)}(x) + c_2 y^{(2)}(x) + c_1 y^{(1)}(x) + c_0 y(x) = 0, \tag{28}$$

where the coefficients c_i hardly obtained by symbolic MAPLE computation, are at most of degree eight in x. It would take several pages to write them so we will give the results only in the following particular cases.

4.1. Laguerre case limit

The limit giving the CAL polynomial case is obtained by the replacement

$$\left.\begin{array}{l} x \to 1 - \dfrac{2x}{\beta} \\ \mu \to +\dfrac{2\mu}{\beta} \end{array}\right\} \quad \beta \to \infty . \tag{29}$$

The representation (16) is the more suitable to obtain the form of the CAL polynomials studied in [8]. Using the Kummer transformation [3, p. 253] for one of the ${}_2F_1$ and his generalization

$${}_2F_2\left(\begin{matrix} a, e+1 \\ c, e \end{matrix}; x\right) = \mathrm{e}^x {}_2F_2\left(\begin{matrix} c-a-1, \dfrac{e(c-a-1)}{e-a}+1 \\ c, \dfrac{e(c-a-1)}{e-a} \end{matrix}; -x\right) \tag{30}$$

for one of the ${}_2F_2$.

4.2. Limit $c = 0$

In this limit we obtain the co-recursive Jacobi polynomials. An explicit form is

$$\begin{aligned} R_n^{\alpha,\beta}(x;\mu) = (-1)^n \frac{(\beta+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha+\beta+1)_k}{k!(\beta+1)_k} x^k \\ \times \left\{ 1 + \frac{\mu(k-n)(n+k+\alpha+\beta+1)}{2(k+1)(\beta+k+1)} \right. \\ \left. \times {}_4F_3\left(\begin{matrix} k-n+1, n+k+\alpha+\beta+2, \beta+1, 1 \\ k+2, \beta+k+2, \alpha+\beta+2 \end{matrix}; 1\right)\right\}, \end{aligned} \tag{31}$$

and the spectral measure is given by

$$\phi'(x) = (1-x)^\alpha x^\beta \left| 1 + \frac{\mu}{2x} {}_2F_1\left(\begin{matrix} 1, \beta+1 \\ \alpha+\beta+2 \end{matrix}; \frac{\mathrm{e}^{\mathrm{i}\pi}}{x}\right)\right|^{-2}. \tag{32}$$

The limit $\mu = 0$ leads back to the Jacobi polynomials.

The fourth-order differential equation satisfied by the co-recursive Laguerre polynomials can be factorized in the limit $c = 0$ to obtain as in [11] the factorized (2 + 2) differential equation

$$\begin{aligned} 0 = {} & [(1-x^2)A(x)\mathrm{D}^2 + \{(\beta-\alpha-(\alpha+\beta+4)x)A(x) - (1-x^2)B(x)\}\mathrm{D} \\ & + \{n(n+\alpha+\beta+1) - (\alpha+\beta+2)\}A(x) + \{\beta-\alpha-(\alpha+\beta+2)x\}B(x) + C(x)] \\ & \times [(1-x^2)\mathrm{D}^2 + \{(\alpha+\beta-2)x+\alpha-\beta\}\mathrm{D} + n(n+\alpha+\beta+1) + \alpha+\beta]\, R_n^{\alpha,\beta}(x;\mu), \end{aligned} \tag{33}$$

where

$$\begin{aligned}A(x) = {} & 2(\alpha+\beta)^2(2n+1)(n+\alpha+\beta+\tfrac{1}{2})x^2 + 2(\alpha+\beta)(4n(n+\alpha+\beta+1) \\ & \times(-\mu(1+\alpha+\beta)+\alpha-\beta)+(1+\alpha+\beta)(-\mu(\alpha+\beta+2)+2\alpha-2\beta))x \\ & + 4n(n+\alpha+\beta+1)(-\mu(1+\alpha+\beta)+\alpha-\beta)^2-(\alpha+\beta)(-2\mu(1+\alpha+\beta) \\ & \times(\beta-\alpha)-2(\beta-\alpha)^2-3\alpha-3\beta), \\ B(x) = {} & -(\alpha+\beta)((\alpha+\beta)(8n(n+\alpha+\beta+1)+3\alpha+3\beta)x+8n(n+\alpha+\beta+1) \\ & \times(-\mu(1+\alpha+\beta)+\alpha-\beta)-2(\alpha+\beta+2)(1+\alpha+\beta)\mu+(\alpha-\beta)(3\alpha+3\beta+4)), \\ C(x) = {} & -(\alpha+\beta)((\alpha+\beta)(\alpha+\beta+2)(2n(n+\alpha+\beta+1)+\alpha+\beta-1)x^2 \\ & + 2(n(n+\alpha+\beta+1)(-\mu(\alpha+\beta+1)(\alpha+\beta-4)+2(\alpha-\beta)(\alpha+\beta-2)) \\ & + (\alpha+\beta)(\alpha-\beta)(\alpha+\beta-1))x - 2n(n+\alpha+\beta+1)(-\mu(\alpha+\beta+1)(\beta-\alpha) \\ & - (\alpha-\beta)^2+6\alpha+6\beta)+(\alpha+\beta)((\alpha-\beta)^2-3\alpha-3\beta-6)).\end{aligned}$$

4.3. Limit $c = -\alpha-\beta$

Due to the $\mathscr{T}$ invariance of (3) we obtain in this limit the special case of CAJ polynomials for which

$$R_n^{\alpha,\beta}(x; -\alpha-\beta, \mu) = R_n^{-\alpha,-\beta}(x;\mu). \tag{34}$$

All the results are obtained from Section 4.2 by changing α to $-\alpha$ and β to $-\beta$.

4.4. Limit $c = -\beta$

The explicit form (19) simplifies in the same way as in the case $c = 0$. One obtains

$$\begin{aligned}R_n^{\alpha,\beta}(x; -\beta, \mu) = {} & (-1)^n \frac{(\alpha-\beta+1)_n}{(\alpha+1)_n} \sum_{k=0}^{n} \frac{(-n)_k (n+\alpha-\beta+1)_k}{k!(1-\beta)_k} x^k \\ & \times \left\{ 1 + \frac{\mu(k-n)(n+k+\alpha-\beta+1)}{2(k+1)(1-\beta+k)} \right. \\ & \left. \times {}_4F_3\left(\begin{matrix} k-n+1, n+k+\alpha-\beta+2, 1-\beta, 1 \\ k+2, 2-\beta+k, \alpha-\beta+2 \end{matrix} ; 1 \right) \right\}.\end{aligned} \tag{35}$$

Comparing this form with the explicit form of the co-recursive Jacobi polynomials (32) one sees that

$$R_n^{\alpha,\beta}(x; -\beta, \mu) = \frac{n!(\alpha-\beta+1)_n}{(\alpha+1)_n(1-\beta)_n} R_n^{\alpha,-\beta}(x;\mu). \tag{36}$$

The spectral measure and the fourth-order differential equation are obtained from (32) and (33) by changing β to $-\beta$.

4.5. Limit $c = -\alpha$

This case is the $\mathscr{T}$ transform of the preceding case. All the results are obtained from Section 4.4 by changing α to $-\alpha$ and β to $-\beta$.

4.6. Limit $\mu = 0$

In this limit we obtain the associated Jacobi polynomials studied in [13]. The form [13, Eq. (28)] may be obtained directly using (18) but a slightly different form is

$$R_n^{\alpha,\beta}(x;c) = (1+\mathscr{T})\frac{(c+\alpha)(c+\alpha+\beta)}{\alpha(2c+\alpha+\beta)}\frac{(c+\alpha+1)_n}{(c+1)_n}$$

$$\times {}_2F_1\left(\begin{matrix}1-c-\alpha-\beta, c\\ 1-\alpha\end{matrix};1-x\right){}_2F_1\left(\begin{matrix}-n-c, n+c+\alpha+\beta+1\\ 1+\alpha\end{matrix};1-x\right). \quad (37)$$

The explicit form [13, Eq. (19)] is easily obtained starting from (19) with $G = 2c+\alpha+\beta$, the ${}_5F_4$ reducing to a ${}_4F_3$. Obviously the limit $c = 0$ leads back to the Jacobi polynomials.

The coefficients of the differential equation (28) satisfied by the associated Jacobi polynomials are

$$c_4 = x^2(x-1)^2, \qquad c_3 = 5x(x-1)(2x-1), \quad (38)$$

$$c_2 = (24-(n+1)^2-A)x(x-1)-Bx-\beta^2+4, \quad (39)$$

$$c_1 = -\tfrac{3}{2}((3(n+3)(n-1)+3A)(2x-1)+B), \quad (40)$$

$$c_0 = n(n+2)A, \quad (41)$$

with

$$A = (C+n+1)(C+n-1), \qquad B = (\alpha-\beta)(\alpha+\beta), \qquad C = 2c+\alpha+\beta. \quad (42)$$

This result was also first given by Hahn [5, Eq. (20)]. Note the $\mathscr{T}$ invariance of A, B and C which implies the invariance of the c_i. Our expressions make this invariance more obvious than in [13, Eqs. (47) and (48)].

4.7. Limit $\mu = [2c(c+\alpha)]/[(\beta+\alpha+2c)(\beta+\alpha+2c+1)]$

In this limit the symmetry $\mathscr{T}$ is broken. We obtain the zero-related Jacobi polynomials studied in [7]. An explicit form is

$$\mathscr{R}_n^{\alpha,\beta}(x;c) = (-1)^n\frac{(2c+\alpha+\beta+1)_n(\beta+c+1)_n}{n!(c+\alpha+\beta+1)_n}\sum_{k=0}^{n}\frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k}$$

$$\times {}_4F_3\left(\begin{matrix}k-n, n+k+2c+\alpha+\beta+1, c, c+\beta+1\\ c+k+1, c+\beta+k+1, 2c+\alpha+\beta+1\end{matrix};1\right)x^k. \quad (43)$$

The limit $c = 0$ leads back to the Jacobi polynomials.

The spectral measure is

$$\phi'(x) = (1 - x)^{\alpha} x^{\beta + 2c} \left| {}_2F_1\left(\begin{matrix} c, c + \beta + 1 \\ 2c + \alpha + \beta + 1 \end{matrix}; \frac{e^{i\pi}}{x} \right) \right|^{-2}. \tag{44}$$

The coefficients of the differential equation (28) satisfied by the polynomials $\mathfrak{R}_n^{\alpha,\beta}(x; c)$ are

$$c_4 = x^2(x-1)^2(Ax + D), \tag{45}$$

$$c_3 = x(x-1)(8Ax^2 - 3(A - 3D)x - 4D), \tag{46}$$

$$\begin{aligned} c_2 = & -\tfrac{1}{2}A(A + 2C^2 - 29)x^3 + (\tfrac{1}{2}A(A + 2C^2 - 2B - 23) - D(C^2 - 19))x^2 \\ & -\tfrac{1}{4}(A(D+1)(D-3) - 2D(2C^2 - 2B + D - 35))x - \tfrac{1}{4}D(D^2 - 9), \end{aligned} \tag{47}$$

$$\begin{aligned} c_1 = & -A(A + 2C^2 - 5)x^2 + \tfrac{1}{4}(A(A + 2C^2 - 2B - 5D - 5) - 3D(4C^2 - 11))x \\ & + \tfrac{1}{4}D((D+3)A + 6C^2 - 6B + 3D - 15), \end{aligned} \tag{48}$$

$$c_0 = 2n(n+1)(C+n)(C+n+1)(Ax + 3D), \tag{49}$$

where B and C are defined in (42) and

$$A = (2n+1)(1 + 2C + 2n), \qquad D = 1 + 2\beta. \tag{50}$$

4.8. Limit $\mu = [2(c+\beta)(c+\alpha+\beta)]/[(\beta+\alpha+2c)(\beta+\alpha+2c+1)]$

This case is the $\mathscr{T}$ transform of Section 4.7. The explicit form is

$$\begin{aligned} \mathfrak{R}_n^{\alpha,\beta}(x; c) = & (-1)^n \frac{(2c+\alpha+\beta+1)_n(\alpha+c+1)_n}{n!(c+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+\alpha+\beta+1)_k(c+\alpha+1)_k} \\ & \times {}_4F_3\left(\begin{matrix} k-n, n+k+2c+\alpha+\beta+1, c+\alpha+\beta, c+\alpha+1 \\ c+\alpha+\beta+k+1, c+\alpha+k+1, 2c+\alpha+\beta+1 \end{matrix}; 1 \right) x^k. \end{aligned} \tag{51}$$

The limit $c = 0$ yields the co-recursive Jacobi polynomials with $\mu = 2\beta/(\alpha+\beta+1)$. The spectral measure is

$$\phi'(x) = (1 - x)^{\alpha} x^{\beta + 2c} \left| {}_2F_1\left(\begin{matrix} c + 1, c + \beta \\ 2c + \alpha + \beta + 1 \end{matrix}; \frac{e^{i\pi}}{x} \right) \right|^{-2}. \tag{52}$$

The coefficients of the differential equation (28) satisfied by the polynomials $\mathfrak{R}_n^{\alpha,\beta}(x; c)$ are obtained from (45)–(50) by changing $D = 1 + 2\beta$ by $D = 1 - 2\beta$.

4.9. Limit $\mu = -[2c(c+\beta)]/[(2c+\alpha+\beta)(2c+\alpha+\beta+1)]$

The symmetry $\mathscr{T}$ is also broken. We obtain a new simple case of CAJ polynomials. An explicit form is

$$\begin{aligned} \mathfrak{R}_n^{\alpha,\beta}(x; c) = & (-1)^n \frac{(2c+\alpha+\beta+1)_n(\beta+c+1)_n}{n!(c+\alpha+\beta+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k} \\ & \times {}_4F_3\left(\begin{matrix} k-n, n+k+2c+\alpha+\beta+1, c, c+\beta \\ c+k+1, c+\beta+k+1, 2c+\alpha+\beta+1 \end{matrix}; 1 \right) x^k \end{aligned} \tag{53}$$

and the spectral measure is

$$\phi'(x) = (1-x)^{\alpha} x^{\beta+2c} \left| {}_2F_1\left(\begin{matrix} c, c+\beta \\ 2c+\alpha+\beta+1 \end{matrix}; \frac{e^{i\pi}}{x} \right) \right|^{-2}. \tag{54}$$

The coefficients of the differential equation (28) satisfied by the polynomials $\tilde{\mathscr{R}}_n^{\alpha,\beta}(x;c)$ are

$$c_4 = x^2(x-1)^2(A(x-1)-D), \tag{55}$$

$$c_3 = x(x-1)(8Ax^2-(13A+9D)x+5A+5D), \tag{56}$$

$$\begin{aligned} c_2 = & -\tfrac{1}{2}A(A+2C^2-29)x^3+(A(A+2C^2-B-32)+D(C^2-19))x^2 \\ & -\tfrac{1}{2}(A(A+2C^2+2\beta^2-2B-43)+D(2C^2-2B-D-41))x \\ & +(\beta^2-4)(A+D), \end{aligned} \tag{57}$$

$$\begin{aligned} c_1 = & -2A(A+2C^2-5)x^2 \\ & +\tfrac{1}{2}(A(7A+14C^2-2B+5D-35)+D(3C^2-33))x \\ & -A(\tfrac{3}{2}A+3C^2-B-2\alpha^2-7)-3D(C^2-B-\tfrac{1}{2}D-3), \end{aligned} \tag{58}$$

$$c_0 = n(n+1)(C+n)(C+n+1)(A(x-1)-3D), \tag{59}$$

where B and C are defined in (42) and

$$A = (2n+1)(1+2C+2n), \qquad D = 1+2\alpha. \tag{60}$$

4.10. Limit $\mu = -[2(c+\alpha)(c+\alpha+\beta)]/[(2c+\alpha+\beta)(2c+\alpha+\beta+1)]$

This case is the $\mathscr{T}$ transform of Section 4.9. The explicit form is

$$\begin{aligned} \mathfrak{R}_n^{\alpha,\beta}(x;c) = & (-1)^n \frac{(2c+\alpha+\beta+1)_n(\alpha+c+1)_n}{n!(c+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+\alpha+\beta+1)_k(c+\alpha+1)_k} \\ & \times {}_4F_3\left(\begin{matrix} k-n, n+k+2c+\alpha+\beta+1, c+\alpha+\beta, c+\alpha \\ c+\alpha+\beta+k+1, c+\alpha+k+1, 2c+\alpha+\beta+1 \end{matrix}; 1 \right) x^k \end{aligned} \tag{61}$$

and the spectral measure

$$\phi'(x) = (1-x)^{\alpha-2} x^{\beta+2c+2} \left| {}_2F_1\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\alpha+\beta+1 \end{matrix}; \frac{e^{i\pi}}{x} \right) \right|^{-2}. \tag{62}$$

The coefficients of the differential equation (28) satisfied by the polynomials $\mathfrak{R}_n^{\alpha,\beta}(x;c)$ are obtained from (55)–(60) by changing only $D = 1+2\alpha$ by $D = 1-2\alpha$.

5. Conclusion

We end with some brief remarks. In this note we have presented some properties of the co-recursive associated Jacobi polynomials.

(a) For three values of the associativity parameter we obtain polynomial families for which the results are of the same complexity as the corresponding co-recursive polynomials.

(b) For four values of the co-recursivity parameter we obtain polynomial families for which the results are of the same complexity as the corresponding associated polynomials.

(c) In some cases the fourth-order differential equations satisfied by the polynomials studied above are factorizable (co-recursive and associated of order one) but we do not find factorization for the co-recursive associated polynomials nor for the associated one, except for the three cases in (a).

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References

[1] W.N. Bailey, *Generalized Hypergeometric Series* (Cambridge Univ. Press, Cambridge; reprinted by Hafner, New York, 1972).

[2] T.S. Chihara, On co-recursive orthogonal polynomials, *Proc. Amer. Math. Soc.* **8** (1957) 899–905.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Vol. I* (McGraw-Hill, New York, 1953).

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Vol. II* (McGraw-Hill, New York, 1953).

[5] W. Hahn, *Über Orthogonalpolynome mit drei Parametern*, Vol. 5, Deutsche. Math. (1940–41).

[6] M.E.H. Ismail, J. Letessier, G. Valent and J. Wimp, Two families of associated Wilson polynomials, *Canad. J. Math.* **42** (1990) 659–695.

[7] M.E.H. Ismail and D.R. Masson, Two families of orthogonal polynomials related to Jacobi polynomials, *Rocky Mountain J. Math.* **21** (1991) 359–375.

[8] J. Letessier. On co-recursive associated Laguerre polynomials, *J. Comput. Appl. Math.* **49** (1993) 127–136.

[9] Y.L. Luke, *The Special Functions and Their Approximations, Vol. I* (Academic Press, New York, 1969).

[10] W.McF. Orr, On the product $J_m(x)J_n(x)$, *Proc. Cambridge Philos. Soc.* **10** (1990) 93–100.

[11] A. Ronveaux and F. Marcellán, Co-recursive orthogonal polynomials and fourth-order differential equation, *J. Comput. Appl. Math.* **25**(1) (1989) 105–109.

[12] J.A. Shohat and J.D. Tamarkin, The problem of moments, in: Math. Surveys **1** (Amer. Math. Soc., Providence, RI, 1950).

[13] J. Wimp, Explicit formulas for the associated Jacobi polynomials and some applications, *Canad. J. Math.* **39** (1987) 983–1000.