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Fréchet–Urysohn for finite sets ☆

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Abstract

E. Reznichenko and O. Sipacheva called a space X "Fréchet–Urysohn for finite sets" if the following holds for each point $x \in X$: whenever P is a collection of finite subsets of X such that every neighborhood of x contains a member of P, then P contains a subfamily that converges to x. We continue their study of this property. We also look at analogous notions obtained by restricting to collections P of bounded size, we discuss connections with topological groups, the α_i-properties of A.V. Arhangel'skii, and with a certain topological game.

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1. Introduction

For a space X and a point $x \in X$, a family P of subsets of X is said to be a π-network at x if for each open U containing x, there is $p \in P$ such that $p \subseteq U$. We will say that an infinite family P of subsets of X converges to x if for each open U containing x,

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$\{p \in P\colon\ p \not\subseteq U\}$ is finite. If P consists of singleton sets, then P converges to x if the sequence formed by any enumeration of the singletons converges to x.

E. Reznichenko and O. Sipacheva defined a space X to be *Fréchet–Urysohn for finite sets*, which we will denote by $\mathrm{FU}_{\mathrm{fin}}$, if for each $x \in X$ and each $P \subset [X]^{<\aleph_0}$, if P forms a π-network at x, then P contains a subfamily that converges to x (see [16]). This notion has appeared earlier in the literature (it is called *groupwise Fréchet* in [4]), but [16] is its first systematic study.

We will say that X is FU_n if for each $x \in X$, and each $P \subset [X]^n$, if P forms a π-network at x, then P contains a subfamily that converges to x. We will say that X is *boundedly*-$\mathrm{FU}_{\mathrm{fin}}$ if it is FU_n for all $n \in \omega$.

Clearly,

first-countable $\to \mathrm{FU}_{\mathrm{fin}} \to$ boundedly-$\mathrm{FU}_{\mathrm{fin}} \to$ Fréchet–Urysohn.

Also, it is clear that for every $n \in \omega \setminus \{0\}$

boundedly-$\mathrm{FU}_{\mathrm{fin}} \to \mathrm{FU}_{n+1} \to \mathrm{FU}_n \to$ Fréchet–Urysohn.

By taking the topological sum of countably many convergent sequences and forming the quotient space by identifying the limit points of each sequence, one obtains the Fréchet–Urysohn fan S_ω. It is not hard to see that S_ω is Fréchet–Urysohn, but not FU_2.

The one-point compactification of an uncountable discrete space is an uncountable $\mathrm{FU}_{\mathrm{fin}}$ space that is not first-countable. This example also has a number of other strong convergence properties (e.g., it is α_1—see below). For this reason we restrict our study to countable $\mathrm{FU}_{\mathrm{fin}}$ spaces. In this note we show that even in the class of countable spaces none of the above implications can be reversed, at least in ZFC. In addition, the relationship between these properties and the α_i-convergence properties of Arhangel'skii is considered. The following fundamental question concerning $\mathrm{FU}_{\mathrm{fin}}$ spaces is left open:

Question 1. Is there, in ZFC, a countable $\mathrm{FU}_{\mathrm{fin}}$ space that is not first-countable?

This question was motivated by the following question (see [2] and [13]):

Question 2 (*Malykhin*). Is there a countable Fréchet–Urysohn topological group that is not metrizable?

The existence of a non-metrizable separable topological group has a number of equivalent formulations (see [16]):

Proposition 1. *The existence of a countable Fréchet–Urysohn topological group that is not metrizable is equivalent to each of the following*:

(1) *The existence of a countable Fréchet–Urysohn topological group that is not first-countable.*
(2) *The existence of a separable Fréchet–Urysohn topological group that is not metrizable.*

The connection between $\mathrm{FU}_{\mathrm{fin}}$ spaces and Fréchet–Uryshon groups is given by the following construction. Let $X = \omega \cup \{\infty\}$ be a space with a single nonisolated point ∞. Let

$G = [\omega]^{<\omega}$ and define $F_0 * F_1 = F_0 \setminus F_1 \cup F_1 \setminus F_0$. Then G with this operation is a group with identity element $\emptyset$. To each open neighborhood U of ∞ let $V_U = \{F \in G\colon\ F \subseteq U\}$. This defines a neighborhood base at $\emptyset$ making G a topological group. Note that X is first-countable if and only if G is first-countable. Moreover, Reznichenko and Sipacheva proved the following theorem:

Proposition 2 [16]. *X is* $\mathrm{FU}_{\mathrm{fin}}$ *if and only if G is Fréchet–Uryshon.*

Thus, there is a countable $\mathrm{FU}_{\mathrm{fin}}$ space that is not first-countable implies that there is a countable Fréchet–Uryshon topological group that is not metrizable. We do not know if the converse holds:

Question 3. Does the existence of a separable non-metrizable Fréchet–Urysohn topological group imply the existence of a countable $\mathrm{FU}_{\mathrm{fin}}$ space that is not first countable?

Arhangel'skii proved that there are countable Fréchet–Urysohn topological groups which are not first-countable assuming $\mathrm{MA} + \neg\mathrm{CH}$. Nyikos showed that there is such an example assuming either $\mathfrak{p} > \omega_1$ or $\mathfrak{p} = \mathfrak{b}$ see [13] and [16]. Both of these examples of Nyikos are $\mathrm{FU}_{\mathrm{fin}}$ (see [16]).

Two essentially different examples of non-metrizable topological groups can be obtained from an uncountable γ-set of reals. An open cover of a space X is said to be an ω-cover if each finite subset of X is contained in an element of the cover. An open cover is said to be a γ-cover if each point of the space is contained in all but finitely many elements of the cover. A space is said to be a γ-space if each ω-cover has a γ-subcover. Gerlits and Nagy introduced this class of spaces and proved that X is a γ-space if and only if $C_p(X)$ is Fréchet–Urysohn [7]. In fact, the same proof shows that $C_p(X)$ is $\mathrm{FU}_{\mathrm{fin}}$ if and only if $C_p(X)$ is Fréchet–Uryshon. Therefore for any γ-set $X \subseteq \mathbb{R}$, $C_p(X)$ is a separable non-metrizable $\mathrm{FU}_{\mathrm{fin}}$ topological group. Another example, the space T_X defined below, is a $\mathrm{FU}_{\mathrm{fin}}$ space if and only if X is a γ-set. This was essentially proved by Nyikos (see [14], although the class of $\mathrm{FU}_{\mathrm{fin}}$ spaces were not explicitly considered there).

It is both consistent with ZFC and independent of ZFC that there exist γ-sets: in fact, $\mathfrak{p}$ is the minimum cardinality of a set of reals that is not a γ-set [7]; the existence of γ-sets contradicts the Borel conjecture: any γ-set has strong measure zero. Therefore, in the Laver model there are no γ-sets [9].

Whether there is a countable non-metrizable Fréchet–Uryshon topological group or even a $\mathrm{FU}_{\mathrm{fin}}$ space that is not first-countable in the Laver model appears to be an open question [15].

Now let us recall the definition of the α_i-spaces, introduced by Arhangel'skii [1]. Let X be a space, and $x \in X$. Suppose that for any countable family $\{A_n\}_{n\in\omega}$ of sequences converging to x, there is a sequence A converging to x such that:

1. $|A_n \setminus A| < \omega$ for every $n \in \omega$, then x is an α_1-point;
2. $|A_n \cap A| = \omega$ for every $n \in \omega$, then x is an α_2-point;
3. $|A_n \cap A| = \omega$ for infinitely many $n \in \omega$, then x is an α_3-point;
4. $|A_n \cap A| \neq \emptyset$ for infinitely many $n \in \omega$, then x is an α_4-point.

Also, if for every *disjoint* collection $\{A_n\}_{n\in\omega}$ of sequences converging to x, there is a sequence A converging to x such that $|A_n \setminus A| < \omega$ for infinitely many $n \in \omega$, then x is an $\alpha_{1.5}$-point. X is an α_i*-space* if every point is an α_i-point.

Reznichenko and Sipacheva proved that $\mathrm{FU}_{\mathrm{fin}}$ spaces are α_2. Among other things, we show FU_2 spaces are α_4, and construct consistent examples showing that there are no other possible implications in ZFC.

2. A boundedly-$\mathrm{FU}_{\mathrm{fin}}$ not $\mathrm{FU}_{\mathrm{fin}}$ space in ZFC

Several spaces in this note are of a similar type, given by the following lemma, which makes them boundedly-$\mathrm{FU}_{\mathrm{fin}}$ and α_3.

Lemma 3. *Let $X = Y \cup \{\infty\}$, where Y is the set of isolated points of X. Suppose Y is contained, as a set, in some compact metric space K, and a subbase for the neighborhood filter at ∞ in X is generated by complements of members of the set*

$$\bigl\{\{y\}\colon\ y \in Y\bigr\} \cup \{S_x\colon\ x \in K\},$$

where S_x is either empty or a sequence of points of Y converging to x in the space K. Then X is boundedly-$\mathrm{FU}_{\mathrm{fin}}$ *and* α_3.

Proof. Suppose $\mathcal{F}$ is a π-net at ∞ of m-element subsets of X. For each $F \in \mathcal{F}$, choose some indexing $\{x_i\colon\ i < m\}$ of F, and let $\vec{F} = (x_i)_{i<m}$ be the corresponding point in K^m, Observe that if a π-net is split into finitely many pieces, one of the pieces must be a π-net. It follows from this and compactness of K^n that there is some $\vec{y} = (y_i)_{i<m} \in K^m$ such that, for every neighborhood U of $\vec{y}$ in K^m, the set $\{F \in \mathcal{F}\colon\ \vec{F} \in U\}$ is a π-net. Thus we can choose $F_n \in \mathcal{F}$ such that the metric distance between $\vec{F_n}$ and $\vec{y}$ is $\leqslant 1/2^n$, and

$$F_n \cap \left[\{y_i\colon\ i < m\} \cup \left(\bigcup_{i<m} S_{y_i}\right) \cup \left(\bigcup_{j<n} F_j\right)\right] = \emptyset.$$

Let us check that $\{F_n\} \to \infty$. If not, there is $x \in K$ such that infinitely many F_n's meet S_x. By the construction of the F_n's, on the one hand x must be y_i for some $i < m$, but on the other hand, no F_n meets S_{y_i}, contradiction.

Now let us check that X is α_3. If $A_n \subseteq Y$ converges to ∞ for each $n \in \omega$, then by compactness of K we may choose $B_n \subseteq A_n$ and $x_n \in K$ such that B_n converges to x_n in the topology of K. Also, B_n still converges to ∞ in the topology on X. Thus we may assume that S_{x_n} (if it exists for x_n) is disjoint from B_n. By compactness of K again, we may find an infinite $M \subseteq \omega$ such that $(x_n\colon\ n \in M)$ converges to some $x \in K$. By removing a finite set from each B_n for $n \in M$ we may assume that $\bigcup\{B_n\colon\ n \in M\}$ is disjoint from S_x. It easily follows that $\bigcup\{B_n\colon\ n \in M\}$ converges to infinity. Thus X is α_3. □

Theorem 4. *There is a boundedly*-$\mathrm{FU}_{\mathrm{fin}}$ *space which is not* $\mathrm{FU}_{\mathrm{fin}}$.

Proof. Let $\mathbb{Q}$ denote the rationals in the unit interval $I = [0, 1]$. Our space X will be $\mathbb{Q} \cup \{\infty\}$, where points of $\mathbb{Q}$ are isolated, and the neighborhood filter of ∞ will be generated by complements of finite subsets of $\mathbb{Q}$, together with complements of certain well-chosen sequences S_x of rationals converging to x, for some points $x \in I$. We will choose at most one S_x for each x; by the previous lemma, this will guarantee the space is boundedly-$\mathrm{FU_{fin}}$. We will make it non-$\mathrm{FU_{fin}}$ by ensuring that a certain collection $\{H_{nm}\colon n, m \in \omega\}$ of finite sets defined at the beginning of our construction is a π-net but has no convergent subsequence. Let $\{H_n\colon n \in \omega\}$ be a collection of finite subsets of I, and for each $x \in \bigcup_{n\in\omega} H_n$, let S_x be a sequence of rationals converging to x, such that the following conditions are satisfied:

1. $H_n \cap H_m = \emptyset$ if $m \neq n$;
2. For each $x \in I$, $d(x, H_n) < 1/2^n$, where d is the usual Euclidean distance;
3. The collection $\{S_x\colon x \in \bigcup_{n\in\omega} H_n\}$ is pairwise-disjoint;
4. $\operatorname{diam}(S_x) < 1/2^n$ for each $x \in H_n$.

Let $\{q_i^x\colon i \in \omega\}$ be a one-to-one enumeration of S_x for $x \in \bigcup_{n\in\omega} H_n$. We let $H_{nm} = \{q_m^x\colon x \in H_n\}$. Since the H_n's become increasingly dense in I and the S_x's have decreasingly small diameter, it is easy to check the following fact:

Fact. For each infinite $A \subset \omega$, for each $f : A \to \omega$, and for each $y \in I$, there are $x_n \in H_n$, $n \in A$, such that $\{q_{f(n)}^{x_n}\}_{n\in A}$ converges to y.

Now let $\{y(\alpha)\colon \alpha < \mathfrak{c}\}$ list any $\mathfrak{c}$-sized subset of $I \setminus \bigcup_{n\in\omega} H_n$, and let f_α, $\alpha < \mathfrak{c}$, list all infinite partial functions from ω to ω. For each α, let $S_{y(\alpha)}$ be a sequence converging to $y(\alpha)$ as in the Fact, with $f = f_\alpha$ and $A = \operatorname{dom}(f_\alpha)$. Then X is the space $\mathbb{Q} \cup \{\infty\}$, where $\mathbb{Q}$ is a set of isolated points and neighborhoods of ∞ are generated by complements of the S_x's, where $x \in \{y(\alpha)\colon \alpha < \mathfrak{c}\} \cup (\bigcup_{n\in\omega} H_n)$.

We already know X is boundedly-$\mathrm{FU_{fin}}$; we need to prove that it is not $\mathrm{FU_{fin}}$. First we show that $\mathcal{H} = \{H_{nm}\colon n, m < \omega\}$ is a π-net. Let K be any finite subset of I; we need to show that $H_{nm} \cap [K \cup (\bigcup_{x\in K} S_x)] = \emptyset$ for some m and n. First find n such that $H_n \cap K = \emptyset$. There are disjoint Euclidean open sets U and V containing H_n and K, respectively. The set $J = [\bigcup_{x\in K} S_x] \setminus V$ is finite. Thus, since the points in H_{nm} converge to the points of H_n as $m \to \infty$, H_{nm} eventually gets inside U and misses J, so there is an $m \in \omega$ as required.

Finally we show that there is no convergent subsequence of $\mathcal{H}$. Since for fixed n, H_{nm} meets $\bigcup_{x\in H_n} S_x$, any convergent sequence of members of $\mathcal{H}$ would have to contain a convergent subsequence of the form $\{H_{nf(n)}\colon n \in \operatorname{dom}(f)\}$ for some infinite partial function $f : \omega \to \omega$. But $f = f_\alpha$ for some α, and by the construction every $H_{nf(n)}$ for $n \in \operatorname{dom}(f)$ meets $S_{y(\alpha)}$, contradiction. □

With the help of CH, we can make the previous example α_1.

Example 5. (CH) There is a boundedly-$\mathrm{FU_{fin}}$ α_1-space which is not $\mathrm{FU_{fin}}$.

Proof. To make the previous example α_1, we will need to list in type $\mathfrak{c}$ the candidates for countable collections of convergent sequences, and at each stage either destroy the fact that one of the sequences is convergent, or find a set almost containing every one of them that is convergent at that stage and that remains convergent throughout the construction.

We will need to be more careful about how we choose the sequences S_y which in the previous example destroyed the $\mathrm{FU_{fin}}$ property, so as not to renege on promises that certain sequences are to remain convergent. To help us do this, we make the H_{nm}'s evenly spaced in $[0,1]$. Then it is an easy exercise to verify the following:

Fact. *Suppose g is an infinite partial function from ω to ω, $A_{nm} \subset H_{nm}$, $\epsilon > 0$, and $|A_{ng(n)}|/|H_{ng(n)}| \geqslant \epsilon$ for all $n \in \mathrm{dom}(g)$. Then the set*

$$\left\{y \in [0,1]\colon\ \exists a_{ng(n)} \in A_{ng(n)} \text{ with } a_{ng(n)} \to y\right\}$$

has Lebesgue measure at least ϵ.

Call a subset A of $\mathbb{Q}$ *small* if

$$\lim_{n \in \mathrm{dom}(g)} \frac{|A \cap H_{ng(n)}|}{|H_{ng(n)}|} = 0$$

for any infinite partial function $g:\omega \to \omega$. We are going to make sure all convergent sequences are small.

Let $\vec{A}_\alpha$, $\alpha < \omega_1$, index all sequences $(A(n))_{n\in\omega}$ of infinite subsets of $\mathbb{Q}$. Recall that S_x for $x \in \bigcup_{n\in\omega} H_n$ is already defined, as in Example 4, in the process of defining the H_{nm}'s. So let us suppose $\alpha < \omega_1$ and we have constructed $y(\beta)$, $S_{y(\beta)}$, and A'_β for $\beta < \alpha$ satisfying the following conditions, where U_β is the filter generated by complements of elements of $\{S_x\colon\ x \in \{y(\delta)\colon\ \delta < \beta\} \cup \bigcup_{n\in\omega} H_n\}$.

(a) $\{y(\beta)\colon\ \beta < \alpha\} \subset [0,1] \setminus \bigcup_{n\in\omega} H_n$;
(b) $S_{y(\beta)}$ is a sequence of rationals converging to $y(\beta)$ (in [0,1]);
(c) If $A_\beta(n)$ is not small for some $n \in \omega$, and k is the least such n, then $|S_{y(\beta)} \cap A_\beta(k)| = \omega$ and $A'_\beta = \emptyset$;
(d) If $A_\beta(n)$ is small and U_β-convergent for every $n \in \omega$, then $S_{y(\beta)} = \emptyset$, A'_β is small, and $A'_\beta {}^* \supset A_\beta(n)$ for every $n \in \omega$;
(e) A'_γ is U_β-convergent for every $\gamma \leqslant \beta + 1$.

We first check that the space is as desired, assuming the construction can be carried out satisfying the above conditions. That X is boundedly-$\mathrm{FU_{fin}}$, and that $\mathcal{H} = \{H_{nm}\colon\ n, m \in \omega\}$ is a π-net is exactly as in Example 4. Let us see that X is not $\mathrm{FU_{fin}}$. If there were some infinite convergent subsequence from $\mathcal{H}$, there would be an infinite partial function $g:\omega \to \omega$ such that the set $A = \bigcup_{n\in\mathrm{dom}(g)} H_{ng(n)}$ is convergent. Note that A is not small. For some α, $\vec{A}_\alpha = (A, A, \ldots)$, but then $S_{y(\alpha)}$ meets A in an infinite set, contradiction. The same argument shows that only small sets are convergent, whence conditions (d) and (e) ensure that X is α_1.

Let us now see how to carry out the induction at step α. We are given $\vec{A}_\alpha = (A_\alpha(n))_{n\in\omega}$. If some $A_\alpha(n)$ is not U_α-convergent, we need not do anything. So suppose these sets are always U_α-convergent. Note that this implies that for any fixed n and m, $A_\alpha(m)$ meets H_{ni} for at most finitely many i. It is then not difficult to check that in case all $A_\alpha(n)$'s are small and U_α-convergent, then there is some A'_α that is also small and U_α-convergent, and almost contains every $A_\alpha(n)$. This gives us condition (d), and part of (e). If not all $A_\alpha(n)$'s are small, let k be the least such that $A_\alpha(k)$ is not small. Then there is a partial function $g:\omega \to \omega$ and $\epsilon > 0$ such that $|A_\alpha(k) \cap H_{ng(n)}|/|H_{ng(n)}| \geqslant \epsilon$ for all $n \in \mathrm{dom}(g)$. As above, there is a set B_α which is small and U_α-convergent, and which almost contains A'_β for every $\beta < \alpha$. Since B_α is small, we have that $|(A_\alpha(k) \setminus B_\alpha) \cap H_{ng(n)}|/|H_{ng(n)}| \geqslant \epsilon/2$ for all sufficiently large $n \in \mathrm{dom}(g)$. By the Fact above, we can find a point $y(\alpha) \notin \{y(\beta): \beta < \alpha\} \cup \bigcup_{n\in\omega} H_n$, and a sequence $S_{y(\alpha)} = \{q_\alpha(j): j \in \mathrm{dom}(g)\}$ of rationals converging to $y(\alpha)$ with $q_\alpha(n) \in (A_\alpha(k) \setminus B_\alpha) \cap H_{ng(n)}$ for almost all $n \in dom(g)$. This ensures the remaining conditions. □

3. Known examples

We consider three related constructions that produce consistent examples of $\mathrm{FU_{fin}}$ spaces.

Example 1. For $F \subseteq 2^\omega$, let τ_F be the topology on $2^{<\omega} \cup \{\infty\}$ generated by taking as a subbase sets of the form

T1. $\{s\}$, for $s \in 2^{<\omega}$ and
T2. U_f for $f \in F$ where

$$U_f = \{\infty\} \cup \big(2^{<\omega} \setminus \{f|n: n \in \omega\}\big).$$

Let X_F denote this space. For a finite set $G \subseteq F$ let $V_G = \bigcap\{U_f: f \in G\}$. Such sets form a local base at the point ∞.

It is known that X_F is always Fréchet–Urysohn that and that it is first-countable if and only if F is countable. This example was considered by Nyikos in [14]. Although the notion of a $\mathrm{FU_{fin}}$ space was not explicitly formulated, Nyikos essentially proved the following (see also [16]):

Theorem 6. *F is a γ-set if and only if X_F is* $\mathrm{FU_{fin}}$.

Notice that the space X_F is of the type constructed in the previous section. Hence X_F is always boundedly-$\mathrm{FU_{fin}}$. Thus, taking F such that F is not a γ-set, gives another construction of a boundedly-$\mathrm{FU_{fin}}$ not $\mathrm{FU_{fin}}$ space (in ZFC).

Example 2. For $F \subseteq \omega^\omega$, let σ_F denote the topology on $\{\infty\} \cup \omega^{<\omega}$ generated by taking as a subbase sets of the form

S1. $\{s\}$ for $s \in \omega^{<\omega}$ and
S2. $\{\infty\} \cup (\omega^{<\omega} \setminus \omega^n)$ for $n \in \omega$ and
S3. sets U_f for $f \in F$ where

$$U_f = \{\infty\} \cup \big(\omega^{<\omega} \setminus \{f|n: n \in \omega\}\big).$$

Let Y_F denote this space. For a finite set $G \subseteq F$ let $V_G = \bigcap\{U_f: f \in G\}$. Such sets form a local base at the point ∞. Note that Y_F is first-countable if and only if F is uncountable. As with X_F, Y_F is always boundedly-$\mathrm{FU_{fin}}$.

Let Ω' denote the family of open ω-covers U of F with the property that each $u \in U$ is the complement of a finite union of basic open subsets of ω^ω. Let us say that F is a *weak γ-set* if every cover from Ω' has a Γ-subcover.

Theorem 7. *Y_F is* $\mathrm{FU_{fin}}$ *if and only if F is a weak γ-set.*

Proof. Fix a weak γ-set $F \subseteq \omega^\omega$. To prove that Y_F is $\mathrm{FU_{fin}}$, fix $P \subseteq [\omega^{<\omega}]^{<\aleph_0}$ a π-network at ∞. Let $u_n = \{x \in F\colon t \nsubseteq x$ for all $t \in p_n\}$. Each u_n is open in ω^ω and is the complement of a finite union of basic open sets. Moreover, since P is a π-network at ∞ in Y_F, $U = \{u_n\colon n \in \omega\}$ is an ω-cover of F. Hence U is in Ω'. By assumption, we may fix $\{u_n\colon n \in A\}$ be a γ-subcover.

Claim 1. *$\{p_n\colon n \in A\}$ converges to ∞.*

Proof. Fix $G \in [F]^{<\aleph_0}$. So V_G is a basic open neighborhood of ∞. There is a $k \in \omega$ such that $G \subseteq u_n$ for all $n \in A \setminus k$. But this means that for each $x \in G$ and each $n \in A \setminus k$, no restriction of x is in p_n. By definition of V_G this means $p_n \subseteq V_G$ for every $n \in A \setminus k$ as required. □

Conversely, suppose that Y_F is $\mathrm{FU_{fin}}$. Fix $U \in \Omega'$ of F. Thus, for each $u \in U$, there is a finite set $p_u \subseteq \omega^{<\omega}$ such that u is the complement of the clopen set $\bigcup\{[s]\colon s \in p_u\}$. Let $P = \{p_u\colon u \in U\}$.

Claim 2. *P is a π-network at ∞.*

Proof. Fix V_G basic open. Fix $u \in U$ such that $G \subseteq u$. Thus $p_u \subseteq V_G$. This proves the claim. □

Since Y_F is $\mathrm{FU_{fin}}$, we may fix $Q \subseteq P$ such that Q converges to ∞. Let $Q = \{p_u\colon u \in V\}$ for some $V \subseteq U$. We claim that V is a γ-cover of F. To see this, fix $x \in F$. Since Q converges to ∞, $p_u \subseteq V_x$ for all but finitely many $u \in V$. Therefore, $x \in u$ for all but finitely many $u \in V$. □

Example 3. For $F \subseteq \omega^\omega$, let γ_F be the topology on $\{\infty\} \cup (\omega \times \omega)$ generated by taking as a subbase sets of the form

G1. $\{(n,m)\}$ for $n,m \in \omega$ and
G2. $\{\infty\} \cup (\omega \times \omega) \setminus (n \times \omega\}$ and
G3. sets U_f for $f \in F$ where

$$U_f = \{\infty\} \cup \big(\omega \times \omega \setminus \big\{\big(n, f(n)\big)\colon n \in \omega\big\}\big).$$

Let Z_F denote this topological space.

In [13], P. Nyikos proved that if $\mathfrak{b} = \mathfrak{p}$ then there is an uncountable $F \subseteq \omega^\omega$ such that Z_F is $\mathrm{FU_{fin}}$ (see also [16]).

Relation among the spaces X_F, Y_F and Z_F. We conjecture that the following are equivalent:

(1) There is a $F \subseteq 2^\omega$ such that X_F is $\mathrm{FU}_{\mathrm{fin}}$.
(2) There is a $F \subseteq \omega^\omega$ such that Y_F is $\mathrm{FU}_{\mathrm{fin}}$.
(3) There is a $F \subseteq \omega^\omega$ such that Z_F is $\mathrm{FU}_{\mathrm{fin}}$.

However, we are only able to show that (1) implies (2) and that in significant cases the spaces Y_F and Z_G may be homeomorphic.

Theorem 8. *If $F \subseteq 2^\omega$ is such that X_F is a* $\mathrm{FU}_{\mathrm{fin}}$ *space, then Y_F is a* $\mathrm{FU}_{\mathrm{fin}}$ *space.*

Note that the statement of the theorem makes sense since $2^\omega \subseteq \omega^\omega$.

Proof. The theorem easily follows from the characterizations given by Theorems 6 and 7. Alternately, we have the following direct proof:

Suppose that $F \subseteq 2^\omega$ is such that X_F is $\mathrm{FU}_{\mathrm{fin}}$. Let $D = \omega^\omega \setminus 2^\omega$. Let $D \cup \{*\}$ be the space where each $s \in D$ is isolated and the family of sets of the form

$$U_n = \{*\} \cup (D \cap \omega^{\omega \setminus n})$$

form a local base at $\{*\}$. Then D is first-countable and Y_F is homeomorphic to the space obtained by identifying the points $*$ and ∞ in the direct sum of D and X_F. By Corollary 21 from the last section below, it follows that Y_F is $\mathrm{FU}_{\mathrm{fin}}$. □

Hence we may conclude that (1) implies (2).

Next we turn our attention to the spaces Y_F and Z_F. As mentioned above, Nyikos proved that $\mathfrak{b} = \mathfrak{p}$ implies that there is an uncountable F such that Z_F is $\mathrm{FU}_{\mathrm{fin}}$. Let us say that a family $F = \{f_\alpha \colon \alpha < \kappa\}$ is an *unbounded scale* if it is an unbounded family in ω^ω with respect to the preorder $<^*$ such that each f_α is increasing and $f_\alpha <^* f_\beta$ for each $\alpha < \beta < \kappa$. Indeed, Nyikos proved that if $F = \{f_\alpha \colon \alpha < \mathfrak{b}\}$ is an unbounded scale and if $\mathfrak{b} = \mathfrak{p}$, then Z_F is $\mathrm{FU}_{\mathrm{fin}}$. We prove the following:

Theorem 9. *There are unbounded scales $G = \{g_\alpha \colon \alpha < \mathfrak{b}\}$ and $F = \{f_\alpha \colon \alpha < \mathfrak{b}\}$ such that Y_F is homeomorphic to Z_G.*

Proof. Fix $H : \omega^{<\omega} :\to \omega \times \omega$ a bijection such that

(a) H maps ω^{n+1} onto $\{n\} \times \omega$.

We claim that there is an unbounded scale $\{g_\alpha \colon \alpha < \mathfrak{b}\} \subseteq \omega^\omega$, such that if $f_\alpha : \omega \to \omega$ is such that $f_\alpha(n)$ is the unique k such that $H(g_\alpha | (n+1)) = (n, k)$, then $\{f_\alpha \colon \alpha < \mathfrak{b}\}$ is an unbounded scale. Indeed, the family $\{g_\alpha \colon \alpha < \mathfrak{b}\}$ is easily constructed by recursion since for any $s \in \omega^n$, the set

$$\left\{H(t) \colon t \in \omega^{n+1} \text{ and } t|n = s\right\}$$

is unbounded in $\{n\} \times \omega$.

Notice that for each $\alpha < \mathfrak{b}$,

(b) H maps the set $\{g|n\colon n > 0\}$ onto the set $\{(n, g_\alpha(n)\colon n \in \omega\}$.

Let $H^*: Y_G \to Z_F$ be the extension of H by defining $H^*(\infty_Y) = \infty_Z$. Then properties (a) and (b) easily imply that H^* is a homeomorphism. □

By the above results, we have the following corollary:

Corollary 10. $\mathfrak{b} = \mathfrak{p}$ *implies that there is a weak* γ*-set in* ω^ω.

The relationship between γ-sets and weak γ-sets is not known. Perhaps $\mathfrak{b} = \mathfrak{p}$ implies the existence of a γ-set. Also we do not know whether there are weak γ-sets in ZFC:

Question 4. Are there weak γ-sets in ZFC?

4. Boundedly-FU$_{\text{fin}}$ and the α_i-properties

FU$_{\text{fin}}$ spaces are α_2 (see [16]). Also, there is a consistent example of a countable Fréchet–Urysohn topological group that is not α_3 [17]. Thus, consistently, it is not the case that every Fréchet–Urysohn topological group is FU$_{\text{fin}}$.

Question 5. Is there a ZFC example of a Fréchet–Urysohn topological group that is not FU$_{\text{fin}}$?

It is easy to see that any space of character less than $\mathfrak{b}$ is α_1, and any space of character less than $\mathfrak{p}$ is FU$_{\text{fin}}$. The example X_F of the previous section is boundedly-FU$_{\text{fin}}$, and can always chosen to be not FU$_{\text{fin}}$ and of character $\mathfrak{p}$. However, we do not know the minimum character of a Fréchet–Uryshon space that is not boundedly FU$_{\text{fin}}$. So it is natural to ask:

Question 6. Is every Fréchet–Urysohn space of character $< \mathfrak{b}$ boundedly-FU$_{\text{fin}}$?

In this section we prove that FU$_2$ spaces are α_4 and construct consistent examples to show that there are no other possible implications in ZFC. In particular, from CH we construct a countable α_1 Fréchet–Urysohn space that is not FU$_2$, and a boundedly-FU$_{\text{fin}}$ space that is not α_3. One other possible implication to consider is whether FU$_{\text{fin}}$ implies α_1. In [4] it is proven to be consistent with ZFC that all $\alpha_{1.5}$ spaces are first-countable. Since $\mathfrak{b} = \mathfrak{p} = \omega_1$ in the model constructed, it follows that there is in this model a FU$_{\text{fin}}$ space that is not α_1. On the other hand, in [3], Dow showed that all α_2 spaces are α_1 in the Laver model. So all FU$_{\text{fin}}$ spaces are α_1 in the Laver model. However, as mentioned earlier, we do not know whether there is a countable FU$_{\text{fin}}$ space that is not first-countable in the Laver model.

We start by showing that there are FU$_{\text{fin}}$ spaces which are not $\alpha_{1.5}$ in any model of CH (again, $\mathfrak{p} = \mathfrak{c}$ suffices).

Theorem 11. (CH) *There is a* $\mathrm{FU}_{\mathrm{fin}}$ *space which is not* $\alpha_{1.5}$.

Proof. Let $X = (\omega \times \omega) \cup \{\infty\}$. Points of $\omega \times \omega$ will be isolated. We intend to make the sets $\{n\} \times \omega$, $n \in \omega$, the collection of covergent sequences which witnesses failure of $\alpha_{1.5}$. We define the neighborhood filter at ∞ by defining a collection $\mathcal{I}$ which generates the co-ideal.

Start by putting $\omega \times \{n\}$ in $\mathcal{I}$ for each $n \in \omega$. Let $\{P_\alpha\colon \alpha < \omega_1\}$ and $\{f_\alpha\colon \alpha < \omega_1\}$ list all colections of finite subsets of $\omega \times \omega$ and all infinite partial functions from ω to ω, respectively.

Let u be any ultrafilter on ω. Call a subset A of $\omega \times \omega$ *small* if its projection $\pi_2(A)$ on the second coordinate is not in u.

Suppose for all $\beta < \alpha$, where $\alpha < \omega_1$, we have defined $F_{\beta n} \in P_\beta$, and infinite partial functions g_β satisfying the following conditions:

(i) Let T_β be the topology generated by all subsets of $\omega \times \omega$ and complements of sets in $\{g_\gamma\colon \gamma < \beta\} \cup \{\omega \times \{n\}\colon n \in \omega\}$. If P_β is a π-net with respect to T_β, then $F_{\beta n} \in P_\beta$ is such that $\bigcup_{n\in\omega} F_{\beta n}$ is small and converges to ∞ in (X, T_β);
(ii) If $\gamma \leqslant \beta$, then $g_\beta \cap (\bigcup_{n\in\omega} F_{\gamma n})$ is finite.
(iii) $\mathrm{dom}(g_\beta) = \mathrm{dom}(f_\beta)$ and $g_\beta(n) \geqslant f_\beta(n)$ for all $n \in \mathrm{dom}(g_\beta)$.

First let us note that if we carry out the induction as above, then X will be as desired. The neighborhood filter at ∞ is by definition generated by complements of members of the set $\mathcal{I} = \{g_\alpha\colon \alpha < \omega_1\} \cup \{\omega \times \{n\}\colon n \in \omega\}$. Then condition (iii) easily guarantees that X will not be $\alpha_{1.5}$. Also, if P is any π-net at ∞, condition (i) guarantees that we will have chosen a subsequence of P at some stage which converged to ∞ in the topology so far, while condition (ii) guarantees that it remains convergent in the end. So X is $\mathrm{FU}_{\mathrm{fin}}$.

Now we check that the induction can be carried out. At step α, we are given P_α. If P_α is a π-net with respect to T_α, since this topology is first-countable we can find $F_{\alpha n} \in P_\alpha$ such that $\{F_{\alpha n}\}_{n\in\omega}$ converges to ∞ in T_α. Since each $\omega \times \{n\} \in \mathcal{I}$, by passing to a subsequence if necessary, we may assume that $\{\pi_2(F_{\alpha n})\colon n \in \omega\}$ is pairwise-disjoint. Now by dividing the sequence into two pieces and choosing the small piece, we may assume that $\bigcup_{n\in\omega} F_{\alpha n}$ is small. So we have (i). Now let S_n, $n \in \omega$, list $\{\bigcup_{n\in\omega} F_{\gamma n}\colon \gamma \leqslant \alpha\}$ and let $\{d_n\colon n \in \omega\}$ list $\mathrm{dom}(f_\alpha)$. Since each S_n is small, we can find $r_n \in (\omega \setminus f_\alpha(d_n)) \setminus \bigcup_{i<n} \pi_2(S_i)$. Let $g_\alpha(d_n) = r_n$; then g_α is as required. □

Theorem 12. *If* X *is* FU_2, *then* X *is* α_4.

Proof. Fix $\{\tau_n\colon n \in \omega\}$ a sequence of convergent sequences in X. Without loss of generality, $\omega \subseteq X$ and the range of each τ_n is contained in ω and each τ_n converges to a point $\infty \in X$. Let $\tau_n = (k_n(i)\colon i \in \omega)$. Let

$$F_i = \big\{\big\{k_0(i), k_i(j)\big\}\colon j < \omega\big\}$$

and let $F = \bigcup\{F_i\colon i \in \omega\}$. It is easy to see that F is a π-network at ∞. So, by FU_2, there are elements

$$x_n = \big\{k_0(i_n), k_{i_n}(j_n)\big\} \in F$$

such that every open set U of ∞ contains all but finitely many of the x_n. Clearly, $\{i_n: n \in \omega\}$ must be infinite, and the sequence $(k_{i_n}(j_n): n \in \omega)$ must converge to ∞. Thus X is α_4. □

Example 13. CH implies the existence of a boundedly-$\mathrm{FU}_{\mathrm{fin}}$ space that is not α_3.

Proof. The underlying set is $\omega \times \omega \cup \{\infty\}$. Points of $\omega \times \omega$ are declared to be clopen and the neighborhood base at ∞ will be constructed recursively. The topology will be constructed so that each column $\{n\} \times \omega$ is a convergent sequence, but there is no convergent sequence hitting infinitely many columns in an infinite set. I.e., the space will not be α_3.

Using CH let $(S_\alpha: \alpha < \omega_1)$ be an enumeration of all sets of the form

$$\bigcup \bigl\{\{n\} \times A_n: n \in X\bigr\},$$

where X is infinite and each A_n is infinite.

Let $\{F_\alpha: \alpha < \omega_1)$ enumerate the collection $\{F: \exists n F \subseteq [\omega \times \omega]^n\}$.

Recursively, we define sets $\{B_\alpha: \alpha < \omega_1\}$ and $\{G_\alpha: \alpha < \omega_1\}$ such that

(a) $B_\alpha \subseteq \omega \times \omega$ is a partial function with infinite domain.
(b) $G_\alpha \subseteq F_\alpha$.

We let U_α be the filter on $\omega \times \omega$ generated by the family of sets

$$\{\omega \times \omega \setminus B_\beta: \beta < \alpha\} \cup \bigl\{\{x\}: x \in \omega \times \omega\bigr\}.$$

We also require our sets to satisfy the following inductive hypotheses:

For each $\beta < \alpha$

(c) $B_\beta \cap S_\beta$ is infinite.
(d) $G_\beta = \emptyset$ in the case that F_β is not a π-net with respect to the filter U_β
(e) If $G_\beta \neq \emptyset$, then G_β converges with respect to the filter U_α.

In order to preserve (e) in the construction we will need the following further inductive hypothesis:

(f) If $G_\beta \neq \emptyset$ then there is a $k < \omega$ such that $f \cap \{n\} \times \omega \neq \emptyset \to g \cap \{n\} \times \omega = \emptyset$ for each $f \neq g$ in G_β and each $n > k$.

Assume that $\alpha < \omega_1$ is a limit and that we have fixed the sets B_β and G_β for $\beta < \alpha$ such that for each $\alpha' < \alpha$, the inductive hypotheses (a)-(f) holds at α'. It is easily follows that it holds also at α. To construct B_α and G_α consider S_α and F_α. Let $g_\beta(n) = \max((\bigcup G_\beta) \cap \{n\} \times \omega))$. By (f) it follows that g_β is a partial function on $\omega \setminus k_\beta$. If we let $B_\alpha \subseteq S_\alpha$ be any partial function which dominates g_β for all $\beta < \alpha$, then it will follow that each G_β still converges with respect to the filter $U_{\alpha+1}$. To define G_α, first note that the filter $U_{\alpha+1}$ is countably generated. So, if F_α is a π-net, then it is easy to extract a convergent sequence.

To extract a convergent sequence satisfying (f) we need to prove the following lemma

Lemma 14. *Suppose that we have any T_1 first-countable topology on $\omega \times \omega \cup \{\infty\}$ (with ∞ the only nonisolated point). Suppose that $F \subseteq [\omega \times \omega]^n$ is any π-net at ∞. Then there is a k and a convergent sequence $G \subseteq F$ such that for all $m > k$ and all $f \neq g$ from G, $f \cap \{m\} \times \omega \neq \emptyset \to g \cap \{m\} \times \omega = \emptyset$.*

Proof. By induction on n. For $n = 1$, since the space is Fréchet–Uryshon, the family of singletons F has a subset which converges to ∞. Either F intersects a column $\{m\} \times \omega$, in an infinite set F' (in which case we can take F' and let $k = m$) or F has finite intersection with each column. In the later case we can thin F out to F' which meets each column in at most 1 point.

Assume the Lemma holds for $n \geqslant 1$ and suppose that $F \subseteq [\omega \times \omega]^{n+1}$ is a π-net at ∞. Order $\omega \times \omega$ lexicographically.

Case 1: There is a k such that $F' = \{x \in F\colon \min(x) \in k \times \omega\}$ forms a π-net. In this case, use first-countability to assume without loss of generality that F' converges to ∞. Then apply the inductive hypothesis to $\{x \setminus \{\min(x)\}\colon x \in F'\}$.

Case 2: Not CASE 1. I.e., for every k the set $\{x \in F\colon x \cap k \times \omega = \emptyset\}$ is a π-net. In this case, it is easy to construct a subset of F convergent to ∞ with the required property satisfied by $k = 0$. □

This completes the proof of the lemma and completes the recursive construction. Let U be the neighborhood filter at ∞ generated by $\bigcup_\alpha U_\alpha$. Clearly the space is not α_3 since no S_α is a convergent sequence ($X \setminus B_\alpha$ is open and misses infinitely many points of S_α). Also, for any n and any π-net $F \subseteq [\omega \times \omega]^n$, there is a β such that $F = F_\beta$. Clearly, F is also a π-net with respect to U_β. So, G_β is not empty and converges to ∞ with respect to all U_α for $\alpha > \beta$. Hence, it converges with respect to U. □

Example 15. (CH) There is a countable Fréchet–Urysohn α_1-space which is not FU_2.

Proof. Let $X = (\omega \times 2) \cup \{\infty\}$. Points of $\omega \times 2$ are declared to be isolated. The base at the point ∞ will be the filter generated by complements of the sets in $\mathcal{I} = \{I(\alpha, e)\colon \alpha < \omega_1,\ e < 2\}$, where $I(\alpha, e)$ is a subset of $\omega \times \{e\}$. We will define these sets by induction. Also, for $\alpha < \omega_1$, we let U_α be the filter generated by complements of the sets in $\{I(\beta, e)\colon \beta < \alpha,\ e < 2\}$. For $A \subset X$, let $\pi(A) = \{n \in \omega\colon \exists e < 2((n, e) \in A\}$. Let p be any p-point in $\beta\omega \setminus \omega$; we will make sure each $\pi(I(\alpha, e))$ is not in p. For convenience, we call a subset A of $X \setminus \{\infty\}$ *p-small* if $\pi(A) \notin p$. Since it may be of some added interest, instead of only making X Fréchet, we will make each subspace $(\omega \times \{e\}) \cup \{\infty\}$ $\mathrm{FU}_{\mathrm{fin}}$. Let W_0 and W_1 be the even and odd countable ordinals, respectively. Let Y_α, $\alpha \in W_0$, and $\vec{A}_\alpha$, $\alpha \in W_1$, index, respectively, all infinite subsets of $[X]^{<\omega}$ and all sequences $(A_\alpha(n))_{n\in\omega}$ of infinite p-small subsets of X. One final bit of notation: for $A \subset X$, we let $A^\perp = \{(n, e)\colon (n, 1 - e) \in A\}$.

Suppose $\alpha < \omega_1$, and for all $\beta < \alpha$ we have constructed sets Z_β, $\beta \in W_0$, B_β, $\beta \in W_1$, and $I(\beta, e)$, $e < 2$, satisfying:

(a) If $\beta \in W_0$, $Y_\beta \subset [\omega \times \{e\}]^{<\omega}$, and Y_β is a π-net at ∞ with respect to U_β, then Z_β is an infinite subset of Y_β whose union is p-small and converges to ∞ with respect to U_β; furthermore, $I(\beta, 1-e) = (\bigcup Z_\beta)^\perp$;
(b) If Y_β consists of singletons, satisfies the conditions of (a), and $\cup Y_\beta$ is not p-small, then $I(\beta, e)$ is the union of an infinite p-small subset of Y_β, disjoint from Z_β, such that $I(\beta, e)^\perp$ is U_β-convergent;
(c) If $\beta \in W_1$, and $e < 2$ is such that, for each n, $A_\beta(n)$ is a U_β-convergent subsequence of $\omega \times \{e\}$, then B_β is p-small, U_β-convergent, and $B_\beta \,{}^* \supset A_\beta(n)$ for every n; furthermore, $I(\beta, 1-e) = B_\beta^\perp$.

If a set Z_α, B_α, or $I(\alpha, e)$ does not need to be defined because the hypotheses of the relevant condition (a), (b), or (c) are not satisfied, then simply define the set to be the empty set.

A key feature that is easily noted from the induction hypotheses is that for any $\alpha < \omega_1$ and $e < 2$, $I(\alpha, e)^\perp$ is convergent w.r.t. U_α. Let us suppose we have completed the inductive construction satisfying these conditions, and check that the space X is as desired.

We first show that each subspace $(\omega \times \{e\}) \cup \{\infty\}$ is $\mathrm{FU}_{\mathrm{fin}}$, which implies X is Fréchet. Suppose Y is a π-net at ∞ consisting of finite subsets of $\omega \times \{e\}$. Then $Y = Y_\alpha$ for some α, and by (a) above, Z_α is a subset of Y_α whose union is convergent in U_α. We need to see that this convergence is not destroyed at some later stage. Suppose $\beta > \alpha$ and $I(\beta, e) \cap (\bigcup Z_\alpha)$ is infinite. Then so is $I(\beta, e)^\perp \cap (\bigcup Z_\alpha)^\perp = I(\beta, e)^\perp \cap I(\alpha, 1-e)$, contradicting $I(\beta, e)^\perp$ convergent w.r.t. U_β. It easily follows from the inductive condition (b) that all sequences in X which converge to ∞ are p-small. Thus in the listing of the $\vec{A}$'s, we only needed to consider, as we did, those $\vec{A}$'s in which the terms $A(n)$ (i.e., the potential convergent sequences) were p-small. With this observation, α_1 follows easily from the inductive condition (c). Preservation of convergence works the same as in the previous paragraph.

Finally, let us check that X is not FU_2. Consider the collection

$$\mathcal{F} = \bigl\{\bigl\{(n,0),(n,1)\bigr\}\colon n \in \omega\bigr\}.$$

That $\mathcal{F}$ is a π-net follows from the fact that all of the $I(\alpha, e)$'s are p-small.

Now suppose A is an infinite subset of α such that $\{\{(n,0),(n,1)\}\colon n \in A\} = A \times 2$ is convergent. Then $\{\{n\}\colon n \in A\} = Y_\alpha$ for some α, and is U_α-convergent, so Z_α is an infinite subset of Y_α. But then $I(\alpha, 1-e) = Z_\alpha^\perp \subset A \times \{1-e\}$, contradicting that $A \times \{1-e\}$ is convergent.

Now let us check that the conditions (a)–(c) can be satisfied. Suppose $\alpha \in W_0$. Then we are given Y_α and we need to show that (a), and (b) too if relevant, may be satisfied. First choose an infinite subset Y'_α of Y_α that converges w.r.t. U_α; this is possible since U_α is countably generated. Then some infinite subsequence Z_α of Y'_α will have p-small union; this Z_α will satisfy (a). If (b) needs to be satisfied as well, then since $\bigcup Y_\alpha$ is not p-small, while every $I(\beta, f)$ for $\beta < \alpha$ and $f < 2$ is p-small, we can pass to a subsequence Y''_α of Y_α such that both $\bigcup Y''_\alpha$ and $(\bigcup Y''_\alpha)^\perp$ converge w.r.t. U_α. Then let Z_α and Z'_α be disjoint infinite subsequences of Y''_α, and let $I(\alpha, e) = \bigcup Z'_\alpha$. Finally, suppose $\alpha \in W_1$ and the hypotheses of (c) are satisfied. Recall that each $A_\alpha(n)$ is p-small. Since p is a p-point, there is a p-small set B'_α which almost contains every $A_\alpha(n)$. Since each $A_\alpha(n)$ is U_α-convergent, and U_α is countably generated, there exists a U_α convergent B''_α which almost contains each $A_\alpha(n)$. Then take $B_\alpha = B'_\alpha \cap B''_\alpha$. $\square$

5. An FU_n not FU_{n+1} space from CH

Sipacheva [19] noted that a point x in a space X is FU_n at x iff X^n is Fréchet at $(x, x, \ldots, x)$. This gives another way to see the result of the previous section that FU_2 spaces are α_4, since $X \times Y$ Fréchet is known to imply that X and Y are α_4. It also follows that a construction of K. Tamano [20] under Martin's Axiom of a space X such that X^n is Fréchet but X^{n+1} is not Fréchet is also a (consistent) example of a space that is FU_n but not FU_{n+1}. In this section we give another construction, assuming CH ($\mathfrak{p} = \mathfrak{c}$ would do), of a space that is FU_n but not FU_{n+1}. Except for the Fréchet fan, which is FU_1, i.e., Fréchet, but not FU_2, there apparently are no known ZFC examples of this phenomenon.

Question 7. Is there a ZFC example of a FU_2 not FU_3 space?

Example 16. (CH) For every $n \in \omega \setminus \{0\}$, there is an FU_n space which is not FU_{n+1}.

Proof. For each $i < n+1$ let $\omega^i = \{m^i \colon m \in \omega\}$ be the copy $\{i\} \times \omega$ of ω and let $X = \bigcup\{\omega^i \colon i < n+1\}$. And let $Y = X \cup \{\infty\}$. Points of X will be isolated and the neighborhood filter at ∞ will be constructed recursively.

For any $A \subseteq X$, let $\pi(A) = \{m \colon \exists i < n+1\ m^i \in A\}$. Enumerate the power set of $[X]^n$ by $\{T_\alpha \colon \alpha < \omega_1\}$.

By recursion on $\alpha < \omega_1$ we construct sets $C_\alpha \subseteq X$ and $S_\alpha \subseteq T_\alpha$. For $\alpha < \omega_1$ we will let U_α be the filter generated by $\{X \setminus C_\beta \colon \beta < \alpha\}$. For each $\alpha < \omega_1$ we require the sets to satisfy the following inductive hypotheses:

(a) For all $\beta < \alpha$, $S_\beta \neq \emptyset$ implies that S_β converges with respect to U_α.
(b) There is $(k_i \colon i < n)$ (depending on α and not all necessarily distinct) such that each $x \in S_\alpha$ is of the form $\{x(i) \colon i < n\}$ where $x(i) \in \omega^{k_i}$.
(c) For all $i \neq j$ either $\pi(x(i)) = \pi(x(j))$ for all $x \in S_\alpha$, or $\pi(x(i)) \neq \pi(x(j))$ for all $x \in S_\alpha$.
(d) $\{\pi(x) \colon x \in S_\alpha\}$ is pairwise disjoint family of sets. Moreover, for all $x \neq y$ from S_α, either $\max \pi(x) < \min \pi(y)$ or $\max \pi(y) < \min \pi(x)$.
(e) For each $i < n$ and for all $\beta < \alpha$, either $\{x(i) \colon x \in S_\alpha\}$ is almost disjoint from $\bigcup S_\beta$, or $\{x(i) \colon x \in S_\alpha\} \subseteq \{x(j) \colon x \in S_\beta\}$ for some $j < n$.

Let I be the set of $i < n$ such that $\{x(i) \colon x \in S_\alpha\}$ is almost disjoint from $\bigcup S_\beta$ for all $\beta < \alpha$. And let $S'_\beta = \{x(i) \colon i \in I \text{ and } x \in S_\beta\}$. Then

(f) C_α is the largest subset of X such that $C_\alpha \cap S'_\alpha = \emptyset$ and $\pi(C_\alpha) = \pi(S'_\alpha)$.
(g) $\{\pi(C_\beta) \colon \beta < \alpha\}$ is an almost disjoint family.

Suppose first that α is a limit and $\{S_\beta\colon \beta<\alpha\}$ and $\{C_\beta\colon \beta<\alpha\}$ have been constructed so that for all $\alpha'<\alpha$ the inductive hypotheses are satisfied at α'. It is easy to check that they are also satisfied at α.

It suffices to explain how to choose S_α and C_α preserving the inductive hypotheses at $\alpha+1$. Consider T_α. If it is not a π-net with respect to neighborhood filter U_α, then let $S_\alpha=C_\alpha=\emptyset$. Otherwise, first fix $S\subseteq T_\alpha$ so that S converges with respect to U_α. For each $x\in S$, order x lexicographically and let $x=\{x(0),\dots,x(n-1)\}$ be its increasing enumeration. Let $\bar{k}^x=(k_0^x,\dots,k_{n-1}^x)$ be such that $x(i)\in\omega^{k_i^x}$ for each $i<n$. Since ${}^n(n+1)$ is finite, by taking an infinite subset of S we may assume that there is a $\bar{k}=(k_i\colon i<n)$ such that $\bar{k}^x=\bar{k}$ for all $x\in S$.

Thus, any subset of S will satisfy inductive hypothesis (b). Since α is countable, it is easy to see that we may find $S_\alpha\subseteq S$ satisfying the inductive hypotheses (d) and (e) (for (e) it suffices to shrink S countably many times and take S_α a pseudointersection of the resulting sequence of subsets).

Inductive hypothesis (f) forces us to define

$$C_\alpha=\bigl\{m^j\colon j<n+1 \text{ and } \exists i\,(m^i\in S'_\alpha)\bigr\}\setminus S'_\alpha.$$

Notice that all the inductive hypotheses except (a) and (g) hold directly by construction. To verify that the other inductive hypotheses hold at $\alpha+1$ we need to prove the following lemmas:

Claim 1. $C_\alpha\cap\bigcup S_\alpha=\emptyset$.

Claim 2. $C_\alpha\cap\bigcup S_\beta$ *is finite for all* $\beta<\alpha$.

Claim 3. $\pi(C_\alpha)\cap\pi(C_\beta)$ *is finite for all* $\beta<\alpha$.

Note that Claim 1 assures that S_α converges with respect to $U_{\alpha+1}$. And for each $\beta<\alpha$, Claim 2 assures that S_β converges with respect to $U_{\alpha+1}$. Hence inductive hypothesis (a) holds. Claim 3 assures that inductive hypothesis (g) holds.

Proof of Claim 1. Suppose that $m^l\in C_\alpha\cap\bigcup S_\alpha$ for some $m<\omega$ and $l<n+1$. $m^l\in\bigcup S_\alpha$ means that there is $x_0\in S_\alpha$ and a j such that $x_0(j)=m^l=m^{k_j}$. Also, by definition of C_α there is a $x_1\in S_\alpha$ and an $i\neq j$ such that $x_1(i)=m^{k_i}\in S'_\alpha$ (and moreover, since $C_\alpha\cap S'_\alpha$, $k_i\neq k_j$). By (d) it follows that $x_0=x_1$. So by (c) it follows that $\pi(x(i))=\pi(x(j))$ for all $x\in S_\alpha$. Also, by definition of S'_α, it follows that $\{x(i)\colon x\in S_\alpha\}\subseteq S'_\alpha$, and hence $\{x(i)\colon x\in S_\alpha\}$ is almost disjoint from each S_β with $\beta<\alpha$. On the other hand, since $x_0(j)\in C_\alpha$ and $x_0(j)\notin S'_\alpha$, there is a $\beta_0<\alpha$ with

$$\bigl\{x(j)\colon x\in S_\alpha\bigr\}\subseteq\bigl\{x(i')\colon x\in S_{\beta_0}\bigr\}.$$

Assume β_0 to be minimal with this property. By minimality, it follows that $\{x(i')\colon x\in S_{\beta_0}\}\subseteq S'_{\beta_0}$ (otherwise, $\{x(i')\colon x\in S_{\beta_0}\}$ would be a subset of a smaller S_β and in turn so would $\{x(j)\colon x\in S_\alpha\}$, contradicting the minimality of β_0). It follows that $\{x(j)\colon x\in S_\alpha\}\subseteq S'_{\beta_0}$. Thus, by definition of C_{β_0} and since $\pi(x(i))=\pi(x(j))$ for each $x\in S_\alpha$, either $\{x(i)\colon x\in S_\alpha\}\subseteq C_{\beta_0}$, contradicting that S_α converges with respect to U_α, or $\{x(i)\colon x\in$

$S_\alpha\} \subseteq S'_{\beta_0}$, contradicting $\{x(i)\colon x \in S_\alpha\}$ is almost disjoint from $\bigcup S_\beta$. In either case we reach a contradiction. □

Proof of Claim 2. Suppose not. Let β_0 be the minimal β satisfying $C_\alpha \cap \bigcup S_\beta$ is infinite. Thus there is an i and $R_{\beta_0} \subseteq S_{\beta_0}$ infinite such that $x(i) \in C_\alpha$ for all $x \in R_{\beta_0}$. By minimality of β_0 we have that $\{x(i)\colon x \in S_{\beta_0}\}$ is almost disjoint from $\bigcup S_\beta$ for all $\beta < \beta_0$. Therefore,

$$\{x(i)\colon x \in S_{\beta_0}\} \subseteq S'_{\beta_0}.$$

Let $A = \{\pi(x(i))\colon x \in R_{\beta_0}\}$. There is a $k < n+1$ such that $\{m^k\colon m \in A\} = \{x(i)\colon x \in R_{\beta_0}\} \subseteq C_\alpha$. By choice of C_α, there is a j such that $\{m^j\colon m \in A\} \subseteq \bigcup S_\alpha$ and such that $\{m^j\colon m \in A\}$ is almost disjoint from all previous $\bigcup S_\beta$ in particular almost disjoint from $\bigcup S_{\beta_0}$. However, since

$$\{m^k\colon m \in A\} \subseteq \{x(i)\colon x \in S_{\beta_0}\} \subseteq S'_{\beta_0},$$

it follows by choice of C_{β_0} that

$$\{m^j\colon m \in A\} \subseteq^* C_{\beta_0}.$$

But this contradicts that S_α converges with respect to U_α. □

Proof of Claim 3. Suppose not and take β_0 minimal such that $\pi(C_\alpha) \cap \pi(C_{\beta_0})$ is infinite. Let $A \subseteq \omega$ be infinite and let i and j be given such that $\{m^i\colon m \in A\} \subseteq C_\beta$ and $\{m^j\colon m \in A\} \subseteq C_\alpha$. By definition of C_{β_0}, for every i', if $\{m^{i'}\colon m \in A\}$ is almost disjoint from $\bigcup S_{\beta_0}$ then $\{m^{i'}\colon m \in A\} \subseteq C_{\beta_0}$. Also, by definition of C_α, there is a j' and a k_α such that $\{m^{j'}\colon m \in A\} \subseteq \{x(k_\alpha)\colon x \in S_\alpha\}$. And $\{x(k_\alpha)\colon x \in S_\alpha\}$ is almost disjoint from all previous $\bigcup S_\beta$. So in particular it is almost disjoint from $\bigcup S_{\beta_0}$. Thus by the previous observation it follows that $\{x(k_\alpha)\colon x \in S_\alpha\}$ has infinite intersection with C_{β_0}. This contradicts that S_α converges with respect to U_α. □

This completes the recursive construction. Moreover, it is clear from the construction that the space is FU_n. To complete the proof we need the following final claim:

Claim 4. $\{\{m^i\colon i < n+1\}\colon m < \omega\}$ *is a* π*-net with no convergent subsequence.*

Proof. To see that it is a π-net note that the neighborhood base at ∞ is generated by $\{X \setminus C_\alpha\colon \alpha < \omega_1\}$ and the family of $\pi(C_\alpha)$'s form an almost disjoint family (although some of the sets may be empty). Being a π-net is equivalent to saying that ω is not covered by finitely many of the sets $\pi(C_\alpha)$. So it suffices to verify that infinitely many of the C_α's are not empty. It can be easily arranged that the first ω many sets $\{C_m\colon m < \omega\}$ are all not empty by arranging $T_m = \{\{k^i\colon i < n\}\colon k \in A_m\}$ where A_m is some disjoint infinite family of sets.

To see that it has no convergent subset, suppose that A is infinite and let's show that $S = \{\{m^i\colon i < n+1\}\colon m \in A\}$ is not a convergent sequence. If it were, then $T = \{\{m^i\colon i < n\}\colon m \in A\}$ would also be a convergent sequence. And there is an α such that $T = T_\alpha$. In this case, A is almost disjoint from all sets $\pi(C_\beta)$ for $\beta < \alpha$. So S_α was chosen at this stage and $S_\alpha = \{\{m^i\colon i < n\}\colon m \in B\}$ for some infinite $B \subseteq A$. It is easy to check that in this case $C_\alpha = \{m^n\colon m \in B\}$. And that S does not converge to ∞ is witnessed by the open set $X \setminus C_\alpha$. Thus X is not FU_{n+1}. $\square$

6. Games and products

In this section, we show that $\mathrm{FU}_{\mathrm{fin}}$-spaces have an interesting game characterization, analogous to a game characterization of a similar property called the "Moving Off Property", or MOP, in [10], which is similar to $\mathrm{FU}_{\mathrm{fin}}$ but with finite sets replaced by compact sets. Related to the game characterization of $\mathrm{FU}_{\mathrm{fin}}$ are characterizations involving sequences of π-nets, and there are applications concerning when the product of a $\mathrm{FU}_{\mathrm{fin}}$-space and another space is $\mathrm{FU}_{\mathrm{fin}}$. The characterizations involving sequences of π-nets are also reminiscent of a similar characterization of γ-sets involving sequences of ω-covers instead of just one ω-cover.

Let X be a space and $x \in X$. In [5], the following game $G_{O,P}(X,x)$ was introduced. At the nth play, O chooses an open neighborhood O_n of x, and P responds by choosing a point $x_n \in O_n$. O *wins* the game if $\{x_n\colon n \in \omega\}$ converges to x. A space in which O has a winning strategy was called a *W-space*, and a space in which P fails to have a winning strategy was called a *w-space*. Clearly, first-countable spaces are W-spaces, and it turns out separable W-spaces must be first-countable. A prototypical non-first-countable W-space is the one-point compactification of an uncountable discrete space. On the other hand, separable or even countable w-spaces need not be first-countable; in fact it was essentially shown by P.L. Sharma [18] that w-spaces are the same as Fréchet–Urysohn α_2-spaces.

Now suppose we modify the game $G_{O,P}(X,x)$ by allowing P to choose a finite set of points at each play instead of just one point (with O winning if the union of P's sets is a sequence converging to x); denote this game by $G^{\mathrm{fin}}_{O,P}(X,x)$. It was noted in [5] that this game is equivalent for O in the sense that O has a winning strategy in $G^{\mathrm{fin}}_{O,P}(X,x)$ iff O has a winning strategy in $G_{O,P}(X,x)$. However, it is not equivalent, at least consistently, for player P. As noted above, P has no winning strategy in $G_{O,P}(X,x)$ iff x is an Fréchet α_2-point, while the next theorem shows that P has no winning strategy in $G^{\mathrm{fin}}_{O,P}(X,x)$ iff X is $\mathrm{FU}_{\mathrm{fin}}$ at x. Incidentally, this gives another way of obtaining Reznichenko and Sipacheva's result that $\mathrm{FU}_{\mathrm{fin}}$ implies α_2, because if P has no winning strategy in $G^{\mathrm{fin}}_{O,P}(X,x)$, P has none in $G_{O,P}(X,x)$ either.

Theorem 17. [1] *Let X be a space and $x \in X$. The following are equivalent*:

(i) *X is $\mathrm{FU}_{\mathrm{fin}}$ at x*;
(ii) *For each sequence $(P_n)_{n\in\omega}$ of π-nets at x consisting of finite sets, for infinitely many $n \in \omega$ there are $F_n \in P_n$ such that $\{F_n\colon n \in \omega\}$ converges to x*;

[1] This result should be compared with Theorem 2.3 in [6], which has similar form with collections of finite sets which are π-nets at x replaced by collections of compact sets which are π-nets at a point at infinity whose neighborhoods are complements of compact subsets of X.

(iii) *For each sequence $(P_n)_{n\in\omega}$ of π-nets at x consisting of finite sets, for each $n \in \omega$ there are $F_n \in P_n$ such that $\{F_n\colon n \in \omega\}$ converges to x;*
(iv) *P has no winning strategy in the game $G^{\mathrm{fin}}_{O,P}(X, x)$.*

Proof. That (iii) implies (ii) is obvious, and that (ii) implies (i) is easy: just apply (ii) with $P_n = P$ for each n, where P is some π-net at x. Reznichenko and Sipcheva [16] show that (i) implies (iii). So we have that (i)–(iii) are equivalent. Now suppose (iv) holds. Let P_n, $n \in \omega$, be a sequence of π-nets at x consisting of finite sets. Then P can choose $F_n \in P_n$ at his nth play. Since this strategy can't always win, there must be a sequence of such F_n's converging to x. This shows (iv) implies (iii).

It remains to prove (i)–(iii) implies (iv). Suppose s is a strategy for P in $G^{\mathrm{fin}}_{O,P}(X, x)$; we need to show that s can be defeated. Let $\mathcal{S}_\emptyset$ be the set of all first moves of P using the strategy s. Note that $\mathcal{S}_\emptyset$ is a π-net at x. By (i), there is a sequence $F^\emptyset_n$, $n \in \omega$, of elements of $\mathcal{S}_\emptyset$ converging to x. For each m, let $\mathcal{S}_{\langle m\rangle}$ be the set of all responses by P using s to O's second move, after some first move by O where P's response was the set $F^\emptyset_m$. Then choose $\{F^{\langle m\rangle}_n\colon n \in \omega\} \subset \mathcal{S}_{\langle m\rangle}$ converging to x. In general, if F^σ_n has been defined for all $\sigma \in \omega^k$ and $n \in \omega$, let $\mathcal{S}_{\sigma^\frown\langle m\rangle}$ be the set of all responses by P using s to O's next move, where O's previous moves led to P playing $F^\emptyset_{\sigma(0)}, F^{\sigma|1}_{\sigma(1)}, F^{\sigma|2}_{\sigma(2)}, \ldots, F^{\sigma|(k-1)}_{\sigma(k-1)}, F^\sigma_m$, and choose a sequence $F^{\sigma^\frown\langle m\rangle}_n$, $n \in \omega$, of elements of $\mathcal{S}_{\sigma^\frown\langle m\rangle}$ converging to x. Noting that $\{F^\sigma_j\colon j \in \omega\}$ is a π-net for each $\sigma \in \omega^{<\omega}$, by (iii) there are $j(\sigma) \in \omega$ such that $\{F^\sigma_{j(\sigma)}\colon \sigma \in \omega^{<\omega}\}$ converges to x. Then we can find $\tau \in \omega^\omega$ such that $\{F^{\tau|n}_{j(\tau|n)}\colon \sigma \in \omega^{<\omega}\}$ is a subsequence of $\{F^\sigma_{n(\sigma)}\colon n \in \omega\}$; but $\{F^{\tau|n}_{j(\tau|n)}\colon n \in \omega\}$ is the result of a play of the game with P using s. So s is not a winning strategy for P. □

Nyikos noted that the Cantor tree space over F, which we denoted X_F in Section 3, is a w-space, i.e., Fréchet α_2, if F is a λ'-set in the Cantor set (which means that for every countable subset A of the Cantor set, A is G_δ in $F \cup A$). Since X_F is $\mathrm{FU}_{\mathrm{fin}}$ iff F is a γ-set, taking F to be a λ'-set which is not a γ-set provides an example of a space in which P has no winning strategy in $G_{O,P}(X, x)$ but, by Theorem 17, P does have a winning strategy in $G^{\mathrm{fin}}_{O,P}(X, x)$. There are λ'sets in ZFC (see, e.g., [11]), so there are many models in which there are λ'-sets which are not γ. However, A. Miller [12] has shown that in the standard model of $\mathrm{MA}_{\sigma\text{-centered}} + \mathfrak{c} = \omega_2$, there are no λ'-sets of cardinality $\mathfrak{c} = \omega_2$, so, since $\mathfrak{p} = \omega_2$ here, it follows that every λ'-set in this model is also a γ-set. Hence the Cantor tree type spaces do not appear to give ZFC examples in which the games are inequivalent for P, and indeed we do not know of any. In an equivalent form, this is the following question:

Question 8. Is there in ZFC a Fréchet α_2-space which is not $\mathrm{FU}_{\mathrm{fin}}$?[2]

The analogue of the equivalence of (i) and (ii), or (i) and (iii), in Theorem 17 for FU_n is false. Indeed, condition (ii) for π-nets of singletons is equivalent to Fréchet α_4, which is

[2] In a sequel [8] to this paper, we show that a certain space obtained from a Hausdorff gap provides a positive answer, in ZFC, to this question.

stronger than $\mathrm{FU}_1 = $ Fréchet, and (iii) for singletons is equivalent to Fréchet α_2. However, we do have the following:

Theorem 18. *Let X be a space, $x \in X$.*

(1) *If $k \in \omega$, and X is FU_{k+1}, then for any sequence P_n, $n \in \omega$, of π-nets at x consisting of k-element sets, for infinitely many $n \in \omega$ there are $F_n \in P_n$ such that $\{F_n\colon n \in \omega\}$ converges to x;*
(2) *X is boundedly $\mathrm{FU}_{\mathrm{fin}}$ iff for any k and for any sequence P_n, $n \in \omega$, of π-nets at x consisting of k-element sets, for infinitely many $n \in \omega$ there are $F_n \in P_n$ such that $\{F_n\colon n \in \omega\}$ converges to x.*

Proof. (1) Suppose X is FU_{k+1}, and P_n, $n \in \omega$, is a sequence of π-nets at x consisting of k-element sets. Take any non-trivial sequence x_n, $n \in \omega$, converging to x. Consider the collection $\{\{x_n\} \cup F\colon n \in \omega$ and $F \in P_n\}$. It is easy to check that this collection is a π-net consisting of sets of cardinality $\leqslant k + 1$. Since any convergent subsequence of this collection has only finitely many terms of the form $\{x_n\} \cup F$ for fixed n, there is an infinite subset A of ω such that $\{\{x_n\} \cup F_n\}_{n \in A}$ is a convergent subsequence. Then the collection $\{F_n\colon n \in A\}$ is the desired convergent selection from infinitely many of the P_n's.

(2) The "if" part of (2) is easy, and the "only if" part is immediate from (1). $\square$

Remark. Part (1) of the above theorem for $k = 1$ gives another proof that FU_2 implies α_4.

We now turn to applications of the above results to products. The part of Theorem 20 below about the $\mathrm{FU}_{\mathrm{fin}}$ property generalizes a corresponding result of Reznichenko and Sipacheva, who proved it in the case y has countable character in Y or if Y is the one-point compactification of a discrete space.

First, it will be helpful to have the following lemma which shows that a finite-set version of countably tight is preserved by products with W-spaces. For the standard version of countably tight, this was proved in [5]. (By "y is a W-point in Y", we mean "O has a winning strategy in $G_{O,P}(Y, y)$".)

Lemma 19. *Let X and Y be spaces, and $(x, y) \in X \times Y$. Suppose that y is a W-point in Y, and that every π-net at x in X consisting of finite sets (resp., $\leqslant k$-element sets for some $k \in \omega$) contains a countable π-net at x. Then every π-net at (x, y) in $X \times Y$ consisting of finite sets (resp., $\leqslant k$-element sets) contains a countable π-net at (x, y).*

Proof. Let σ be a winning strategy for O at y in Y in the game $G^{\mathrm{fin}}_{O,P}(Y, y)$ (recall that, for O, this game is equivalent to $G_{O,P}(Y, y)$). That is, σ is a function which assigns to each finite sequence $H_0, H_1, \ldots, H_n$ of finite subsets of Y an open neighborhood $\sigma(H_0, H_1, \ldots, H_n)$ of y such that if O plays $\sigma(H_0, H_1, \ldots, y_n)$ whenever P has played $H_0, H_1, \ldots, H_n$, then $\{H_n\colon n \in \omega\}$ converges to y.

Consider an arbitrary π-net $\mathcal{F}$ at (x, y) consisting of finite sets (or $\leqslant k$-element sets). Let $\mathcal{M}$ be a countable elementary submodel (of some sufficiently large fragment of the universe) containing all relevant objects (X, Y, $\mathcal{F}$, etc.). We claim that $\mathcal{M}\cap\mathcal{F}$ is a countable π-net at (x, y).

To see this, suppose (x, y) is in the open set $U \times V$. Let

$$\mathcal{F}(\emptyset) = \{\pi_X(F)\colon\ F \in \mathcal{F} \text{ and } \pi_Y(F) \subset \sigma(\emptyset)\}.$$

Then $\mathcal{F}(\emptyset)$ is in $\mathcal{M}$ and is a π-net at x, so there is, in $\mathcal{M}$, a countable subset $\mathcal{C}(\emptyset)$ of $\mathcal{F}(\emptyset)$ which is a π-net at x and is also in $\mathcal{M}$. Hence there is $F_0 \in \mathcal{F}\cap\mathcal{M}$ such that $\pi_X(F_0) \subset U$ and $\pi_Y(F_0) \subset \sigma(\emptyset)$. By the same argument, if $F_i \in \mathcal{F}\cap\mathcal{M}$ have been defined for $i < n$, we can find $F_n \in \mathcal{F}\cap\mathcal{M}$ such that $\pi_X(F_n) \subset U$ and $\pi_Y(F_n) \subset \sigma((\pi_Y(F_i))_{i<n})$. Then the sequence $(\pi_Y(F_n))_{n\in\omega}$ is the result of a play of the game $G^{\text{fin}}_{O,P}(Y, y)$ with O using σ, hence for some n we must have $\pi_Y(F_n) \subset V$. Then F_n is contained in $U \times V$ and is a member of $\mathcal{F}\cap\mathcal{M}$. □

Theorem 20. *Suppose X and Y are spaces, $x \in X$, and $y \in Y$. If x is a* (*boundedly*) FU_{fin} *point in X and y is a W-point in Y, then (x, y) is a* (*boundedly*) FU_{fin} *point in $X \times Y$.*

Proof. We prove the boundedly FU_{fin} case, the other being similar. Suppose $\mathcal{F}$ is a π-net at (x, y) consisting of k-element sets. By the previous lemma, we may assume $\mathcal{F}$ is countable. Since countable W-spaces are first-countable, there is a decreasing neighborhood base $\{U_n\}_{n\in\omega}$ at y relative to the subspace $\{y\} \cup (\bigcup\{\pi_Y(F)\colon\ F \in \mathcal{F}\})$ of Y. Let $\mathcal{F}_n = \{\pi_X(F)\colon\ F \in \mathcal{F} \text{ and } \pi_Y(F) \subset U_n\}$. Then $\mathcal{F}_n$, $n \in \omega$, is a sequence of π-nets at x consisting of $\leqslant k$-element sets, so by Theorem 18, for infinitely many n there are $F_n \in \mathcal{F}$ with $\pi_Y(F_n) \subset U_n$ such that the $\pi_X(F_n)$'s converge to x. Then the F_n's converge to (x, y). □

Corollary 21. *Suppose that X is* FU_{fin} *at the point x, and that $y \in Y$. If $\chi(y, Y) = \aleph_0$, or more generally if y is a W-point in Y, then the quotient space $X \oplus Y/\{x, y\}$ obtained by taking the topological sum of X and Y and identifying the points x and y is* FU_{fin} *at $\{x, y\}$.*

Proof. This quotient space is homeomorphic to a subspace of $X \times Y$, so the result follows from Theorem 20. □

Remarks. It is consistent that Corollary 21 does not hold if one only assumes that y is a FU_{fin} point in Y. Indeed, it follows from CH that there are two γ sets X and Y such that $X \oplus Y$ is not a γ-set. Thus the corresponding FU_{fin} spaces T_X and T_Y do not satisfy the conclusion of Corollary 21. In addition, the subspaces $(\omega\times\{0\})\cup\{\infty\}$ and $(\omega\times\{1\})\cup\{\infty\}$ of Example 15 provides a consistent example where the conclusion fails badly: the quotient space obtained by identifying the two ∞ points is not even FU_2. So, some strengthening of the FU_{fin} property is needed for these results. Since any space of character $< \mathfrak{p}$ is FU_{fin} and α_1, we are led to ask:

Question 9. Do Corollary 21 or Theorem 20 hold assuming only $\chi(y, Y) < \mathfrak{p}$?

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