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On separating families of bipartitions

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ABSTRACT

We focus on families of *bipartitions*, i.e. set partitions consisting of at most two components. A family of bipartitions is a *separating family* for a set if every two elements in the set are separated by some bipartition. In this paper we enumerate separating families of arbitrary size. We furthermore enumerate inclusion-wise minimal separating families of minimum and maximum sizes.

1. Introduction

A *bipartition* of a set S is $\{S\}$ or an unordered pair $\{U, V\}$ of nonempty subsets of S such that $U \cap V = \emptyset$ and $U \cup V = S$. Note that we allow $\{S\}$ as a bipartition, because it corresponds to the case where the ground set S is divided into S and $\emptyset$. A collection of bipartitions of S is a *separating family* for S if every two elements in S are separated by some bipartition in the collection, that is, they are contained in different components of some bipartition. A separating family for S is *minimal* if no proper subfamily is a separating family for S.

Example 1. Let $S = \{1, 2, 3, 4\}$ and let P_1, P_2, Q_1, Q_2, Q_3 be the bipartitions given as

$$P_1 = \{\{1, 2\}, \{3, 4\}\}, \qquad Q_1 = \{\{1\}, \{2, 3, 4\}\},$$
$$P_2 = \{\{1, 3\}, \{2, 4\}\}, \qquad Q_2 = \{\{1, 2\}, \{3, 4\}\},$$
$$Q_3 = \{\{1, 2, 3\}, \{4\}\}.$$

The family of bipartitions $\{P_1, P_2\}$ is a minimal separating family of minimum size for S, while $\{Q_1, Q_2, Q_3\}$ is a minimal separating family of maximum size. Here the size of a separating family denotes its cardinality.

The concept of separating families appears in the following search problem. Suppose that we are given a finite set S and a collection $\{P_1, \ldots, P_m\}$ of bipartitions of S. For an unknown element x in S, we choose a bipartition P_i and we are allowed to ask which component of P_i contains x, thereby narrowing down the range containing x. The goal is to locate the unknown element x by asking a series of such questions. One can easily observe that for every element in S there exists a series of questions which leads to finding it if and only if $\{P_1, \ldots, P_m\}$ is a separating family for S. Rényi [9] initiated the study of the search problem described above, although he did not employ bipartitions but subsets of S as questions. Since then, many authors have studied combinatorial problems related to finding the minimum size of a separating family under various

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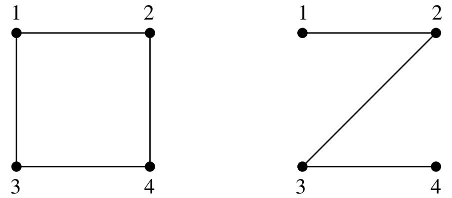


Fig. 1. The graphs induced by the minimal separating families $\{P_1, P_2\}$ and $\{Q_1, Q_2, Q_3\}$ given in Example 1.

constraints (see [1,6,7] for a survey). In this paper we deal with bipartitions since we do not want to distinguish a subset of S from its complements.

Motivation and contribution. As mentioned above, separating families can be considered as sets of questions which allow to locate an arbitrary unknown element in the ground set. It is natural to ask how many such sets of questions exist for a given ground set. Since one is often interested in inclusion-wise minimal sets, we first concentrate on separating families which are minimal and enumerate those of maximum size. This is done by obtaining a bijection from the set of minimal separating families of maximum size for a set S to the set of spanning trees on S. This result partially answers our enumeration question in the minimal case. In a next step we enumerate separating families of arbitrary size, which need not be minimal. This result includes minimal separating families of minimum size as a special case.

This paper is organized as follows. In Section 2 we enumerate all minimal separating families of maximum size for a finite set. In Section 3 we extend this analysis and enumerate all separating families of arbitrary size, which need not be minimal.

2. Minimal separating families of maximum size and spanning trees

In this section we enumerate all minimal separating families of maximum size for a finite set S by obtaining a bijection onto the set of all spanning trees on S.

Proposition 2.1. *The maximum size of a minimal separating family for an n-element set is $n - 1$.*

Proof. Let S be an n-element set and let a be a fixed element in S. It is easy to see that the collection of all bipartitions of the form $\{\{x\}, S \setminus \{x\}\}$ for any x other than a is a minimal separating family of size $n - 1$, for $n > 2$. To see that the maximum size is at most $n - 1$, let $\mathcal{P}$ be any minimal separating family. Let us remove members from $\mathcal{P}$ one by one, as shown below, where Q_1, Q_2, Q_3 are the bipartitions given in Example 1.

$$\{1\}, \{2\}, \{3\}, \{4\} \xrightarrow{Q_1} \{1, 2\}, \{3\}, \{4\} \xrightarrow{Q_2} \{1, 2, 3\}, \{4\} \xrightarrow{Q_3} \{1, 2, 3, 4\}.$$

The first step shows that $\{1\}$ and $\{2\}$ are merged by removing Q_1, because neither Q_2 nor Q_3 separate them. From the minimality of $\mathcal{P}$ it follows that in each step some isolated components get merged; in other words, the number of isolated components decreases by at least one. Since there are n isolated components in the initial state, the size of $\mathcal{P}$ is at most $n - 1$. □

Definition 1. Let S be a nonempty finite set. For each minimal separating family $\mathcal{P}$ for S we associate a graph denoted by $\Phi_S(\mathcal{P})$ such that the vertices of the graph correspond to the elements in S and two vertices are adjacent if the corresponding elements in S are separated by exactly one bipartition in $\mathcal{P}$.

This definition induces a correspondence between the bipartitions in $\mathcal{P}$ and the edges in $\Phi_S(\mathcal{P})$. One can easily observe that each edge of $\Phi_S(\mathcal{P})$ corresponds to a unique bipartition in $\mathcal{P}$, while each bipartition may correspond to multiple edges (see the left graph in Fig. 1).

Theorem 2.2. *Let S be a nonempty finite set. The mapping $\Phi_S : \mathcal{P} \mapsto \Phi_S(\mathcal{P})$ is a bijection from the set of all minimal separating families of maximum size for S to the set of all spanning trees on S.*

Proof. Proposition 2.1 states that the maximum size of a minimal separating family for an n-element set S is $n - 1$. Let G be a maximal subgraph of $\Phi_S(\mathcal{P})$ such that the vertex set of G is S and no two edges in G are separated by the same bipartition in $\mathcal{P}$. Here an edge is *separated* by a bipartition if the end-vertices of the edge are separated by this bipartition.

Such a subgraph G exists, because the empty graph on S trivially satisfies the two conditions above, thus a desired maximal subgraph is obtained by adding edges of $\Phi_S(\mathcal{P})$ as long as the second condition holds. By the definition of $\Phi_S(\mathcal{P})$, each edge in G corresponds to a unique bipartition in $\mathcal{P}$ and by the second condition no two edges in G correspond to the same bipartition. On the other hand each bipartition in $\mathcal{P}$ corresponds to some edge in G, because otherwise, we could further extend G by adding one of the edges in $\Phi_S(\mathcal{P})$ corresponding to the bipartition. Therefore, the edges in G are in one-to-one correspondence with the bipartitions in $\mathcal{P}$.

Next suppose for contradiction that G contains a cycle. Consider any bipartition $P \in \mathcal{P}$ separating some edge in a cycle. As argued above P only separates its own edge e_P of G. This is however impossible because there are two ways to connect the end-vertices of e_P in the cycle and thus P separates not only e_P but also some edge in the other path connecting the end-vertices of e_P. Therefore, we conclude that G is a forest. Since the vertex set of G is of size n and the edge set of G is of size $n-1$, it follows that G is a spanning tree on S.

To prove that $\Phi_S(\mathcal{P})$ is a spanning tree on S, suppose $G \neq \Phi_S(\mathcal{P})$. Thus there is an edge f in $\Phi_S(\mathcal{P})$ which is not in G and adding f to G yields a cycle. Let e and e' be two distinct edges in the cycle other than f. The second condition implies that the bipartitions P_e and $P_{e'}$ corresponding to e and e' do not separate any edges in G other than e and e', respectively. Thus both P_e and $P_{e'}$ must separate f. Since each edge of $\Phi_S(\mathcal{P})$ corresponds to a unique bipartition, we obtain $P_e = P_{e'}$, which is a contradiction. Therefore, we obtain $G = \Phi_S(\mathcal{P})$ and $\Phi_S(P)$ is indeed a spanning tree on S. It is straightforward to see that Φ_S is a one-to-one and onto mapping. □

Since Cayley's formula [3] states that the number of spanning trees in the complete graph on n labeled vertices is n^{n-2}, we immediately obtain the following enumeration result.

Theorem 2.3. *The number of minimal separating families of maximum size for an n-element set is* n^{n-2}.

3. Enumerating separating families of arbitrary size

In this section we shift our attention to separating families which need not be minimal and enumerate separating families of arbitrary size by representing them as matrices whose entries are 0 or 1. This result includes the enumeration of minimal separating families of minimum size as a special case.

Throughout this section we assume without loss of generality that $S = \{1, \ldots, n\}$.

Definition 2. For each bipartition P of S, we define $b(P)$ to be the vector of length n whose i-th coordinate is given by

$$b_i(P) = \begin{cases} 1 & \text{if } P \text{ separates } 1 \text{ and } i \\ 0 & \text{otherwise.} \end{cases}$$

Any k-tuple $(P_1, \ldots, P_k)$ of bipartitions can then be encoded as an $n \times k$ matrix $M_{(P_1,\ldots,P_k)}$ whose j-th column vector is $b(P_j)$.

We distinguish between a family of bipartitions and a tuple of bipartitions by denoting the former by $\mathcal{P}$, $\mathcal{Q}$, etc. and the latter by P, Q, etc. We call vectors and matrices whose entries are 0 or 1 (0, 1)-*vectors* and (0, 1)-*matrices*, respectively.

Example 2. The matrix-representations of the tuples (P_1, P_2) and (Q_1, Q_2, Q_3) whose entries are given in Example 1 are

$$M_{(P_1,P_2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad M_{(Q_1,Q_2,Q_3)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The following lemma is straightforward to verify.

Lemma 3.1. *The mapping* $P \mapsto M_P$ *is a bijection from the set of all k-tuples of bipartitions of S to the set of all* (0, 1)*-matrices of size* $n \times k$ *such that the entries in the first row are all* 0.

Note that column vectors whose entries are all 0 correspond to the bipartition $\{S\}$.

Lemma 3.2. *Let* $\mathcal{P}$ *be a family of bipartitions of S and let P be a tuple obtained from* $\mathcal{P}$ *by some ordering of its members. It holds that* $\mathcal{P}$ *is a separating family for S if and only if every two row vectors of* M_P *are distinct.*

Proof. It is easy to see that for every two elements $i, j \in S$ there is a bipartition in $\mathcal{P}$ separating them if and only if the i-th row vector and the j-th row vector of M_P are distinct. From this observation this lemma immediately follows. □

The following result is known (see for example Lemma 1 in [4] and Theorem 4 in [7]), but for the completeness we prove it.

Proposition 3.3. *The minimum size of a separating family for an n-element set is* $\lceil \log_2 n \rceil$.

Proof. Suppose for contradiction that the minimum size m of a separating family is less than $\lceil \log_2 n \rceil$. Thus $2^m < n$ and since the number of all (0, 1)-vectors of length m is 2^m, at least two row vectors in an $n \times m$ matrix have to coincide. This is a contradiction according to Lemma 3.2.

To see that the minimum size is at most $\lceil \log_2 n \rceil$, we construct a tuple P of bipartitions in such a way that the row vectors of the corresponding matrix M_P are all distinct. To achieve this, a row length of $\lceil \log_2 n \rceil$ is sufficient and by Lemma 3.2 the family consisting of the entries of P is a separating family. □

On the other hand, the maximum size of a separating family for an n-element set is 2^{n-1} because a separating family of maximum size contains all possible bipartitions.

Let us denote by $\left\{ {k \atop i} \right\}$ the number of partitions of a k-element set into i nonempty subsets. This number is known as a *Stirling number of the second kind* (see [5, Section 6.1]).

Example 3. As shown below, there are 6 ways to partition the set $\{1, 2, 3, 4\}$ into three nonempty subsets, yielding $\left\{ {4 \atop 3} \right\} = 6$.

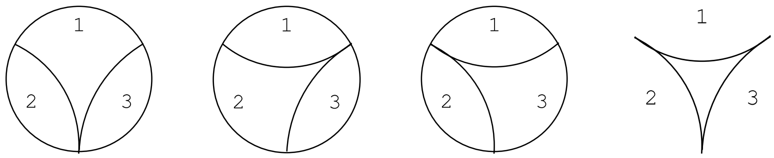
$$\begin{array}{lll} \{\{1\}, \{2\}, \{3, 4\}\} & \{\{1\}, \{2, 3\}, \{4\}\} & \{\{1\}, \{2, 4\}, \{3\}\} \\ \{\{1, 2\}, \{3\}, \{4\}\} & \{\{1, 3\}, \{2\}, \{4\}\} & \{\{1, 4\}, \{2\}, \{3\}\} \end{array}$$

Lemma 3.4. *The number of sequences of length k containing each element from a set of i symbols at least once is $i! \left\{ {k \atop i} \right\}$.*

Proof. When we identify distinct positions in a sequence of length k with k distinct objects, there are $\left\{ {k \atop i} \right\}$ many ways to distribute k distinct objects into i indistinguishable boxes, with no box left empty. If the boxes are distinguishable, then there are $i! \left\{ {k \atop i} \right\}$ many possibilities. □

Let us denote by $\tau_{n,i}$ the number of separating families of size i for an n-element set.

Example 4. The following diagrams illustrate $\tau_{3,3} = 4$, where in the leftmost diagram the circle, the left arc, and the right arc represent the bipartitions $\{\{1, 2, 3\}\}$, $\{\{1, 3\}, \{2\}\}$, and $\{\{1, 2\}, \{3\}\}$, respectively.



Lemma 3.5. *The following equation holds*

$$\sum_{i=1}^{k} i! \left\{ {k \atop i} \right\} \tau_{n,i} = (2^k - 1)(2^k - 2) \cdots (2^k - n + 1),$$

where $2 \le n$ and $1 \le k \le 2^{n-1}$.

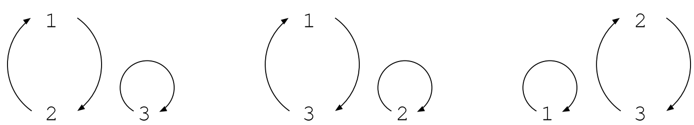
Proof. It is easy to verify the case $k < \lceil \log_2 n \rceil$. For the case $k \ge \lceil \log_2 n \rceil$, we prove the equation by counting in two ways the number of $(0, 1)$-matrices of size $n \times k$ whose entries in the first row are all 0 and for which every two row vectors are distinct.

For the first way to count, observe that the desired matrices can be obtained by arranging distinct nonzero $(0, 1)$-vectors of length k from the second row to the n-th row. Since the number of all nonzero $(0, 1)$-vectors of length k is $2^k - 1$, the number of such matrices is $\binom{2^k-1}{n-1} (n-1)!$.

For the second way to count, observe that the column vectors of a desired matrix need not be distinct. For each $i \in \{1, \dots, k\}$ we count the number of all desired $n \times k$ matrices having i distinct column vectors. We can assume $i \ge \lceil \log_2 n \rceil$ because otherwise there are no such matrices. Clearly those i column vectors form a separating family of size i. Conversely, the desired matrices can be obtained from separating families of size i by arranging the column vectors corresponding to the i bipartitions with repetition from the first column to the k-th column in such a way that each vector occurs at least once. Combining this observation with Lemma 3.4, it follows that the number of matrices constructed in this way is $i! \left\{ {k \atop i} \right\} \tau_{n,i}$. By summing over all $i \in \{1, \dots, k\}$, we obtain the left side of the equation stated in this lemma. □

Let us denote by $\left[{k \atop i} \right]$ the number of permutations of k elements which contain exactly i permutation cycles. This number is known as an *unsigned Stirling number of the first kind* (see [5, Section 6.1]).

Example 5. As shown below, there are 3 permutations of the set $\{1, 2, 3\}$ containing two permutation cycles, yielding $\left[{3 \atop 2} \right] = 3$.



Theorem 3.6. *The number $\tau_{n,k}$ of separating families of size k for an n-element set is*

$$\frac{(n-1)!}{k!}\sum_{i=1}^{k}(-1)^{k-i}\begin{bmatrix}k\\i\end{bmatrix}\binom{2^i-1}{n-1},$$

where $2 \le n$ *and* $1 \le k \le 2^{n-1}$.

Proof. The inversion formula for Stirling numbers (see [2, Section 3.1]) states that if two integer sequences $\{a_i\}_{1\le i\le n_0}$ and $\{b_i\}_{1\le i\le n_0}$ satisfy $b_k = \sum_{i=1}^{k}\left\{{k \atop i}\right\} a_i$, then they also satisfy $a_k = \sum_{i=1}^{k}(-1)^{k-i}\left[{k \atop i}\right] b_i$. From Lemma 3.5 we obtain this theorem. □

Proposition 3.7. *The number of minimal separating families of minimum size for an n-element set is*

$$\frac{(n-1)!}{\lceil \log_2 n\rceil !}\binom{2^{\lceil \log_2 n\rceil}-1}{n-1}.$$

Proof. Since every separating family of minimum size is minimal, it suffices to calculate the number of separating families of minimum size. According to Proposition 3.3 their minimum size for an n-element set is $\lceil \log_2 n\rceil$. Since $\binom{2^i-1}{n-1} = 0$ for $i \le \lceil \log_2 n\rceil$, substituting $\lceil \log_2 n\rceil$ for k in the formula of Theorem 3.6 yields the expression stated in this proposition. □

This number corresponds to the number of state assignments of an n-state machine in switching theory (see [8, Chapter 12]).

Let us denote by $\sigma_{n,k}$ the number of separating families of size k for an n-element set such that every bipartition consists of exactly two components, i.e. they do not contain $\{S\}$.

Lemma 3.8. *The equation* $\sigma_{n,k-1} + \sigma_{n,k} = \tau_{n,k}$ *holds for all numbers n and k with* $2 \le n$ *and* $2 \le k \le 2^{n-1}$.

Proof. Observe in Example 4 that the first three diagrams can be identified with separating families of size 2 by excluding $\{S\}$ and that the rightmost diagram is the unique separating family of size 3 in which every bipartition consists of exactly two components. Thus we obtain $\sigma_{3,2} + \sigma_{3,3} = \tau_{3,3}$. In general one obtains that separating families of size k which contain $\{S\}$ are identified by those of size $k-1$ which do not contain $\{S\}$. Thus this lemma follows. □

Lemma 3.9.

$$\begin{bmatrix}k+1\\i+1\end{bmatrix} = k!\sum_{j=i}^{k}\frac{1}{j!}\begin{bmatrix}j\\i\end{bmatrix}.$$

Proof. It holds (see [5, Section 6.1]) that $\left[{k+1 \atop i+1}\right] = k!\sum_{j=0}^{k}\left[{j \atop i}\right]/j!$. Since $\left[{j \atop i}\right] = 0$ for $0 \le j < i$, we obtain the equation. □

Proposition 3.10. *The number $\sigma_{n,k}$ of separating families of size k for an n-element set such that every bipartition consists of exactly two components is*

$$\frac{(n-1)!}{k!}\sum_{i=1}^{k}(-1)^{k-i}\begin{bmatrix}k+1\\i+1\end{bmatrix}\binom{2^i-1}{n-1},$$

where $2 \le n$ *and* $1 \le k < 2^{n-1}$.

Proof. Using Lemmas 3.8 and 3.9, we can calculate $\sigma_{n,k}$ as follows.

$$\begin{aligned}\sigma_{n,k} &= \sum_{j=1}^{k}(-1)^{k-j}\tau_{n,j}\\ &= (n-1)!\sum_{j=1}^{k}\sum_{i=1}^{j}\frac{(-1)^{k-i}}{j!}\begin{bmatrix}j\\i\end{bmatrix}\binom{2^i-1}{n-1}\\ &= (n-1)!\sum_{i=1}^{k}\sum_{j=i}^{k}\frac{(-1)^{k-i}}{j!}\begin{bmatrix}j\\i\end{bmatrix}\binom{2^i-1}{n-1}\\ &= (n-1)!\sum_{i=1}^{k}\frac{(-1)^{k-i}}{k!}\begin{bmatrix}k+1\\i+1\end{bmatrix}\binom{2^i-1}{n-1}. \quad \square\end{aligned}$$

Lemma 3.11. *The equation* $\sigma_{n,k-1}\,(k-1)! = \sigma_{k,n-1}\,(n-1)!$ *holds for all numbers n and k with* $2 \le n$ *and* $2 \le k \le 2^{n-1}$.

Proof. Consider (0, 1)-matrices whose entries in the first row and in the first column are all 0 and for which every two row vectors and every two column vectors are distinct. Since both conditions are symmetric with respect to rows and columns, the matrix transpose operation induces a bijection between the set of (0, 1)-matrices of size $n \times k$ satisfying the conditions and the set of those of size $k \times n$ satisfying the conditions. Observe for example that the following matrices which are transposes of each other satisfy the conditions.

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad M^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Furthermore, every (0, 1)-matrix of size $n \times k$ satisfying the conditions can be obtained by arranging the bipartitions in a separating family of size $k - 1$ which does not contain $\{S\}$ from the second column to the k-th column. Thus the number of such matrices is $\sigma_{n,k-1}(k-1)!$. Since the same argument applies to (0, 1)-matrices of size $k \times n$, this lemma follows. □

From Proposition 3.10 and Lemma 3.11, one obtains the following proposition.

Proposition 3.12. *The number $\sigma_{n,k}$ of separating families of size k for an n-element set such that every bipartition consists of exactly two components is*

$$\sum_{i=1}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i+1 \end{bmatrix} \binom{2^i - 1}{k},$$

where $2 \le n$ *and* $1 \le k < 2^{n-1}$.

Proof.

$$\begin{aligned} \sigma_{n,k} &= \frac{(n-1)!}{k!} \sigma_{k+1,n-1} \\ &= \frac{(n-1)!}{k!} \frac{k!}{(n-1)!} \sum_{i=1}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i+1 \end{bmatrix} \binom{2^i - 1}{k}. \quad \square \end{aligned}$$

From Lemma 3.8 and Proposition 3.12, one obtains the following theorem.

Theorem 3.13. *The number $\tau_{n,k}$ of separating families of size k for an n-element set is*

$$\sum_{i=1}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i+1 \end{bmatrix} \binom{2^i}{k},$$

where $2 \le n$ *and* $2 \le k < 2^{n-1}$.

Proof.

$$\begin{aligned} \tau_{n,k} &= \sigma_{n,k} + \sigma_{n,k-1} \\ &= \sum_{i=1}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i+1 \end{bmatrix} \left\{ \binom{2^i - 1}{k} + \binom{2^i - 1}{k-1} \right\} \\ &= \sum_{i=1}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i+1 \end{bmatrix} \binom{(2^i - 1) + 1}{k}. \quad \square \end{aligned}$$

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