K₂ of finite abelian group algebras

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1. Introduction

Let F be a finite field of characteristic p and G a finite abelian group. A general formula for K₂(FG) has been given in [7], Theorem 6.7; it is the quotient of G ⊗ A G with A the unramified p-ring such that F ≅ A/p. An upper bound for the order of K₂(FG) was also given in [7]. However, it is not so easy to determine the structure of K₂(FG) directly from this quotient, even its precise order. The result of Dennis and Stein in [6] (Corollary 4.4(a)) implies that for a cyclic group Cₙ, K₂(FCₙ) = 1. For an elementary abelian p-group G, Dennis, Keating and Stein [1] proved that K₂(FG) is an elementary abelian p-group whose precise rank is also given. In Magurn [2], when F is a characteristic 2, K₂(F[G × Z₂]) is isomorphic to the direct sum of K₂(FG) and an elementary abelian 2-group whose rank is determined. Using this result, Magurn calculated K₂(FG) when G is a finite abelian group with 4-ranking ≤ 1 and F is of characteristic 2. In Section 3 of this paper, we extend Magurn's results to the case when p is an odd prime. In Section 4, it will be shown that to get the precise order of K₂(FG), the only thing we need to know is the orders of kernels of \(\Omega^1_{FG/\mathbb{Z}} \to K_2(F\mathbb{G}/(t^{k^2}), (t^{k}))\), \(k \equiv 1 \mod p\), \(k > p\), where F is of odd characteristic p, G is a finite abelian p-group. We determine the de Rham cohomology group H₁^{dr}(FG) for arbitrary abelian p-groups G, and show how this cohomology group can be used to compute the above kernels in case G is an elementary abelian p-group.

2. Preliminaries

Suppose k is a commutative ring, A is a k-algebra, for an A-module M, a k-derivation from A to M is a k-linear map \(d: A \to M\) such that

\[d(ab) = (da)b + a(db), \quad (a, b \in A).\]

The set of all such derivations Derₖ(A, M) is an A-module, which is functorial in M. A universal k-derivation \(d: A \to \Omega^1_{A/\mathbb{K}}\) is defined by taking \(\Omega^1_{A/\mathbb{K}}\) to be the A-module defined by generators \(da \ (a \in A)\), and relations

\[d(ab) = adb + bda, \quad d(a + b) = da + dba, \quad b \in A, \quad \text{as well as} \quad dc = 0, \ c \in k.\]

\(\Omega^1_{A/\mathbb{K}}\) is called Kähler differentials of A over k and the universality is expressed by the natural isomorphism \(\text{Hom}_{A\text{-mod}}(\Omega^1_{A/\mathbb{K}}', M) \to \text{Der}_k(A, M)\) sending \(f\) to \(f \circ d\).
The algebra of differential forms over an algebra $A$ is the exterior algebra $\Omega^*_{A/K} = \bigoplus_q \Omega^q_{A/K}$, $\Omega^q_{A/K} = \wedge^q_A \Omega^1_{A/K}$, any element of $\Omega^q_A$ is called a differential form of degree $q$. The morphism $d : \Omega^q_{A/K} \to \Omega^{q+1}_{A/K}$ of degree +1, defined by $d(a_0 a_1 \cdots a_q) = a_0 a_1 \cdots a_q$, changes $\Omega^*_{A/K}$ into a complex. $(\Omega^*_{A/K}, d)$ is called the de Rham complex of $A$ and the cohomology algebra $H^*_D(A)$ is called the de Rham cohomology of $A$ over $k$.

Let $R$ be a commutative ring with identity. $\Phi_1(R) (i \geq 2)$ is defined by the following exact sequence in [1].

$$1 \to \Phi_1(R) \to K_2(R[t]/(t^i)) \to K_2(R[t]/(t^{i-1})) \to 1.$$ 

The following theorem is due to Bloch [5].

**Theorem 2.1.** If $R$ is a commutative local $F_p$-algebra and $p$ is odd, then

$$\Phi_1(R) \approx \begin{cases} \Omega^1_{R/Z} & i \neq 0, 1 \mod p, \\ \Omega^1_{R/Z} \oplus R/R^p & i = mp^r, (p, m) = 1, r \geq 1. \end{cases}$$

Note that there are no formulas for $\Phi_k(R)$ when $k \equiv 1 \mod p$. In fact it is very difficult to determine them. The following result of Magurn [2] gives the structure of $\Omega^1_{FG/Z}$.

**Theorem 2.2.** Suppose $F$ is a finite field of characteristic $p$, $G$ is a finite abelian group, and $\bar{a}_1, \ldots, \bar{a}_i$ is an $F_p$-basis of $G/G^p$, then $\Omega^1_{FG/Z}$ is a free $FG$-module with basis $da_1, \ldots, da_i$.

When $R$ is a commutative ring with identity and $I$ is a radical ideal, $K_2(R, I)$ is the abelian group which has a presentation with generators the Dennis–Stein symbols $(a, b)$ for every $(a, b) \in R \times I \cup I \times R$ and the following relations

- (D1) $(a, b) = -(b, a)$ if $a \in I$;
- (D2) $(a, b) + (a, c) = (a, b + c - abc)$ if $a \in I$ or $b, c \in I$;
- (D3) $(a, bc) = (ab, c) + (ac, b)$ if $a \in I$.

Let $\tilde{R}$ be a ring containing $R$. If $a \in I$ and $b \in R \cap \tilde{R}$, then the image of $(a, b)$ under the map $K_2(R, I) \to K_2(\tilde{R})$ is the Steinberg symbol $[1 - ab, b]$. One can consult [3] to see more about Dennis–Stein symbols.

### 3. Adding $\mathbb{Z}_p$ summands to $G$

Suppose $F$ is a finite field of characteristic $p$, $G$ is a finite abelian group, and $Z_{p^s} = \langle \sigma \rangle$ is a cyclic group of order $p^s$. Let $A = F[G \times Z_{p^s}]$. Then there is a partial augmentation map $\varepsilon : A \to F[G]$ sending $\sigma$ to 1; the kernel of $\varepsilon$ is $I = (1 - \sigma)A$. Since $\varepsilon$ is a split surjective map and $K_\varepsilon$ are functors, we have a split exact sequence which is just a part of the long exact sequence in $K$-theory with respect to the pair $(A, I)$:

$$1 \to K_2(A, I) \to K_2(A) \to K_2(FG) \to 1.$$ 

So obviously $K_2(A) \approx K_2(FG) \oplus K_2(A, I)$. From the isomorphisms

$$A \approx FG[t]/(t^{p^s} - 1) \approx FG[t]/(t^{p^s}),$$

it follows that $K_2(A, I) \approx K_2(FG[t]/(t^{p^s}), (t))$; thus

$$K_2(F[G \times Z_{p^s}]) \approx K_2(FG) \oplus K_2(FG[t]/(t^{p^s}), (t)). \tag{3.1}$$

**Theorem 3.1.** Suppose $F$ is a finite field of characteristic $p$, $G$ is a finite abelian group whose Sylow $p$-subgroups is $G_p$, then $K_2(FG)$ is a finite $p$-group annihilated by the exponent of $G_p$.

**Proof.** Decompose $G$ as the direct sum $G_p \oplus H$ with $G_p$ the Sylow $p$-subgroup of $G$. By Maschke’s Theorem, $FH$ is a semisimple ring. Then the Wedderburn–Artin Theorem implies that $FH \approx \bigoplus_i F_i$, where $F_i$ is a finite field with the same characteristic as $F$. Now $FG \approx (FH)[G_p] \approx \bigoplus_i F_i G_p$, and $K_2(FG) \approx \bigoplus_i K_2(F_i G_p)$.

Decompose $G_p$ as a finite direct sum of cyclic $p$-groups. $G_p \approx C_{p^1} \oplus \cdots \oplus C_{p^r}$, where $p^1 \leq \cdots \leq p^r$. Let $R_j = F_i[C_{p^1} \oplus \cdots \oplus C_{p^j}]$, $1 \leq j \leq r$. Using (3.1) we get the following isomorphism

$$K_2(F_i G_p) \approx \bigoplus_{j=1}^{r-1} K_2(R_j[t]/(t^{p^{j+1}}), (t)) \bigoplus K_2(F_i C_{p^j}).$$

The last summand vanishes; and $K_2(R_j[t]/(t^{p^{j+1}}), (t))$ has exponent $p^{j+1}$ since it is generated by $(a, b)$ with $a \in (t)$, and $p^{j+1} (a, b) = (a^{p^{j+1}}, b^{p^{j+1}} - 1)$, $b = 0$ since $a^{p^{j+1}} \in (t^{p^{j+1}})$. Thus $K_2(F_i C_{p^j})$ has exponent $p^j$ and accordingly $K_2(FG)$ is a $p$-group with exponent $p^r$. \(\square\)
When \( FG \) is not a local \( F_p \)-algebra, one cannot use Theorem 2.5 directly to compute \( \Phi_i(FG) \). In order to be more convenient in concrete calculation, we partially extend Theorem 2.1 to the form we need.

**Theorem 3.2.** Suppose \( F \) is a finite field of odd prime characteristic and \( G \) is a finite abelian group and \( R = FG \). Then Bloch's calculation still works for \( FG \):

\[
\Phi_i(FG) \approx \begin{cases} 
\Omega^1_{FG/Z} & i \neq 0, \text{ mod } p, \\
\Omega^1_{FG/Z} \oplus R/R^p & i = mp^t, (p, m) = 1, r \geq 1. 
\end{cases}
\]

**Proof.** As in the proof of Theorem 3.1, \( FG \approx \bigoplus F_j G_p \), where \( G_p \) is the Sylow \( p \)-subgroup of \( G \) and \( F_j \) has the same characteristic as \( F \). Say \( G_p \cong C_{p^1} \oplus \cdots \oplus C_{p^t} = \langle g_1 \rangle \times \cdots \times \langle g_r \rangle \). By the definition of \( \Phi_i \) we have \( \Phi_1(FG) \approx \bigoplus \Phi_1(F_j G_p) \). Since \( F_j G_p \) is a local \( F_p \)-algebra, then by Theorem 2.1

\[
\Phi_1(F_j G_p) \approx \begin{cases} 
\Omega^1_{F_j G_p/Z} & i \neq 0, \text{ mod } p, \\
\Omega^1_{F_j G_p/Z} \oplus F_j G_p/(F_j G_p)^p & i = mp^t, (p, m) = 1, r \geq 1. 
\end{cases}
\]

By Theorem 2.2, \( \Omega^1_{FC/Z} \) is a free \( FG \)-module with basis \( dg_1, \ldots, dg_r \), \( \Omega^1_{F_j G_p/Z} \) is a free \( F_j G_p \)-module with basis \( dg_1, \ldots, dg_r \), so \( \Omega^1_{FC/Z} \approx \bigoplus \Omega^1_{F_j G_p/Z} \neq G_p \) as abelian groups. Obviously \( FG/(FG)^p \approx \bigoplus (F_j G_p)/(F_j G_p)^p \), the theorem now follows. \( \square \)

The following theorem and corollary extend the results of Magurn [2] (Theorems 4 and 5) to the case when \( p \) is an odd prime.

**Theorem 3.3.** Suppose \( F \) is finite field with \( p' \) elements, \( p \) is an odd prime, \( G \) is a finite abelian group of order \( n \), and \( r \) is the dimension of the \( F_p \)-space \( G/G_p \). Then

\[
K_2(F[G \times \mathbb{Z}_p]/\mathbb{Z}_p) \cong K_2(FG) \oplus \mathbb{Z}_p^{f(n(1/2 + (p-1)r)).}
\]

**Proof.** By (3.1) we have the following isomorphism

\[
K_2(F[G \times \mathbb{Z}_p]) \cong K_2(FG) \oplus K_2(FG/(t^p), (t)).
\]

Since \( p(a, b) = \langle a^p b^{-1}, b \rangle \), \( K_2(FG/(t^p), (t)) \) is an elementary abelian \( p \)-group. By the isomorphism \( F[G \times \mathbb{Z}_p] \cong FG[\mathbb{Z}_p] \cong FG(t)/\langle t^p \rangle \) and the exact sequence

\[
1 \rightarrow \Phi_1(FG) \rightarrow K_2(FG(t)/\langle t^p \rangle) \rightarrow K_2(FG(t)/\langle t^{i-1} \rangle) \rightarrow 1, \quad 2 \leq i \leq p,
\]

we have

\[
|K_2(FG(t)/\langle t^p \rangle, (t))| = \prod_{i=2}^{p} |\Phi_i(FG)|. 
\]

By Theorem 3.2, \( \Phi_1(FG) \approx \Omega^1_{FG/Z}, 2 \leq i < p, \Phi_i(FG) \approx \Omega^1_{FG/Z} \oplus FG/(FG)^p \). \( \Omega^1_{FG/Z} \) is a free \( FG \)-module of rank \( r \), so it has rank \( nfr \) as an \( F_p \)-vector space. The group \( G \) has \( p' \) \( p \)-th power classes, so there are \( n(1 - 1/p) \) elements of \( G \) that are not elements of \( G^p \), hence \( FG/F[G^p] \) has \( F_p \)-dimension \( n(1 - 1/p^t) \). Thus

\[
dim_{F_p} (K_2(FG(t)/\langle t^p \rangle, (t))) = (p - 1)nfr + n^f \left( 1 - \frac{1}{p^t} \right) = n^f \left( 1 - \frac{1}{p^t} + (p - 1)r \right) .
\]

The theorem now follows. \( \square \)

**Corollary 3.4.** Suppose \( F \) is a finite field with \( p' \) elements, and \( G \) is a finite abelian group of order \( n \) with \( p \)-rank \( t \), \( p^2 \)-rank \( \leq 1 \), then

\[
K_2(FG) \cong \mathbb{Z}_p^{nf(t(1)-(p-1))/p}. 
\]

**Proof.** If \( p = 2 \), the theorem is just Theorem 5 in [2]; if \( p \) is an odd prime, repeated use of Theorem 3.3 yields the result. Since the process has been shown in Theorem 5 in [2], we omit the details. \( \square \)

**Example 1.** A direct use of Theorem 3.4 yields \( K_2(F_3[\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \times \mathbb{Z}_2]) \approx \mathbb{Z}_5^{7440}. \)
4. The order of $K_2(F_\varphi G)$

Suppose $F$ is a finite field of odd prime characteristic $p$ and $G$ is a finite abelian group. By the definition of $\Phi_i(R)$ and the isomorphism $F[\eta \times \mathbb{Z}_{p^l}] \cong F[t]/(t^{p^l})$, we have

$$|K_2(F[\eta \times \mathbb{Z}_{p^l}])| = |K_2(FG)| \cdot \prod_{i=2}^{p^l} |\Phi_i(FG)|.$$  

When $i \not\equiv 1 \mod p$, the order of $\Phi_i(FG)$ can be determined by Theorem 3.2. When $i \equiv 1 \mod p$, it is very difficult to determine the precise order of $\Phi_i(FG)$. We have the following commutative diagram, let $R = FG$.

$$
\begin{array}{ccccccc}
1 & \rightarrow & K_2(R[t]/(t^{mp^l} + 1), (t)) & \xrightarrow{f} & K_2(R[t]/(t^{mp^l} + 1)) & \rightarrow & K_2(R) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & K_2(R[t]/(t^{mp^l}), (t)) & \rightarrow & K_2(R[t]/(t^{mp^l})) & \rightarrow & K_2(R) & \rightarrow & 1.
\end{array}
$$

We will use the injectivity of $f$ to determine whether a Dennis–Stein symbol is trivial in $K_2(R[t]/(t^{mp^l} + 1), (t))$. By (1.6) and (1.10) in [3], we have the following exact sequences

$$\Omega_{1,R}^{\eta} \xrightarrow{\psi_{m,r}} K_2(R[t]/(t^{mp^l} + 1), (t)) \rightarrow K_2(R[t]/(t^{mp^l}), (t)) \rightarrow 1,$$

where $\psi_{m,r}(ab) = (at^{mp^l}, b)$. So we have $\Phi_{mp^l+1}(FG) \cong \Omega_{1,R}^{\eta}/\text{Ker} \psi_{m,r}$. If we can determine orders of all $\text{Ker} \psi_{m,r}$, then we can get the order of $K_2(FG)$. By the facts in the proof in Theorem 3.1, we only need to deal with the case when $G$ is a finite abelian $p$-group.

When $R$ is a regular ring, essentially of finite type over a field of positive characteristic $p > 0$, by Theorem 2.5 in [3], $\text{Ker} \psi_{m,r}$ depends only on $r$ and is the subgroup $D_{r,R}$ of $\Omega_{1,R}^{\eta}$ generated by $\{a(d^{p^l-1}-1)da \mid 0 \leq l < r, a \in R\}$. When $R$ is not regular, for example $R = FG$, where $F$ is a finite field of characteristic $p$ and $G$ is a finite abelian $p$-group, by the computations in Lemma 1.10 in [3], $\text{Ker} \psi_{m,r}$ still contains $D_{r,R}$ but does not coincide in general. Here is an example.

**Example 2.** Suppose $R = F_3[\mathbb{Z}_3 \times \mathbb{Z}_3], \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle \sigma \rangle \times \langle \tau \rangle$. Then by Theorem 2.2, $\Omega_{1,R}^{\eta}$ is the free $R$-module with basis $d\sigma, d\tau$. Put $m = 1, p = 3, r = 1$, we have

$$\Omega_{1,R}^{\eta} \xrightarrow{\Omega_{1,R}^{\eta}} K_2(R[t]/(t^4), (t)) \xrightarrow{f} K_2(R[t]/(t^4)).$$

By the definition $D_{1,R} = \langle (d^{p^l-1}-1)da \mid 0 \leq l < 1 \rangle = \{da \mid a \in R\}$. An easy computation shows that $D_{1,R}$ is the free $R$-space over basis $\{d\sigma, d\tau, d\sigma d\tau, d\sigma \tau + d\tau \sigma, 2d\tau d\sigma + \sigma^2d\sigma, 2\sigma d\tau + d\sigma d\tau + \sigma^2d\sigma\}$. Obviously $\sigma^2d\sigma, \tau^2d\tau \notin D_{1,R}$. However

$$f \circ \varphi(\sigma^2d\sigma) = f(\sigma^2d\sigma, \sigma) = (1 - \sigma^3, \sigma) = (1 - t, \sigma^3) = 1.$$

Similarly, $f \circ \varphi(\tau^2d\tau) = 1$. Since $f$ is an injective map, $\varphi(\sigma^2d\sigma) = \varphi(\tau^2d\tau) = 1$. Then $\sigma^2d\sigma, \tau^2d\tau \in \text{Ker} \varphi$. Hence $\text{Ker} \varphi \supsetneq D_{1,R}$.

**Theorem 4.1.** Suppose $R = FG, F$ is a finite field of odd characteristic $p$ and $G$ is a finite abelian group. Let $A = R[t]/(t^{mp^l} + 1)$ with $r \geq 1, (m, p) = 1$. If $\mathfrak{A}_1, \ldots, \mathfrak{A}_r$ is the $F_P$-basis of $G/\mathfrak{P}^r, A^\vee$ is the free $A$-module with generators $da_1, \ldots, da_n$, then there is a map $K_2(A, (t)) \xrightarrow{\psi} \bigwedge^2_A(A^\vee)$.

**Proof.** First we consider the test map

$$d \log : K_2(A, (t)) \rightarrow \bigwedge^2_A \Omega_{A,Z}$$

$$(a, b) \mapsto \frac{da \wedge db}{1 - ab}.$$

Let $D : A \rightarrow \Omega_{A,Z}^{1} \otimes_R A$ be defined by applying the derivative $d : R \rightarrow \Omega_{R,Z}^{1}$ to each coefficient, then according to Section 2 of [4], there is an $A$-module isomorphism

$$\Omega_{A,Z}^{1} \cong \bigwedge_{A,R} \otimes (\Omega_{R,Z}^{1} \otimes_R A)$$

$$df \mapsto \left(\frac{df}{dt}, Df\right).$$

Then $\bigwedge^2_A \Omega_{A,Z}^{1} \cong \bigwedge^2_A \Omega_{A,R}^{1} \oplus (\bigwedge^1_A \otimes (\Omega_{R,Z}^{1} \otimes_R A)) \otimes \bigwedge^2_A (\Omega_{R,Z}^{1} \otimes_R A)$, and by Theorem 2.2, $\Omega_{R,Z}^{1}$ is a free $R$-module with generators $da_1, \ldots, da_n$, so $\bigwedge^1_A \otimes (\Omega_{R,Z}^{1} \otimes_R A)$ is a free $A$-modules with the same generators, now

$$K_2(A, (t)) \xrightarrow{\psi} \bigwedge^2_A \Omega_{A,Z}^{1} \xrightarrow{\pi_3} \bigwedge^2_A (\Omega_{R,Z}^{1} \otimes_R A) \xrightarrow{\psi} \bigwedge^2_A A^\vee.$$  

The $f = \psi \circ \pi_3 \circ d \log$ is the map we want to obtain. \hfill $\Box$
Let $R$ be as above, $\wedge^*_R \Omega^1_{R/\mathbb{Z}}$, the algebra of differential forms over $R$, $(\wedge^*_R \Omega^1_{R/\mathbb{Z}}, d)$ the de Rham complex of $R$, and $H^*_\text{DR}(R)$ the de Rham cohomology of $R$.

**Corollary 4.2.** Let $A$, $R$ and $f$ be as above, $\varphi_m: \Omega^1_{R/\mathbb{Z}} \to K_2(A, (t))$, then the cocycle $Z^1$ of the de Rham complex $(\wedge^*_R \Omega^1_{R/\mathbb{Z}}, d)$ is equal to the $\text{Ker}(f \circ \varphi_{m,r})$.

**Proof.** By the definition, $f \circ \varphi_{m,r}$ is the composite of the following maps

$$
\Omega^1_{R/\mathbb{Z}} \xrightarrow{\varphi_{m,r}} K_2(A, (t)) \xrightarrow{d \log} \wedge^2 \Omega^1_{A/\mathbb{Z}} \xrightarrow{\partial} \wedge^2 \Omega^1_{R/\mathbb{Z} \otimes R} \xrightarrow{\psi} \wedge^2 A^1
$$

where $f \circ \varphi_{m,r}(ab) = \psi \circ \partial \circ \log((at, b)) = \psi \circ \partial \circ \log((at \cdot da \otimes 1) \vee (db \otimes 1)) = \psi(t, da \otimes db)$. By Theorem 2.2, $\Omega^1_{R/\mathbb{Z}}$ is the free $R$-module with basis $da_1, \ldots, da_n$, hence $\Omega^1_{R/\mathbb{Z}}$ is the free $R$-module with basis $da_i \otimes da_j$, $1 \leq i < j \leq s$, and $\wedge^2 A^1$ is a free $A$-module with the same generators. Define an $R$-homomorphism $g: \Omega^1_{R/\mathbb{Z}} \to \wedge^2 A^1$ by

$$
g(da_1 \otimes da_2) = t, da_1 \otimes da_2, \text{obviously } g \text{ is injective. The } g \circ d^1 \text{ is a homomorphism } \Omega^1_{R/\mathbb{Z}} \to \wedge^2 A^1 \text{ such that } g \circ d^1(ab) = t \cdot da_1 \otimes da_2 = f \circ \varphi_{m,r}(ab). \text{ Hence } g \circ d^1 = f \circ \varphi_{m,r}. \text{ Since } g \text{ is an injective map, we have } \text{Ker}(f \circ \varphi_{m,r}) = \text{Ker}(g \circ d^1) = \text{Ker}(d^1) = Z^1. \square$

**Theorem 4.3.** Suppose $F$ is a finite field of odd prime characteristic $p$, $G$ is a finite abelian $p$-group with cyclic decomposition $G = \langle x_1 \rangle \times \cdots \times \langle x_n \rangle$, $\text{ord}(x_i) = p^i$, $1 \leq i \leq n$. Let $G_i = \prod_{j \neq i} \langle x_j \rangle$. Then $H^1_{\text{DR}}(FG)$ is an $F$-space with basis $S = \{ x_i \text{d}x_i | 1 \leq i \leq n, 0 \leq j < p^i, j \equiv -1 \text{ mod } p, g \in G_i \}$.

**Proof.** If $T = \sum f_{a_1, \ldots, a_n} x_1^{a_1} \cdots X_n^{a_n} \in F[X_1, \ldots, X_n]$, the polynomial ring in $X_1, \ldots, X_n$ over $F$, the formal partial derivative $\frac{\partial T}{\partial X_i}$ is defined by

$$
\frac{\partial T}{\partial X_i} = \sum \alpha_{a_1, \ldots, a_n} x_1^{a_1} \cdots X_i^{a_i-1} \cdots X_n^{a_n}.
$$

If $x \in FG$, then $x = H(x_1, \ldots, x_n)$ for some polynomial $H(X_1, \ldots, X_n)$ in $F[X_1, \ldots, X_n]$, thus $dx = \sum_{i=1}^n \frac{\partial H}{\partial x_i}(X_1, \ldots, X_n) dx_i \in \Omega^1_{FG/\mathbb{Z}}$. By Theorem 2.2, $\Omega^1_{FG/\mathbb{Z}}$ is a free $FG$-module with basis $dx_1, \ldots, dx_n$. If $v \in \Omega^1_{FG/\mathbb{Z}}$, then

$$
v = \sum_{j=1}^n H_j(x_1, \ldots, x_n) dx_j,
$$

where $H_j(x_1, \ldots, x_n) \in F[X_1, \ldots, X_n]$, $j = 1, \ldots, n$. Then

$$
dv = \sum_{i,j} \left( \frac{\partial H_j}{\partial X_i} - \frac{\partial H_i}{\partial X_j} \right) (x_1, \ldots, x_n) dx_i \wedge dx_j.
$$

Since $\Omega^2_{FG/\mathbb{Z}}$ is the free $FG$-module with basis $dx_i \wedge dx_j$, $1 \leq i < j \leq n$, then $dv = 0$ if and only if

$$
\frac{\partial H_j}{\partial X_i} = \frac{\partial H_i}{\partial X_j}, \quad 1 \leq i < j \leq n. \quad (4.1)
$$

We can write $H_1(X_1, \ldots, X_n)$ in the following

$$
H_1(X_1, \ldots, X_n) = \sum_{i=0}^{p^1-1} Q_i(X_2, \ldots, X_n) X_1^i,
$$

with $Q_i \in F[X_2, \ldots, X_n]$. Let $w = \sum_{i \equiv 1 \text{ mod } p} (i + 1)^{-1} Q_i(X_2, \ldots, X_n) x_1^{i+1}$. Then

$$
v - dw = H_1'(x_1, \ldots, x_n) dx_1 + \sum_{j=2}^n H_j'(x_1, \ldots, x_n) dx_j,
$$

where $H_j' = \sum_{i=1}^n Q_i$. Since $d(v - dw) = dv - ddw = 0$, then $v - dw \in Z^1$, by (4.1) we have

$$
\frac{\partial H_j'}{\partial X_j} = \frac{\partial H_j'}{\partial X_i}, \quad 2 \leq j \leq n. \quad (4.2)
$$
Suppose \( H'_j(X_1, \ldots, X_n) = \sum_{i=1}^{p^j-1} Q_i(X_2, \ldots, X_n)X_1^i, \ j \geq 2 \) then by (4.2)
\[
\sum_{i = 1 \mod p} \frac{\partial Q_i(X_2, \ldots, X_n)}{\partial X_j} X_1^i = \sum_{i = 1}^{p^j-1} iQ_i(X_2, \ldots, X_n)X_1^{i-1}.
\]
By comparing the degree of \( X_1 \) of the two polynomials above, we conclude that both sides are equal to 0, so we have
\[
Q_i(X_2, \ldots, X_n) = P_i(X_2^p, \ldots, X_n^p), \ i \equiv -1 \mod p ;
\]
\[
H'_j(X_1, \ldots, X_n) = P'_j(X_1^p, X_2, \ldots, X_n), \ j = 2, \ldots, n.
\]
Now we have found \( v_1 \in Z^1 \) with \( \bar{\omega}_1 = \bar{v} \) in \( H^1_{\text{DR}}(FG) \), and
\[
v_1 = \left( \sum_{i = 1}^n P_i(x_2, \ldots, x_n)x_1^i \right) dx_1 + \sum_{j=2}^n P'_j(x_1, x_2, \ldots, x_n)dx_j.
\]
Now using induction and repeating the above process we can eventually find \( v_n \in Z^1 \) such that \( \bar{\omega}_n = \bar{v} \) in \( H^1_{\text{DR}} \) and
\[
v_n = \sum_{i = 1}^n \left( \sum_{j = 1}^n T_i(x_2^p, \ldots, x_n^p)x_j^i \right) dx_i.
\]
Obviously \( v_n \) can be generated by \( S, S \subseteq Z^1, S \cap B^1 = \{0\} \), and \( S \) is an \( F \)-independent set since \( \Omega^1_{FG} \) is an \( F \)-space with basis \( \{dg/g \in G, 1 \leq i \leq n\} \). So \( S \) is an \( F \)-basis of \( H^1_{\text{DR}}(FG) \).

**Corollary 4.4.** Let \( F \) be as above, \( G \) an elementary abelian \( p \)-group with independent generators \( x_1, \ldots, x_n \). Then \( H^1_{\text{DR}}(FG) \) is an \( n \)-dimensional \( F \)-vector space with basis \( \{x_i^{p-1}dx_i\} 1 \leq i \leq n \).

**Proof.** Since \( G \) is an elementary abelian \( p \)-group, \( G_i^p = 1, i = 1, \ldots, n \). Now the conclusion follows immediately from Theorem 4.3. \( \Box \)

**Proposition 4.5.** Let \( F \) and \( G \) be as in Theorem 4.3, then the coboundary \( B^1 \) of \( \left( \Omega^1_{FG/Z}, d \right) \) has basis \( S = \{dg/g \in G - G^p\} \) as an \( F \)-vector space.

**Proof.** Suppose \( x \in FG \), then \( x = H(x_1, \ldots, x_n) \), where \( H(x_1, \ldots, x_n) \) is a polynomial in \( F[X_1, \ldots, X_n] \). Thus \( dx = \sum_{i = 1}^n \frac{\partial H}{\partial x_i}(x_1, \ldots, x_n)dx_i \). Hence \( dx = 0 \) if and only if \( \frac{\partial H}{\partial x_i}(x_1, \ldots, x_n) = 0, 1 \leq i \leq n \). This implies \( H(x_1, \ldots, x_n) = H_1(x_1^p, \ldots, x_n^p) \) for some polynomial \( H_1(x_1, \ldots, x_n) \). If \( g_1, \ldots, g_m \in G - G^p \), \( \sum_{i=1}^m f_idg_i = 0, f_i \in F \), that is \( d \left( \sum_{i=1}^m f_ig_i \right) = 0 \), so
\[
\sum_{i=1}^m f_ig_i = H'(x_1^p, \ldots, x_n^p), \ H' \in F[X_1, \ldots, X_n].
\]
We conclude that \( f_i = 0, 1 \leq i \leq m \), \( S \) is an \( F \)-independent set. Obviously \( B^1 \) is generated by \( S \), now the proposition is proved. \( \Box \)

**Theorem 4.6.** Suppose \( F \) is a finite field of odd prime characteristic \( p \), \( G \) is an elementary abelian \( p \)-group with independent generators \( g_1, \ldots, g_n \). Set \( \varphi_{m, r} : \Omega^1_{FG/Z} \to K_2(FG[t]/(t^{mp^r-1}), (t)), (m, p) = 1, r \geq 1 \). Then \( \text{Ker} \varphi_{m, r} \) has basis \( S = \{dg_i, g_i^{p^r-1}dg_i/g \in G - \{1\}, 1 \leq i \leq n \} \) as an \( F \)-vector space.

**Proof.** By Theorem 4.3, \( \text{Ker} \varphi_{m, r} \subseteq (S) \), where \( (S) \) is the \( F \)-vector space generated by \( S \). By Lemma (1.10) in [3], \( \{dg/g \in G - \{1\}\} \subseteq \text{Ker} \varphi_{m, r} \). Since \( f : K_2(FG[t]/(t^{mp^r-1}), (t)) \to K_2(FG[t]/(t^{mp^r-1})) \) is injective and
\[
f \circ \varphi(g_i^{p^r-1}dg_i) = f \left( (g_i^{p^r-1}t^{mp^r-1}, g_i) \right) = \{1 - t^{mp^r}, g_i\} = \{1 - t^{mp^r-1}, 1\} = 1,
\]
it implies that \( (S) \subseteq \text{Ker} \varphi_{m, r} \). Thus \( \text{Ker} \varphi_{m, r} = (S) \). The independence of \( S \) follows from Proposition 4.5. \( \Box \)

**Theorem 4.7.** Let \( F \) be a finite field of odd prime characteristic \( p \), \( G \) is an arbitrary finite abelian \( p \)-group. Let \( \bar{S} = \{a^{d-1}da, g^{p^r-1}dg \mid 0 \leq l < r, g \in G, g^{p^r} = 1\} \), then \( \bar{S} \subseteq \text{Ker} \varphi_{m, r} \).
**Proof.** By Theorem 2.5 in [3], $D_{r,FG} = \langle a^{p^{l-1}d} | a \in FG, 0 \leq l < r \rangle \subseteq \text{Ker} \varphi_{m,r}$. Since $f : K_2(FG[t]/(t^{mp^r+1}), (t)) \rightarrow K_2(FG[t]/(t^{mp^r+1}))$ is injective, and

$$f \circ \varphi_{m,r}(g^{p^{l-1}d}) = f((g^{p^{l-1}t^{mp^r}}, g))$$

$$= \{1 - g^{p^r}t^{mp^r}, g\}$$

$$= \{1 - gt^m, g\}$$

$$= \{1 - gt^m, 1\} = 1.$$

Thus $g^{p^{l-1}d} \in \text{Ker} \varphi_{m,r},$ so $\langle \tilde{S} \rangle \subseteq \text{Ker} \varphi_{m,r}$. □

**Remark.** For an arbitrary finite abelian $p$-group $G$, we guess that $\text{Ker} \varphi_{m,r}$ is generated by $\{a^{p^{l-1}d}, g^{p^{l-1}d} | a \in FG, 0 \leq l < r, g \in G, g^{p^r} = 1\}$. By Theorem 4.6, this is true when $G$ is an elementary abelian $p$-group.

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**References**


**Further reading**


