On semi-artinian V-modules

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Abstract


Over a general ring $R$, a quasi-projective right $R$-module $M$ is a semi-artinian V-module if and only if for every $m \in M$ and submodule $N$ of $mR$, $mR/N$ contains a nonzero $M$-injective submodule.

Introduction

One consequence of the Osofsky–Smith Theorem [9] is the following fact: over an arbitrary ring $R$, a right $R$-module $M$ is semisimple if and only if every cyclic subquotient of $M$ is $M$-injective. Semisimple modules $M$ are semi-artinian (i.e. every nonzero homomorphic image of $M$ has nonzero socle), and are V-modules (i.e. every simple module is $M$-injective). The aim of this paper is to prove the following theorem:

**Theorem.** Let $R$ be any ring. Then the following statements are equivalent for a quasi-projective right $R$-module $M$:

(i) $M$ is a semi-artinian V-module.

(ii) Every nonzero cyclic subquotient of $M$ contains a nonzero $M$-injective submodule.

We have been unable to derive this theorem from the Osofsky–Smith Theorem. Rather, we show that a module $M$ satisfying (ii) has a nonzero submodule $N$.

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whose endomorphism ring $S$ is semiprime artinian. This is done by way of a
lemma essentially due to Osofsky [7] and used by her in her original proof that
rings for which every cyclic module is injective, are semiprime artinian. In order
to use $S$, we need the hypothesis that $M$ be quasi-projective, because in this case
certain categorical facts are available. It is natural to ask whether the theorem can
be extended to general modules.

1. Semi-artinian modules

All rings are assumed to have identity elements and all modules are unital right
modules.

Let $R$ be a ring and $M$ an $R$-module. Then the socle series of $M$ is the ascending
chain of submodules

$$0 = S_0(M) \subseteq S_1(M) \subseteq \cdots \subseteq S_{\alpha}(M) \subseteq S_{\alpha+1}(M) \subseteq \cdots,$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}(M)/S_{\alpha}(M)$ is the socle of the module
$M/S_{\alpha}(M)$, and if $\alpha$ is a limit ordinal, then

$$S_{\alpha}(M) = \bigcup_{0 \leq \beta < \alpha} S_{\beta}(M).$$

The following result is well known.

**Proposition 1.** The following statements are equivalent for a module $M$:

(i) Every nonzero homomorphic image of $M$ has essential socle.

(ii) Every nonzero homomorphic image of $M$ has nonzero socle.

(iii) $S_{\rho}(M) = M$ for some ordinal $\rho \geq 0$.

(iv) There exists an ascending chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\alpha} \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_{\tau} = M,$$

such that $M_{\alpha+1}/M_{\alpha}$ is semisimple for each $0 \leq \alpha < \tau$, and $M_{\alpha} = \bigcup_{0 \leq \beta < \alpha} M_{\beta}$ if $\alpha$ is
a limit ordinal. $\square$

Modules which satisfy the equivalent conditions of Proposition 1 are called semi-artinian. Clearly, artinian modules are semi-artinian. Moreover, the class $\mathcal{A}'$
of semi-artinian $R$-modules is closed under taking submodules, factor modules
and arbitrary direct sums. Let $M$ be any module. Let $\sigma[M]$ denote the full
subcategory of $\text{mod-}R$ generated by $M$ (see [11, 12]). The above remarks about
$\mathcal{A}'$ show that, if $M$ is any semi-artinian module, then any module in $\sigma[M]$ is
semi-artinian. We mention one further fact about semi-artinian modules, namely
that the class $\mathcal{A}'$ is closed under extensions. Indeed much more is true, as the
following result shows. We give a proof for completeness.
Proposition 2. Let $M$ be a module such that there exists a chain of submodules

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\alpha \subseteq L_{\alpha + 1} \subseteq \cdots \subseteq L_\nu = M,$$

such that $L_{\alpha + 1}/L_\alpha$ is semi-artinian for all ordinals $0 \leq \alpha < \nu$, and, for all limit ordinals $\alpha$, $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$. Then $M$ is semi-artinian.

Proof. Let $N$ be a proper submodule of $M$. There exists a least ordinal $\alpha$ such that $L_\alpha \not\subseteq N$. It is clear that $\alpha$ is not a limit ordinal and hence $\alpha - 1$ exists. Note that $L_{\alpha - 1} \subseteq N$. It follows that $(N + L_\alpha)/N$ is a homomorphic image of the semi-artinian module $L_\alpha/L_{\alpha - 1}$. By Proposition 1, $(N + L_\alpha)/N$ has nonzero socle. Thus $M/N$ has nonzero socle. By Proposition 1, $M$ is semi-artinian. \(\square\)

Note finally in this section that the class $\mathcal{A}'$ is a torsion class in the sense of [10]. For any module $M$, any sum of semi-artinian submodules of $M$ is semi-artinian, and thus $M$ contains a unique maximal semi-artinian submodule (the torsion submodule of $M$ in this torsion theory) and it is not difficult to see that this submodule is none other than the union of the socle series of $M$.

2. Quasi-projective modules

In this section we make some observations about quasi-projective modules which will be required in the proof of the theorem. A submodule $N$ of a module $M$ will be called fully invariant if $\varphi(N) \subseteq N$ for every endomorphism $\varphi$ of $M$.

Lemma 3. Let $N$ be a fully invariant submodule of a quasi-projective module $M$. Then $M/N$ is quasi-projective.

Proof. Let $K$ be any submodule of $M$ containing $N$ and consider the diagram

$$
\begin{array}{ccc}
M/N & \overset{\theta}{\longrightarrow} & 0 \\
\downarrow & & \\
M/N & \overset{\chi}{\longrightarrow} & M/K & \longrightarrow & 0 & \text{exact}
\end{array}
$$

where $\theta$ and $\chi$ are homomorphisms. Let $\pi : M \rightarrow M/N$ denote the canonical epimorphism. Form the diagram

$$
\begin{array}{ccc}
M & \overset{\pi}{\longrightarrow} & M/N & \overset{\theta}{\longrightarrow} & 0 & \text{exact}.
\end{array}
$$

By hypothesis, there exists a homomorphism $\varphi : M \rightarrow M$ such that $\theta \pi = \chi \pi \varphi$. 
Now \( \varphi(N) \subseteq N \), so that \( \varphi \) induces a homomorphism \( \varphi' : M/N \rightarrow M/N \) defined by

\[
\varphi'(m + N) = \varphi(m) + N \quad (m \in M).
\]

Now, for any \( m \in M \),

\[
\chi \varphi'(m + N) = \chi \pi \varphi(m) = \theta \pi(m) = \theta(m + N).
\]

Thus \( \chi \varphi' = \theta \). It follows that \( M/N \) is quasi-projective.  

**Lemma 4.** For any module \( M \), any submodule \( \text{Soc}_\alpha(M) \) in the socle series of \( M \) is fully invariant.

**Proof.** Suppose that the result is false. Then there exists a least ordinal \( \alpha \) such that \( \text{Soc}_\alpha(M) \) is not fully invariant. Note that \( \alpha > 0 \). Clearly, \( \alpha \) is not a limit ordinal. Let \( \varphi \) be any endomorphism of \( M \). Then \( \varphi(\text{Soc}_{\alpha-1}(M)) \subseteq \text{Soc}_{\alpha-1}(M) \).

Thus \( \varphi \) induces a homomorphism \( \varphi' : M' \rightarrow M' \), where \( M' = M/(\text{Soc}_{\alpha-1}(M)) \).

Now \( \varphi'(\text{Soc}_\alpha(M')) \subseteq \text{Soc}_\alpha(M') \). Thus \( \varphi(\text{Soc}_\alpha(M)) \subseteq \text{Soc}_\alpha(M) \). It follows that \( \text{Soc}_\alpha(M) \) is fully invariant, a contradiction. The result follows.  

**Corollary 5.** Let \( M \) be a quasi-projective module. Then \( M/(\text{Soc}_\alpha(M)) \) is quasi-projective for all ordinals \( \alpha \geq 0 \).

**Proof.** By Lemmas 3 and 4.  

We do not require the next lemma in its fully generality, but include it for the sake of interest.

**Lemma 6.** Let \( M \) be a quasi-injective module with endomorphism ring \( S \). Suppose that every nonzero submodule of \( M \) contains a nonzero \( M \)-projective submodule. Then \( S \) is a von Neumann regular right self-injective ring.

**Proof.** Let \( \varphi \in S \) and suppose that the kernel \( K \) of \( \varphi \) is an essential submodule of \( M \). Suppose that \( \varphi \neq 0 \). By hypothesis, \( \varphi(M) \) contains a nonzero \( M \)-projective submodule \( N \). There exists a submodule \( L \) of \( M \), properly containing \( K \), such that \( L/K \equiv N \). Thus \( L/K \) is \( M \)-projective, and hence \( L \)-projective (see, for example, [1, Proposition 16.12]). It follows that \( L = K \oplus K' \) for some submodule \( K' \) of \( L \). But \( K \) is essential in \( M \), so that \( K' = 0 \), \( L = K \) and \( N = 0 \), a contradiction. Thus \( \varphi = 0 \). Now by [8, Lemma 7 and Theorem 12] (or see [2, Theorem 19.27]), the result follows.  

We shall require the following corollary of Lemma 6 in the proof of the theorem.
Corollary 7. Let $M$ be a quasi-projective, quasi-injective module with endomorphism ring $S$. Suppose that every nonzero submodule of $M$ contains a nonzero $M$-injective submodule. Then $S$ is a von Neumann regular right self-injective ring.

Proof. By Lemma 6, because every $M$-injective submodule of $M$ is a direct summand and hence is $M$-projective. □

3. A lemma of Osofsky

The following lemma of Osofsky [7] plays a crucial role in her proof that rings are semiprime artinian if their cyclic modules are injective.

Lemma 8. Let $R$ be a right self-injective von Neumann regular ring and let \{$e_i; i \in I$\} be an infinite set of orthogonal idempotents in $R$. Then the right $R$-module $R/(\bigoplus_i e_iR)$ is not injective. □

Corollary 9. Let $R$ be a right self-injective von Neumann regular ring and let $E$ be a right ideal of $R$ which is countably, but not finitely, generated. Then the right $R$-module $R/E$ is not injective.

Proof. Suppose that $E = a_1R + a_2R + a_3R + \cdots$, where $a_i \neq 0 \ (i \geq 1)$. Because $R$ is regular, we can suppose that $a_i$ is idempotent for all $i \geq 1$. By [4, p. 68, Lemma] we can suppose that $a_1a_2 = a_2a_1 = 0$, and that

$$a_1R + a_2R = a_1R \oplus a_2R = (a_1 + a_2)R,$$

where $a_1 + a_2$ is idempotent. Again by [4] we can suppose that $(a_1 + a_2)a_3 = a_3(a_1 + a_2) = 0$. Thus $a_1$, $a_2$, $a_3$ are orthogonal idempotents. In this way, without loss of generality, we can suppose that \{$a_i; i \geq 1$\} is a set of orthogonal idempotents. By Lemma 8, $R/E$ is not an injective $R$-module. □

Using Corollary 9 we can now prove the key lemma in our investigation.

Lemma 10. Let $R$ be a right self-injective ring such that every nonzero right $R$-module contains a nonzero injective submodule. Then $R$ is semiprime artinian.

Proof. It is clear that every nonzero right ideal contains an idempotent and hence $R$ has zero Jacobson radical. Thus $R$ is von Neumann regular (see, for example, [2, Corollary 19.28] or [7, Lemma 7]). Suppose that $R$ is not semiprime artinian. Then $R$ contains a countably, but not finitely, generated right ideal $A$. Now $A$ is essential in the right ideal $eR$, for some idempotent $e$. Let $B = A \oplus (1 - e)R$. Then $B$ is a countably, but not finitely, generated essential right ideal of $R$. 
Because \( B \neq R \), the module \( R/B \) contains a nonzero injective submodule \( C/B \), for some right ideal \( C \) containing \( B \). Now \( C/B \) is a direct summand of \( R/B \), and hence there exists a right ideal \( D \) containing \( B \) such that \( R/B = (C/B) \oplus (D/B) \). Note that \( R/D \cong C/B \), so that \( R/D \) is injective. Moreover, \( D/B \) is cyclic, so that \( D \) is countably generated. By Corollary 9, \( D \) is finitely generated. There exists an idempotent \( f \) such that \( D = fR \). Then \( B \subseteq fR \). But \( B \) is essential, so that \( f = 1 \) and \( D = R \), a contradiction. \( \square \)

4. The theorem

By a subquotient of a module \( M \) we mean a module of the form \( N/K \), where \( N \) and \( K \) are submodules of \( M \) with \( K \subseteq N \).

**Lemma 11.** The following statements are equivalent for a module \( M \):

(i) Every nonzero cyclic subquotient of \( M \) contains an \( M \)-injective submodule.

(ii) Every nonzero module in \( \sigma[M] \) contains a nonzero \( m \)-injective submodule.

**Proof.** (ii) \( \Rightarrow \) (i) Clear.

(i) \( \Rightarrow \) (ii) Let \( X \) be any nonzero submodule in \( \sigma[M] \). In order to prove that \( X \) contains a nonzero \( M \)-injective subquotient, we can suppose, without loss of generality, that \( X \) is cyclic. There exists a nonempty index set \( I \) and a module \( M' = \bigoplus_{i \in I} M_i \) such that \( M_i = M \) (\( i \in I \)) and \( X \cong N/K \) for some submodules \( K \subseteq N \) of \( M' \). There exists \( x \in N \) such that \( N = xR + K \) and hence \( X \cong N/K = (xR + K)/K \cong xR/(xR \cap K) \). Thus we can suppose, without loss of generality, that \( N \) is cyclic. There exists a finite subset \( J \) of \( I \) such that \( N \subseteq \bigoplus_{j \in J} M_j \). Thus we can suppose that \( N \subseteq M_{j_1} \oplus \cdots \oplus M_{j_n} \), where \( n \) is a positive integer and \( M_i = M \) (\( 1 \leq i \leq n \)).

We shall prove that \( X \) contains a nonzero \( M \)-injective subquotient by induction on \( n \). If \( n = 1 \), then this is clear by (i). Suppose that \( n \geq 2 \). Let \( L = M_{j_1} \oplus \cdots \oplus M_{j_n} \). If \( K \cap L \neq N \cap L \), then

\[
(N \cap L)/(K \cap L) \cong [(N \cap L) + K]/K \nsubseteq N/K \cong X.
\]

By induction on \( n \), \( (N \cap L)/(K \cap L) \), and hence \( X \), contains a nonzero \( M \)-injective subquotient.

Now suppose that \( N \cap L = K \cap L \). Then \( N + L \neq K + L \), and hence \((K + L)/L \nsubseteq (N + L)/L \) are distinct submodules of the module \( M/L \cong M \). Thus, by (i), \((N + L)/(K + L)\) contains a nonzero \( M \)-injective subquotient. But

\[
(N + L)/(K + L) \cong N/[K + (N \cap L)] = N/K \cong X.
\]

Thus \( X \) contains a nonzero \( M \)-injective submodule. \( \square \)
Given a ring $R$, a right $R$-module $M$ is called a $V$-module if every simple right $R$-module is $M$-injective. Note the following elementary fact.

**Lemma 12.** A module $M$ is a $V$-module if and only if every simple subquotient of $M$ is $M$-injective.

**Proof.** The necessity is clear. Now suppose that every simple subquotient of $M$ is $M$-injective. Let $U$ be any simple module. Let $N$ be a submodule of $M$ and $\varphi : N \to U$ a homomorphism. If $\varphi = 0$, then $\varphi$ lifts to $M$ trivially. If $\varphi$ is nonzero, then $U \cong N/K$, where $K = \ker \varphi$. By hypothesis, $N/K$, and hence $U$, is $M$-injective. $\square$

We are now in a position to prove the following theorem:

**Theorem 13.** Let $M$ be a quasi-projective module. Then $M$ is a semi-artinian $V$-module if and only if every nonzero cyclic subquotient of $M$ contains a nonzero $M$-injective submodule.

**Proof.** The necessity is clear.

Conversely, suppose that every nonzero cyclic subquotient of $M$ contains a nonzero $M$-injective submodule. In particular, every simple subquotient of $M$ is $M$-injective. By Lemma 12, $M$ is a $V$-module. Moreover, for any submodule $N$ of $M$, every nonzero cyclic subquotient of $M/N$ contains a nonzero $(M/N)$-injective submodule (see, for example, [1, Proposition 16.13]).

It remains to prove that $M$ is semi-artinian. To do so, it is sufficient to prove that $M$ has nonzero socle. For, assume that this is the case. By Corollary 5 and the above remark, it will follow that, for all ordinals $\alpha \geq 0$, $M/(\text{Soc}_\alpha(M))$ has nonzero socle if $M \neq \text{Soc}_\alpha(M)$. Thus the socle series of $M$ must terminate in $M$.

By Proposition 1, $M$ is semi-artinian.

Let $0 \neq m \in M$. Then $mR$ contains a nonzero $M$-injective submodule $N$. Thus $N$ is a direct summand of $M$, so that $N$ is cyclic, quasi-projective and quasi-injective. Let $S$ denote the endomorphism ring of $N$. Because every nonzero cyclic subquotient of $N$ contains a nonzero $M$-injective (and hence $N$-injective) submodule, Corollary 7 gives that $S$ is von Neumann regular right self-injective.

By Lemma 11, every nonzero module in $\sigma[N]$ contains a nonzero $N$-injective submodule. But $N$ is a generator in $\sigma[N]$, by [12, 23.8]. Thus the category $\sigma[N]$ is equivalent to the category of right $S$-modules (see [12, 46.2] or [3]). It follows that every nonzero right $S$-module contains a nonzero injective submodule. By Lemma 10, $S$ is semiprime artinian.

Now consider $N$. Because $S$ is semiprime artinian, we know that $N$ has finite Goldie dimension. Thus $N$ contains a (nonzero) uniform submodule $U$. Let $0 \neq u \in U$. Then $uR$ contains a nonzero direct summand of $U$, by hypothesis. Thus $U = uR$. It follows that $U$ is simple. Therefore, $U$ is a simple submodule of $M$. Thus $M$ has nonzero socle, as required. $\square$
5. Examples

In this section we give a method for producing many examples of semi-artinian V-modules. A ring \( R \) is called right semi-artinian if the right \( R \)-module \( R \) is semi-artinian. On the other hand, the ring \( R \) is a right \( V \)-ring if every simple right \( R \)-module is injective, equivalently the right \( R \)-module is a \( V \)-module. For a discussion of right \( V \)-rings see [5]. The next result is clear.

**Proposition 14.** Let \( R \) be a ring.

(i) If \( R \) is right semi-artinian, then every right \( R \)-module is semi-artinian.

(ii) If \( R \) is a right \( V \)-ring, then every right \( R \)-module is a \( V \)-module.

In view of Proposition 14, in order to find examples of semi-artinian \( V \)-modules it is sufficient to produce examples of right semi-artinian right \( V \)-rings, and this is what we shall do. Note that right semi-artinian right (or left) \( V \)-rings are von Neumann regular (see [6, Corollary 4.3]).

Let \( K \) be a field and \( R \) a \( K \)-algebra. We shall consider \( K \) as a subring of \( R \) in the usual way. For each positive integer \( i \), let \( R_i = R \). Let \( R^* \) denote the subring of the complete direct product \( \prod R_i \), consisting of all elements \((r_1, r_2, r_3, \ldots)\) such that there exists \( n \geq 1 \) with

\[ r_n = r_{n+1} = r_{n+2} = \cdots \quad \text{and} \quad r_n \in K. \]

Clearly, \( I = \bigoplus R_i \) is an ideal of \( R^* \).

**Lemma 15.** With the above notation,

\[ \text{Soc}_1(R^*) = \text{Soc}_1(R_1) \oplus \text{Soc}_1(R_2) \oplus \text{Soc}_1(R_3) \oplus \cdots. \]

**Proof.** Let \( U \) be any minimal right ideal of the ring \( R^* \). Suppose that \( 0 \neq r = (r_1, r_2, r_3, \ldots) \in U \). There exists a positive integer \( n \geq 1 \) such that \( r_n \neq 0 \). Let \( e = (e_1, e_2, e_3, \ldots) \in R^* \), where \( e_n = 1 \) and \( e_m = 0 \) \( (m \neq n) \). Then \( 0 \neq re = (0, \ldots, 0, r_n, 0, 0, \ldots) \in U \). Thus \( U = reR^* \subseteq I \). It follows that \( \text{Soc}_1(R^*) = \text{Soc}_1(I) = \bigoplus \text{Soc}_1(R_i) \), by [1, Proposition 9.19].

**Corollary 16.** With the above notation, for any ordinal \( \alpha \geq 0 \),

\[ \text{Soc}_\alpha(R^*) = \text{Soc}_\alpha(R_1) \oplus \text{Soc}_\alpha(R_2) \oplus \text{Soc}_\alpha(R_3) \oplus \cdots. \]

**Proof.** By transfinite induction using Lemma 15.
the socle length of $M$. In particular, if $R$ is right semi-artinian, then the socle length of $R$ is defined to be the socle length of the right $R$-module $R$.

**Lemma 17.** With the above notation, if $R$ is a right semi-artinian ring of socle length $\alpha \geq 1$, then $R^*$ is a right semi-artinian ring of socle length $\alpha + 1$.

**Proof.** By Corollary 16, $\text{Soc}_\alpha(R^*) = I \neq R$. Moreover, $R/I \cong K$ gives $\text{Soc}_{\alpha+1}(R^*) = R^*$. By Proposition 1, $R^*$ is right semi-artinian of socle length $\alpha + 1$. □

**Lemma 18.** Let $K$ be a field. For any nonlimit ordinal $\alpha \geq 0$, there exists a right semi-artinian $K$-algebra of socle length $\alpha$.

**Proof.** Suppose not. Then there exists at least a nonlimit ordinal $\alpha \geq 1$ such that no right semi-artinian $K$-algebra of socle length $\alpha$ exists. If $\alpha - 1$ is not a limit ordinal, then there exists a right semi-artinian $K$-algebra $R$ of socle length $\alpha - 1$. By Lemma 17, $R^*$ is a right semi-artinian $K$-algebra of socle length $\alpha$.

Now suppose that $\alpha - 1$ is a limit ordinal. We adapt the above construction. For each ordinal $1 \leq \beta < \alpha - 1$, there exists a right semi-artinian $K$-algebra $R_\beta$ of socle length $\beta$. Again we think of $K$ as a subring of $R_\beta$ for all $\beta$. Let $R^+$ denote the subring of the direct product of the rings $R_\beta$ consisting of all elements $r = \{r_\beta\}$ such that $r_\beta \in R_\beta (1 \leq \beta < \alpha)$ and there exists an element $k, \in K$ such that $r_\beta = k,_{\beta}$ for all but a finite number of ordinals $1 \leq \beta < \alpha$. By adapting the earlier proofs, it can be checked that $\text{Soc}_{\alpha-1}(R^+) = \bigoplus_\beta \text{Soc}_{\alpha-1}(R_\beta) = \bigoplus R_\beta = I$ (say). But $R^+/I \cong K$, so that $\text{Soc}_\alpha(R^+) = R^*$. This contradiction proves the result. □

Note that if $\alpha$ is a limit ordinal, then a right semi-artinian ring $R$ cannot have socle length $\alpha$. For, in this case $R = \text{Soc}_\alpha(R) = \bigcup \text{Soc}_\beta(R)$, where the union is taken over all ordinals $\beta$ with $1 \leq \beta < \alpha$, and this gives $1 \in \text{Soc}_\beta(R)$ and hence $R = \text{Soc}_\beta(R)$ for some $1 \leq \beta < \alpha$. This problem disappears for modules. In fact, in the proof of Lemma 18 we saw how to deal with limit ordinals, because $I$ is a semi-artinian right $R^+$-module of socle length the limit ordinal $\alpha - 1$. Thus we have the following result:

**Theorem 19.** Let $K$ be a field. For any ordinal $\alpha \geq 0$ there exists a $K$-algebra $R$ and a semi-artinian right $R$-module $M$ such that $M$ has socle length $\alpha$. □

Of course, there are other (and easier) constructions of right semi-artinian rings. We mention one of these. Let $K$ be a field and $R$ a $K$-algebra. Let $S$ denote the subring of the ring of $2 \times 2$ matrices with entries in $R$ which consists of all matrices of the form

$$
\begin{bmatrix}
a & b \\
0 & c
\end{bmatrix}
$$
where \( a \in K \) and \( b, c \in R \). It is easy to check that if \( R \) is right semi-artinian of socle length \( \alpha \), for some ordinal \( \alpha \geq 0 \), then \( S \) is right semi-artinian of socle length \( \alpha + 1 \). However, right V-rings are semiprime (see [5, Corollary 2.2] or [6, Corollary 4.3]), so that these matrix examples will not give right semi-artinian right V-rings. No such problem arises with our earlier construction, as we now prove.

**Theorem 20.** With the above notation, if the \( K \)-algebra \( R \) is a right V-ring, then \( R^* \) is also a right V-ring.

**Proof.** Let \( A \) be a proper right ideal of the ring \( R^* \). Let \( r \in R^* \), \( r \not\in A \). We shall show that there exists a maximal right ideal \( P \) of \( R^* \) such that \( A \subseteq P \), but \( r \not\in P \). Let \( r = (r_1, r_2, r_3, \ldots) \in R^* \). Suppose that \( r_n \not\in A \cap R_n \) for some \( n \geq 1 \). By [5, Theorem 2.1], there exists a maximal right ideal \( P_n \) of \( R_n \) such that \( A \cap R_n \subseteq P_n \) but \( r_n \not\in P_n \). Let

\[
P = \{(s_1, s_2, s_3, \ldots) : s_n \in P_n \}.
\]

If \( \pi_n : R^* \rightarrow R_n \) is the ring epimorphism given by projection onto the \( n \)th component, then \( \ker \pi_n \subseteq P \) and \( \pi_n(P) = P_n \), so that \( P \) is a maximal right ideal of \( R^* \). Clearly \( A \subseteq P \) and \( r \not\in P \).

Now suppose that \( r_j \in A \cap R_j \) for all \( j \geq 1 \). There exists a positive integer \( m \) and \( 0 \neq k \in K \) such that \( k = r_{m+1} = r_{m+2} = r_{m+3} = \cdots \). It follows that \( r \not\in I \). Suppose that \( A \not\subseteq I \). Let \( a = (a_1, a_2, a_3, \ldots) \in A \), \( a \not\in I \). Clearly, we can suppose, without loss of generality, that there exists \( s \geq m \) such that \( k = a_{s+1} = a_{s+2} = a_{s+3} = \cdots \). Now

\[
(0, \ldots, 0, k, k, \ldots) = a(0, \ldots, 0, 1, 1, \ldots) \subseteq A ,
\]

and hence

\[
r = (r_1, \ldots, r_s, 0, 0, \ldots) + (0, \ldots, 0, k, k, \ldots) \in A ,
\]

a contradiction. Thus \( A \subseteq I \). But, in this case, \( I \) is a maximal right ideal of \( R^* \) such that \( A \subseteq I \) but \( r \not\in I \).

Now we apply [5, Theorem 2.1] to conclude that \( R^* \) is a right V-ring. \( \Box \)

Note that if \( R \) is a semiprime artinian ring, then \( R \) is a right semi-artinian right V-ring. Combining this fact with Theorem 20 and our earlier discussion we now conclude the following:

**Corollary 21.** Let \( K \) be a field.

(i) For any nonlimit ordinal \( \alpha \geq 0 \), there exists a \( K \)-algebra \( R \) such that \( R \) is a right semi-artinian right V-ring of socle length \( \alpha \).

(ii) For any ordinal \( \alpha \), there exists a \( K \)-algebra \( R \) and a right \( R \)-module \( M \) such that \( M \) is a semi-artinian V-module of socle length \( \alpha \). \( \Box \)
6. Categories

We conclude the paper with a comment about Grothendieck categories. Let $\mathcal{C}$ be a locally finitely generated Grothendieck category. It is well known that if every object of $\mathcal{C}$ is injective, then every object of $\mathcal{C}$ is semisimple (see, for example, [10, Chapter V, Proposition 6.7]). In view of Theorem 13, it would be interesting to investigate the categories $\mathcal{C}$ with the property that every nonzero object contains a nonzero injective subobject, and, in particular, to determine whether every nonzero object of such a category must contain a simple subobject. More specifically, does Theorem 13 extend to modules which are not quasi-projective?

Note added in proof

It has been brought to our attention that K. Ohtake in a paper entitled ‘Commutative rings of which all radicals are left exact’ (Comm. Algebra 8 (1980) 1505–1512), has proved that a commutative ring $R$ is a semi-artinian $V$-ring if and only if every nonzero $R$-module contains a nonzero injective submodule.

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