Well-posedness and blow-up phenomena for a higher order shallow water equation

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1. Introduction

In this paper, we consider the Cauchy problem for the following higher order shallow water equation

\[
\begin{align*}
    & y_t + a u_x y + b u y_x = 0, & t > 0, & x \in \mathbb{R}, \\
    & u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{align*}
\]

(1.1)

where \( a, b \) are positive constants, the notation \( y := \Lambda^{2k} u \equiv (I - \partial_x^2)^k u \) with \( k \in \mathbb{N} \).

In order to motivate our results, we recall some classical results related to (1.1). In 1895, Korteweg and de Vries [45] discovered an interesting phenomenon in water channels, which is the appearance...
of waves with length much greater than the depth of the water. Then they started with the mathematical theory of this phenomenon and derived a model describing unidirectional propagation of waves of the free surface of shallow layer of water, which led to the well-known KdV equation

\[ u_t - 6uu_x + u_{xxx} = 0, \]  

(1.2)

where \( u \) describes the free surface of water (the physical derivation of this equation can be found in [1]). The beautiful structure behind the KdV equation initiated a lot of mathematical investigations. For instance, the KdV equation is completely integrable and its solitary waves are solitons (see [55]). The local and global existence of the solutions to (1.2) were proven in [61]. The well-posedness and scattering results for the generalized Korteweg–de Vries equation are studied via the contraction principle (see [44]). It is also observed that the KdV equation does not accommodate wave breaking (by wave breaking we mean that the wave remains bounded but its slope becomes unbounded in finite time) (cf. [64]).

In 1993, Camassa and Holm [5] proposed a new model to describe the unidirectional propagation of shallow water waves over a flat bottom

\[ u_t - c_0 u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \]  

(1.3)

where the variable \( u(t, x) \) represents the fluid velocity at time \( t \) and in the spatial direction \( x \), and \( c_0 \) is a nonnegative parameter related to the critical shallow water speed (see also [6]). The Camassa–Holm equation (1.3) is also a model for the propagation of axially symmetric waves in hyperelastic rods (cf. [21]). It is well known that Eq. (1.3) has also a bi-Hamiltonian structure (see [33,47]) and is completely integrable (see [5,9] and the in-depth discussion in [3,16]). Its solitary waves are smooth if \( c_0 > 0 \) and peaked in the limiting case \( c_0 = 0 \) (cf. [6]). The orbital stability of the peaked solitons is proved in [19], and the stability of the smooth solitons is considered in [20]. It is worth pointing out that solutions of this type are not mere abstractions: the peakons replicate a feature that is characteristic for the waves of great height—waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves (cf. [10,15,62]). The explicit interaction of the peaked solitons is given in [2]. The Camassa–Holm equation in comparison with the KdV equation (1.2) lies in the fact that the Camassa–Holm equation has peaked solitons and models the peculiar wave breaking phenomena (see [6,12]).

In 1999, Degasperis and Procesi [26] derived a nonlinear dispersive equation

\[ u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \]  

(1.4)

which can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the Camassa–Holm shallow water equation, and the Degasperis–Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. This follows from the formal considerations made in [27,28], and the rigorous considerations made in [17]. Degasperis, Holm and Hone [24] proved the formal integrability of this equation by constructing a Lax pair. They also showed that this equation has a bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa–Holm peakons. Lundmark and Szmigielski [54] presented an inverse scattering approach for computing \( n \)-peakon solutions, and the direct and inverse scattering approach pursued recently in [22]. The traveling wave solutions for the Degasperis–Procesi equation was studied by Vakhnenko and Parkes in [63]. The Cauchy problem for the Degasperis–Procesi equation has been studied extensively. Local well-posedness of this equation is established in [66] for initial data \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \). Similar to the Camassa–Holm equation, the precise blow-up scenario and a blow-up result were derived
in [30,51,66,67], the global existence of strong solutions and global weak solutions to Eq. (1.4) are studied in [30,51,66]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ (see [30,66]) and global entropy weak solutions belonging to the class $L^2(\mathbb{R}) \cap BV(\mathbb{R})$ and to the class $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ (see [23]).

Notice that in (1.3) (with $c_0 = 0$) and (1.4), the coefficient of $u_{xx}$ is equal to the coefficient of $u_{xxx}$ plus the coefficient of $u_{xxxx}$, that is, $3 = 2 + 1, 4 = 3 + 1$. Indeed, this relationship among the coefficients plays an important role to study the essential dynamical properties of the single Camassa–Holm and Degasperis–Procesi equations. Although the Degasperis–Procesi equation is similar to the Camassa–Holm equation in several aspects (e.g., bi-Hamiltonian structure [24,33,47], infinite speed of propagation [35–37], peakon solutions of the form $u(x, t) = ce^{-|x-ct|}$ [51,52]), these two equations are actually different. One of the novel features of the Degasperis–Procesi differs from the Camassa–Holm equation is that it has not only peakon solutions (see [24]) and periodic peakon solutions (see [31]), but also shock peakons (see [53]) and the periodic shock waves (see [31]). Another is that the conservation laws are also different (see [34,52]). For this reason, some authors considered the following generalized shallow water equation

$$u_t - u_{xxx} + (A + B)u_{xx} = Au_x u_{xx} + Bu_{xxxx}, \quad t > 0, x \in \mathbb{R}. \tag{1.5}$$

Obviously, if $A = b, B = 1$, Eq. (1.5) becomes a b-equation, which can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated (see [27–29]). The solutions of the b-equation were studied numerically for various values of $b$ in [38,39], where $b$ was taken as a bifurcation parameter. The necessary conditions for integrability of the b-equation were investigated in [57]. The KdV equation, the Camassa–Holm equation and the Degasperis–Procesi equation are the only three integrable equations in the b-equation, which was shown by using Painlevé analysis in [25,26,40]. The b-equation also admits peakon solutions for any $b \in \mathbb{R}$ (see [25,38,39]). The well-posedness, blow-up phenomena and global solutions for the b-equation were shown in [32,59].

On the other hand, taking $A = 1 - \theta, B = \theta$ in (1.5) we find the $\theta$-equation, which was derived by Liu [49]. It was identified in his study of model equations for some dispersive schemes to approximate the Hopf equation. This new kind of $\theta$-equation admits blow-up phenomenon and infinite propagation speed like the Camassa–Holm equation [5] and the Degasperis–Procesi equation [26], which have been proved in Ni and Zhou’s recent work [58]. In [50], Liu and Yin investigated both global regularity of solutions and wave breaking phenomenon for $\theta \in \mathbb{R}$. It was shown that the regularity of solutions improves as $\theta$ increases. Moreover, if the momentum of initial data had a definite sign, then for any $\theta \in \mathbb{R}$ global smoothness of the corresponding solution of [50] was proved. Recently, Lai and Wu [46] studied the global solutions and blow-up phenomena to Eq. (1.5). Indeed, it is easy to see that the b-equation, $\theta$-equation and Eq. (1.5) are equivalent to each other.

For the higher order Camassa–Holm equation, Mclachlan and Zhang [56] considered the modified Camassa–Holm equations derived as the Euler–Poincaré differential equation on the Bott–Virasoro group with respect to the $H^k$ metric, i.e.,

$$\begin{cases}
y_t + 2u_x y + uy_x = 0, & t > 0, \ x \in S, \\
u(x, 0) = u_0(x), & x \in S, \tag{1.6}
\end{cases}$$

where $S$ is the unit circle, and $y = x^{2k} \equiv (1 - a^2 x^2)^k u$ with $k \geq 2$ a positive integer. This equation, with $k = 0, 1$, corresponds to the KdV equation and the Camassa–Holm equation respectively (see [41,56]). In [7], Coclite, Holden and Karlsen considered higher order Camassa–Holm equations (1.6) describing exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle. They establish the existence of global weak solutions, and also present some invariant spaces under the action of the equation.

Motivated by the results mentioned above, this paper deals with the global existence and blow-up phenomenon for the high order shallow water equation. To keep the presentation short, details are
presented for the case $k=2$ only ($k=1$ corresponds to Eq. (1.5)). First, we use Kato’s Theorem to obtain the existence and uniqueness for Eq. (1.1).

**Theorem 1.1.** Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 7/2$. Then there exists a maximal $T = T(\|u_0\|_{H^s(\mathbb{R})})$, and a unique solution $u(x, t)$ to the problem (1.1) such that

$$u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$u_0 \rightarrow u(\cdot, u_0) : H^s(\mathbb{R}) \rightarrow C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$$

is continuous.

We show as well that if a classical solution $u$ of (1.1) starts out having compact support, then this property will be inherited by $y$ at all times $t \in [0, T)$.

**Theorem 1.2.** Assume that $u_0 \in H^s(\mathbb{R})$ (with $s \geq 7/2$) and $y_0 = (I - \partial_x^2)^2 u_0$ has compact support. Let $T = T(u_0) > 0$ be the maximal existence time of the unique solution $u(x, t)$ to (1.1) with initial data $u_0(x)$. Then for any $t \in [0, T)$ the $C^1$ function $x \mapsto y(x, t)$ has compact support.

Next we give some sufficient conditions for global existence and blow-up of the solutions to the problem (1.1).

**Theorem 1.3.** Let $u_0 \in H^s(\mathbb{R}) \cap W^{k, b}(\mathbb{R})$ with $s \geq 7/2$ and $b \geq a$. Then the solution of the problem (1.1) remains smooth for all time.

**Theorem 1.4.** Let $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s \geq 7/2$ and $y_0 = (I - \partial_x^2)^2 u_0 \geq 0$ for all $x \in \mathbb{R}$ (or equivalently $y_0 = (I - \partial_x^2)^2 u_0 \leq 0$ for all $x \in \mathbb{R}$). Then we have the conservation laws

$$\int \frac{y}{x} \, dx = \int y_0 \, dx = \int u \, dx = \int u_0 \, dx,$$

$$\int \frac{y}{x} \, dx = \int y_0 \, dx = \int y_0 \, dx.$$

Moreover, we obtain that $\|u(t)\|_{H^s(\mathbb{R})}$ is finite for any $0 < t < \infty$.

**Theorem 1.5.** Let $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s \geq 7/2$, and $T$ be the maximal time of the solution $u(x, t)$ to the problem (1.1) with the initial data $u_0$. If $b = 2a$, then every solution to (1.1) remains regular globally in time. If $b < 2a$, then the corresponding solution blows up in finite time if and only if

$$\lim_{t \to T^-} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty.$$

**Remark 1.1.** For the case of $b > 2a$ and $k=1$ in Eq. (1.5), some authors have shown that the solution blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time (see [32,50]). But for the case $k=2$, we can’t obtain this result. Therefore it is still an open problem.

Finally, we consider a uniqueness of global weak solution to the problem (1.1) provided the initial data $y_0$ satisfies certain sign conditions.
**Theorem 1.6.** Let $u_0 \in H^2(\mathbb{R})$ and $y_0 = (I - \partial_x^2)^2 u_0$ be a positive Radon measure on $\mathbb{R}$. Then there exists a unique global weak solution $u \in W^{2,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^2(\mathbb{R}))$. Moreover, we obtain that $y = \Lambda^4 u$ is a positive Radon measure on $\mathbb{R}$ whose total variation on $\mathbb{R}$ is uniformly bounded for $t \geq 0$.

The plan of this paper is organized as follows. In the next section, the local well-posedness for the problem (1.1) is established, and Theorem 1.1 is proved. We deal with the global existence and blow up phenomena of solutions to the problem (1.1), and prove Theorems 1.2–1.5 in Section 3. In the last section, the weak solution for the problem (1.1) is considered, and Theorem 1.6 is proved.

2. Local well-posedness for the case $k = 2$

**Proof of Theorem 1.1.** When $k = 2$, then $y = \Lambda^4 u \equiv (I - \partial_x^2)^2 u = (l - 2\partial_x^2 + \partial_x^4) u$. To prove well-posedness we apply Kato’s semigroup approach [42]. For this, we rewrite (1.1) as follows

$$
\begin{cases}
  u_t + bu_x = -\partial_x \Lambda^{-4} \left[ \frac{a}{2} u^2 + \frac{6b - 2a}{2} u_x^2 + (a - 5b) u_x \partial_x^2 u - \frac{a + 5b}{2} u_{xx}^2 \right], & t > 0, \ x \in \mathbb{R}, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}.
\end{cases}
$$

(2.1)

Let $A(u) := bu \partial_x$, \( f(u) := -\Lambda^{-4} \partial_x \left[ \frac{a}{2} u^2 + \frac{6b - 2a}{2} u_x^2 + (a - 5b) u_x \partial_x^2 u - \frac{a + 5b}{2} u_{xx}^2 \right] \). \( Y = H^s(\mathbb{R}), X = H^{s-1}(\mathbb{R}) \) and \( Q = \Lambda \). Following closely the considerations made in [11, 56, 67], we obtain the statement of Theorem 1.1.

**Corollary 2.1.** If Theorem 1.1 yields the maximal time interval of existence is \((0, T)\), then we have

$$
T = +\infty \quad \text{or} \quad \lim_{t \to T^{-}} \|u(\cdot, t)\|_{H^s(\mathbb{R})} = +\infty \quad \text{if} \ T < \infty.
$$

**Proof.** By Theorem 1.1, we have

$$
T = +\infty, \quad \text{or} \quad \lim_{t \to T^{-}} \left( \|u(\cdot, t)\|_{H^s(\mathbb{R})} + \|u_t(\cdot, t)\|_{H^{s-1}(\mathbb{R})} \right) = +\infty \quad \text{if} \ T < \infty.
$$

On the other hand, from the proof of Theorem 1.1 and Eq. (1.1), we have

$$
\|u_t(\cdot, t)\|_{H^{s-1}(\mathbb{R})} \leq c \|u(\cdot, t)\|_{H^s(\mathbb{R})},
$$

which completes the proof of Corollary 2.1.

3. Global existence and blow-up phenomena

In this section, we deal with the global existence and blow-up phenomena for solution to Eq. (1.1). Let \( u_0 \in H^s(\mathbb{R}) \) with \( s \geq 7/2 \), then Theorem 1.1 ensures that there exists a unique solution \( u \) to problem (1.1) and

$$
u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))
$$

with the maximal existence time \( T > 0 \).

Next, we consider the following differential equation

$$
\begin{align*}
p_t &= bu(t, p(t, x)), & t \in [0, T), \ x \in \mathbb{R}, \\
p(0, x) &= x, & x \in \mathbb{R},
\end{align*}
$$

(3.1)
where \( u \) denotes the solution to the problem (1.1). Applying classical results in the theory of ordinary differential equations, one can obtain the following results on \( p(x, t) \) which are crucial in the proof on blow-up results.

**Lemma 3.1.** Let \( u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) \) with \( s \geq 7/2 \), there exists a unique solution \( p \in C([0, T) \times \mathbb{R}, \mathbb{R}) \) to the problem (3.1). Moreover, the map \( p(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
p_x(t, x) = \exp \left( \int_0^t b u_x(s, p(s, x)) \, ds \right) > 0, \tag{3.2}
\]

for all \((t, x) \in [0, T) \times \mathbb{R}\).

**Proof.** The proof can be found in [46,67], we omit it here. \( \square \)

**Lemma 3.2.** Let \( u_0 \in H^s(\mathbb{R}) \) with \( s \geq 7/2 \), and \( T > 0 \) be the maximal existence time of the corresponding solution \( u(x, t) \) to the problem (1.1), then we get

\[
y(t, p(t, x)) p^\frac{s}{2}(t, x) = y(0, x) = y_0(x). \tag{3.3}
\]

Moreover, if there exists \( M_1 > 0 \) such that \( u_x \geq -M_1 \) for all \((t, x) \in [0, T) \times \mathbb{R}\), then

\[
\| y(t, \cdot) \|_{L^\infty(\mathbb{R})} = \| y(t, p(t, \cdot)) \|_{L^\infty(\mathbb{R})} \leq \exp(aM_1T) \| y_0(\cdot) \|_{L^\infty(\mathbb{R})} \quad \text{for all } t \in [0, T). \tag{3.4}
\]

Furthermore, we get a conservation laws as follows

\[
\frac{d}{dt} \int \| y \|_{\mathbb{H}^s}^2 (t, x) \, dx = 0. \tag{3.5}
\]

**Proof.** Firstly, note that \( \frac{dp_x(t, x)}{dt} = p_x f = bu_x(p(t, p(t, x))p_x(t, x)) \). Differentiating the left-hand side of (3.3) with respect to \( t \), we have

\[
\frac{d}{dt} \left[ y(t, p(t, x)) p^\frac{s}{2}(t, x) \right] = \left[ y_t(t, p) + y_p(t, p) p_t(t, x) \right] p^\frac{s}{2}(t, x) + \frac{a}{b} y(t, p) p^\frac{s-1}{2}(t, x) p_x(t, x)
\]

\[
= \left[ y_t(t, p) + b y_p(t, p) u(t, p) + a y(t, p) u_p(t, p) \right] p^\frac{s}{2}(t, x)
\]

\[
= 0,
\]

which means that \( y(t, p(t, x)) p^\frac{s}{2}(t, x) \) is independent on time \( t \).

On the other hand, due to (3.1) we know \( p_x(x, 0) = 1 \), thus, (3.3) holds.

Secondly, by Lemma 3.1, (3.3), \( p_x(0, x) = 1 \) and the assumptions in Lemma 3.2, we get

\[
\| y(t, \cdot) \|_{L^\infty(\mathbb{R})} = \| y(t, p(t, \cdot)) \|_{L^\infty(\mathbb{R})}
\]

\[
= \left\| \exp \left( -a \int_0^t u_x(s, p(s, x)) \, ds \right) y_0(\cdot) \right\|_{L^\infty(\mathbb{R})}
\]

\[
\leq \exp(aM_1T) \| y_0(\cdot) \|_{L^\infty(\mathbb{R})} \quad \text{for all } t \in [0, T).
\]
By (3.3) and Lemma 3.1, we have

\[
\int_{\mathbb{R}} |y(t, x)|^\frac{b}{\sigma} \, dx = \int_{\mathbb{R}} |y(t, p(t, x))|^\frac{b}{\sigma} \, p_x(t, x) \, dx
\]

\[
= \int_{\mathbb{R}} (|y(t, p(t, x))|^\frac{a}{\sigma} \, p_x(t, x))^\frac{b}{a} \, dx
\]

\[
= \int_{\mathbb{R}} |y_0(x)|^\frac{b}{\sigma} \, dx \quad \text{for all } t \in [0, T).
\]

Therefore, the proof of Lemma 3.2 is complete. \( \square \)

**Remark 3.1.** It follows from Lemma 3.2 that \( y = (I - \partial_x^2)^2 u \geq 0 \) if \( y_0 = (I - \partial_x^2)^2 u_0 \geq 0 \). Since the operator \( (I - \partial_x^2)^{-2} \) preserves positivity, we get \( u \geq 0 \). Similarly, if \( y_0 = (I - \partial_x^2)^2 u_0 \leq 0 \), then \( (I - \partial_x^2)^2 u \leq 0 \) and \( u \leq 0 \).

**Proof of Theorem 1.2.** Since \( u_0 \in H^s(\mathbb{R}) \) (with \( s \geq 7/2 \)) and \( y_0 = (I - \partial_x^2)^2 u_0 \) has compact support. Without loss of general, we assume that \( y_0 \) is supported in the compact interval \([a, b]\). By Lemma 3.1, we have \( p_x(x, t) > 0 \) on \( \mathbb{R} \times [0, T) \). Therefore, by Lemma 3.2, we can conclude that the \( C^1 \) function \( y(x, t) \) has its support in the compact interval \([p(a, t), p(b, t)]\) for any \( t \in [0, T) \). The proof of Theorem 1.2 is complete. \( \square \)

**Proof of Theorem 1.3.** Since \( u_0 \in H^s(\mathbb{R}) \cap W^{4, \frac{b}{\sigma}}(\mathbb{R}) \), \( s \geq 7/2 \) and \( b \geq a \), by Lemma 3.2, we have

\[
\int_{\mathbb{R}} |y(t, x)|^\frac{b}{\sigma} \, dx = \int_{\mathbb{R}} |y_0(x)|^\frac{b}{\sigma} \, dx \leq \|u_0\|_{W^{4, \frac{b}{\sigma}}(\mathbb{R})},
\]

which yields \( y = (I - \partial_x^2)^2 u \in L^{\frac{b}{\sigma}}(\mathbb{R}) \), it follows that \( u \in W^{4, \frac{b}{\sigma}}(\mathbb{R}) \). By the Sobolev imbedding theorem, we have \( W^{4, \frac{b}{\sigma}}(\mathbb{R}) \subset C^1(\mathbb{R}) \). Thus the solution of the problem (1.1) remains smooth for all time. The proof of Theorem 1.3 is complete. \( \square \)

**Lemma 3.3.** Let \( u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R}) \) with \( s \geq 7/2 \) and \( y_0 = (I - \partial_x^2)^2 u_0 \geq 0 \) for all \( x \in \mathbb{R} \) (or equivalently \( y_0 = (I - \partial_x^2)^2 u_0 \leq 0 \) for all \( x \in \mathbb{R} \)), then there exists a constant \( K > 0 \) such that the solution of the problem (1.1) satisfies \( \|u_{xxx}\|_{L^\infty} \leq K \).

**Proof.** Since \( y_0 = (I - \partial_x^2)^2 u_0 \geq 0 \), by Lemma 3.2, we conclude that \( y = (I - \partial_x^2)^2 u \geq 0 \) and \( u \geq 0 \). Integrating directly the equation in (1.1), and using \( u_0 \in L^1(\mathbb{R}) \), we get (see [32] for the details)

\[
\int_{\mathbb{R}} y \, dx = \int_{\mathbb{R}} y_0 \, dx = \int_{\mathbb{R}} u \, dx = \int_{\mathbb{R}} u_0 \, dx.
\]  

(3.6)

Inspired by [25, 34, 50], it follows from (3.5) and (3.6) that another conservation laws as follows

\[
\int_{\mathbb{R}} b^2 y^{-\frac{b}{\sigma}} y_x^2 + a^2 y^{-\frac{b}{\sigma}} \, dx = \int_{\mathbb{R}} b^2 y_0^{-\frac{b}{\sigma}} y_{0x}^2 + a^2 y_0^{-\frac{b}{\sigma}} \, dx.
\]  

(3.7)
By using (3.5)–(3.7), and Sobolev embedding theorem, we have

\[ \|u_x\|_{L^\infty} \leq M. \tag{3.8} \]

On the other hand, since \( y \geq 0 \) and \( u \geq 0 \), then have

\[ 0 \leq \int_{-\infty}^{x} y \, dx = \int_{-\infty}^{x} (u - 2\partial_x^2 u + \partial_x^4 u) \, dx \leq \|u\|_{L^1(\mathbb{R})} - 2u_x + u_{xxx}, \]

\[ \|u\|_{L^1(\mathbb{R})} \geq \int_{-\infty}^{x} y \, dx \geq -\|u\|_{L^1(\mathbb{R})} - 2u_x + u_{xxx}, \]

which implies

\[ \|2u_x - u_{xxx}\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^1(\mathbb{R})}. \tag{3.9} \]

Combining (3.8) and (3.9), we get

\[ \|u_{xxx}\|_{L^\infty(\mathbb{R})} \leq K, \tag{3.10} \]

where the constant \( K \) depending only on the \( L^1 \) norm and \( H^2 \) norm of the initial \( u_0 \). The proof of Lemma 3.3 is complete. \( \square \)

**Proof of Theorem 1.4.** From Lemmas 3.2 and 3.3, we obtain the conservation laws. Applying \((\Lambda^4 u)\Lambda^4\) on both sides of the following equation

\[ u_t = -bu_x - f(u), \]

where \( f(u) := -\Lambda^{-4}\partial_x[\frac{a}{2}u^2 + \frac{6b-2a}{2}u_x^2 + (a - 5b)u_x\partial_x^3 u - \frac{a + 5b}{2}u_{xx}^2]. \) Integrating the new equation with respect to \( x \) by parts, we obtain the equation

\[ \frac{d}{dt} \int_{\mathbb{R}} (\Lambda^4 u)^2 = -2b \int_{\mathbb{R}} (\Lambda^4 u) (uu_x) \, dx - 2 \int_{\mathbb{R}} (\Lambda^4 u) \Lambda^4 f(u) \, dx. \tag{3.11} \]

We will estimate each terms in the right-hand side of (3.11). For the first terms in the right-hand side of (3.11), using integration by parts and Kato–Ponce inequality (see [43]), we get

\[ \int_{\mathbb{R}} (\Lambda^4 u) (uu_x) \, dx \leq c \|u\|_{H^1(\mathbb{R})}^2 \|u_x\|_{L^\infty(\mathbb{R})}. \tag{3.12} \]

By Lemma 3.3, Kato–Ponce inequality and Cauchy inequality, we have

\[ \int_{\mathbb{R}} (\Lambda^4 u) \Lambda^4 f(u) \, dx \leq \|u\|_{H^1(\mathbb{R})} \|f(u)\|_{H^1(\mathbb{R})} \]

\[ \leq C \|u\|_{H^1(\mathbb{R})} \left\| \frac{a}{2}u^2 + \frac{6b-2a}{2}u_x^2 + (a - 5b)u_x\partial_x^3 u - \frac{a + 5b}{2}u_{xx}^2 \right\|_{H^{3-3}(\mathbb{R})}. \]
\[ \leq C \| u \|_{H^s(R)} \left( \| u \|_{H^{s-3}(R)}^2 + \| u_x \|_{H^{s-3}(R)}^2 + \| u_{xx} \|_{H^{s-3}(R)}^2 \right) \]
\[ \leq C \| u \|_{H^s(R)} \left( \| u \|_{L^\infty(R)} \| u \|_{H^{s-3}(R)} + \| u_x \|_{L^\infty(R)} \| u \|_{H^{s-3}(R)} + \| u_{xx} \|_{L^\infty(R)} \| u \|_{H^{s-3}(R)} \right) \]
\[ + \| \partial_x^2 u \|_{L^\infty(R)} \| u_x \|_{H^{s-3}(R)} + \| u_{xx} \|_{L^\infty(R)} \| u_{xx} \|_{H^{s-3}(R)} \right) \]
\[ \leq C \| u \|_{H^s(R)}^2. \tag{3.13} \]

Therefore, by applying the Gronwall’s inequality and (3.11)–(3.13), the proof of Theorem 1.4 is complete. \[ \square \]

**Proof of Theorem 1.5.** By Theorem 1.1, we let \( u \) be the solution to the problem (1.1) with the initial data \( u_0 \in H^s(R) \) and \( s \geq \frac{7}{2} \), and \( T \) be the maximal existence of the solution \( u \). Multiplying the equation in (1.1) by \( y \) and integrating by parts, we have
\[ \frac{d}{dt} \int_R y^2 \, dx = 2 \frac{d}{dt} \int_R y y_t \, dx = -2 \int_R y (ay u_x + by_x u) \, dx = (b - 2a) \int_R y^2 u_x \, dx. \tag{3.14} \]

Clearly, if \( b = 2a \), (3.14) implies that every solution to the problem (1.1) remains regular globally in time.

Next, differentiating the equation in (1.1) with respect to \( x \), and then multiplying the resultant equation by \( y_x \) and integrating by parts, we get
\[ \frac{d}{dt} \int_R y_x^2 \, dx = 2 \int_R y_x y_{xt} \, dx \]
\[ = -2 \int_R y_x (axy_{xx} + (a + b)y_x u_x + by_{xx} u) \, dx \]
\[ = -(2a + b) \int_R y_x^2 u_x \, dx + a \int_R y^2 u_{xxx} \, dx. \tag{3.15} \]

Together (3.14) with (3.15), we get
\[ \frac{d}{dt} \left( \int_R y^2 + y_x^2 \, dx \right) = (b - 2a) \int_R y^2 u_x \, dx - (2a + b) \int_R y_x^2 u_x \, dx + a \int_R y^2 u_{xxx} \, dx. \tag{3.16} \]

If \( b < 2a \), by Lemma 3.2 and our assumption that there exists \( M_1 > 0 \) such that \( u_x \geq -M_1 \) for all \((t, x) \in [0, T) \times R\), we have
\[ \| y(t, \cdot) \|_{L^\infty(R)} = \| y(t, p(t, \cdot)) \|_{L^\infty(R)} \leq \exp(aM_1 T) \| y_0(\cdot) \|_{L^\infty(R)} \text{ for all } t \in [0, T). \tag{3.17} \]

Thus, it follows from (3.16) that
\[ \frac{d}{dt} \left( \int_R y^2 + y_x^2 \, dx \right) \leq c_1 \left( \int_R y^2 \, dx + \int_R y_x^2 \, dx \right) + a \exp(aM_1 T) \| y_0(\cdot) \|_{L^\infty(R)} \int_R y u_{xxx} \, dx \]
\[ \leq c_1 \left( \int_{\mathbb{R}} y^2 \, dx + \int_{\mathbb{R}} y_x^2 \, dx \right) + c_2 \int_{\mathbb{R}} y^2 + u_{xxx} \, dx \]
\[ \leq c_3 \left( \int_{\mathbb{R}} y^2 \, dx + \int_{\mathbb{R}} y_x^2 \, dx \right). \] \hfill (3.18)

where \( c_1, c_2, c_3 \) are positive constants.

By applying the Gronwall’s inequality, we derive that \( \|u\|_{H^5(\mathbb{R})} \) does not blow up in finite time. Furthermore, by Sobolev’s imbedding theorem, the solution \( u \) to the problem (1.1) does not blow up in finite time. This complete the proof of Theorem 1.5. \( \square \)

4. Weak solutions

In this section, we will show that there exists a unique global weak solution to the problem (1.1) provided the initial data \( y_0 \) satisfies certain sign conditions. In fact, the problem (1.1) can be rewritten as

\[ \begin{cases} u_t + F(u)_x = 0, & t > 0, \ x \in \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \] \hfill (4.1)

where

\[ F(u) = \frac{b}{2} u^2 + \Lambda^{-4} \partial_x \left[ \frac{a}{2} u^2 + \frac{6b - 2a}{2} u_x^2 + (a - 5b) u_x \partial_x^3 u - \frac{a + 5b}{2} u_{xx}^2 \right] \]
\[ = \frac{b}{2} u^2 + \Lambda^{-4} \partial_x \left[ \frac{a}{2} u^2 + \frac{6b - 2a}{2} u_x^2 + \frac{5b - 3a}{2} u_{xx}^2 \right] + (a - 5b) \partial_x \Lambda^4 (u_x u_{xx}). \] \hfill (4.2)

**Definition 4.1.** Let \( u_0 \in H^2(\mathbb{R}) \). If \( u \) belongs to \( L^\infty_{\text{loc}} ([0, T); H^2(\mathbb{R})) \) and satisfies the identity

\[ \int_{0}^{T} \int_{\mathbb{R}} (u \psi_t + F(u) \psi_x) \, dx \, dt + \int_{\mathbb{R}} u_0(x) \psi(0, x) = 0 \]

for all \( \psi \in C^\infty_0 ([0, T) \times \mathbb{R}) \), then \( u \) is called a weak solution to (4.1). If \( u \) is a weak solution on \([0, T)\) for every \( T > 0 \), then it is called a global weak solution to (4.1).

**Proposition 4.1.**

(i) Every strong solution is a weak solution.

(ii) If \( u \) is a weak solution and \( u \in C([0, T); H^s(\mathbb{R})) \cap ([0, T); H^{s-1}(\mathbb{R})) \) with \( s > 7/2 \), then it is a strong solution.

**Proof.** The proof is similar to that of Proposition 5.1 in [14], we omit it here. \( \square \)

**Proof of Theorem 1.6.** Combining the proofs on weak solutions in [7,32,50,56], we can prove Theorem 1.6 directly. In detail we omit here. \( \square \)
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