On embeddings of the countable fan space

M.V. Matveev

Department of Higher Mathematics, Bauman Moscow State University of Technology;
2-ja Baumanskaja ul. 5, Moscow 107005, Russia

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Abstract

We consider the possibility to embed the countable fan space into a countably compact or pseudocompact space satisfying certain properties (good separation, countable tightness, countable character at all points except one, etc.). © 1997 Elsevier Science B.V.

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Introduction

The countable fan $V_\omega$ is the space $V_\omega = \{*\} \cup \{a_{m,n}: m, n \in \omega\}$; points $a_{m,n}$ are isolated in $V_\omega$ while the basic neighborhood $U_f$ of the point $*$ (where $f \in {}^\omega \omega$) is the set $U_f = \{*\} \cup \{a_{m,n}: m \in \omega, n > f(m)\}$ (see, for example, [1]). We denote $V_\omega^* = V_\omega - \{*\}$ and $I_n = \{a_{m,n}: n \in \omega\}$.

A.V. Arhangel'skii rose a problem which spaces of countable tightness can be embedded into countably compact or pseudocompact spaces of countable tightness preserving good axioms of separation. This problem is related with similar questions for first countable spaces and for Moore spaces (see [11,2]). A.V. Arhangel'skii noted also that $V_\omega$ is one of the most natural test spaces to spread light onto this general problem.

In this note we obtain a series of results concerning different types of embeddings of $V_\omega$ into Hausdorff or regular pseudocompact or countably compact spaces. We consider:

(1) dense embeddings (i.e., pseudocompactifications and countable compactifications),

(2) closed embeddings into pseudocompact spaces (recall that a closed subspace of a pseudocompact space need not be pseudocompact (see [10] for the survey on closed embeddings into pseudocompact spaces), and, occasionally

(3) arbitrary embeddings. The main question whether $V_\omega$ can be embedded into a regular pseudocompact space remains
unsolved. On the other hand, we consider a related question: can $V_\omega$ be densely embedded into a regular pseudocompact or countably compact space $X$ so that $\chi(\ast, X) = \chi(\ast, V_\omega)$ and $X$ is first-countable at all points except $\ast$. We show that under certain extra set-theoretic assumptions, the answer is "yes".

If the space is not Tychonoff, by pseudocompactness we mean the property (sometimes called feeble compactness or DFCC) that every locally finite (equivalently, every discrete) family of open sets is finite. For Tychonoff spaces, this property is equivalent to pseudocompactness—see [5].

For a set $A$, we denote $[A]^\omega$ the family of all infinite countable subsets of $A$. Recall (see [31]) that $b = \min\{|B|\}: P$ is an unbounded subset of $\langle\omega, \leq^*\rangle$, and $\delta = \min\{|D|: D$ is a dominating subset of $\langle\omega, \leq^*\rangle\}$; dominating means cofinal. The equality $b = \delta$ is known to be equivalent to the statement that $\langle\omega, \leq^*\rangle$ has a scale, i.e., a well-ordered cofinal subset (see [3, Corollary 3.5]). Note that $\delta = \chi(\ast, T)$. Indeed, by [3, Theorem 3.6] $\delta_1 = \delta$ where $\delta_1 = \min\{|D|: D$ is cofinal in $\langle\omega, \leq\rangle\}$. A subfamily $A$ of a family of sets $B$ is called mad (maximal almost disjoint) if $A \cap B$ is finite for any distinct $A, B \in A$ and for any $A \subset B$ there exists a $B \in A$ such that the intersection $A \cap B$ is infinite.

1. Dense embeddings

**Proposition 1.1** [7,91]. $V_\omega$ can be embedded as a dense subspace into a countably tight pseudocompact Hausdorff space $X$ which is regular and first-countable at all points except $\ast$.

**Proof.** To prove this proposition one needs just to complete $\{I_n: n \in \omega\}$ to obtain a mad family of subsets of $V_\omega$. □

**Proposition 1.2.** If $V_\omega$ is embedded as a dense subspace into a Hausdorff space $X$, $A = X - V_\omega$, $B \subset A$ and $|B| < b$ then $\ast \notin \overline{B}$.

**Proof.** For each $b \in B$ choose disjoint open neighborhoods $W_b$ and $V_b$ of $\ast$ and $b$, respectively. Then $W_b \cap V_\omega \supset U_{f_b}$ for some function $f_b \in \omega$. The family $\{f_b: b \in B\}$ is bounded in $\langle\omega, \subseteq\rangle$ by some function $f$. There exists an open set $W$ in $X$ such that $W \cap V_\omega = U_f$. We claim that $W \cap B = \emptyset$ thus ensuring that $\ast \notin \overline{B}$. Indeed, if $b \in B \cap W$ then $V_b \cap W \cap V_\omega$ is infinite while $V_b \cap W_b$ is empty and $(W - W_b) \cap V_\omega$ is finite—a contradiction. □

**Proposition 1.3.** If $V_\omega$ is embedded as a dense subspace into a Hausdorff pseudocompact space $X$ which is regular at $\ast$ then $t(\ast, X) \geq b$.

**Proof.** Denote $A = X - V_\omega$. Since $X$ is regular at $\ast$, and $V_\omega$ is not locally pseudocompact at $\ast$, $\ast \in \overline{A}$. By Proposition 1.2, $\ast \notin \overline{B}$ for any $B \subset A$, $|B| < b$. □
Corollary. (a) $V_\omega$ cannot be embedded into a regular countably tight pseudocompact space as a dense subspace; (b) $V_\omega$ cannot be embedded into a regular countably tight countably compact space.

This corollary was contained in [7,9]. V.I. Malykhin has informed the author that he had obtained this result earlier. Recently, A.V. Arhangel'skii obtained similar results concerning $V_\omega$ and also concerning $\omega \cup \{p\}$ where $p \in \omega^*$.

**Proposition 1.4** ($b = 0$). $V_\omega$ can be embedded as a dense subspace into a zero-dimensional (hence regular) pseudocompact space $X$ which is first-countable at all points except $\ast$ and $\chi(\ast, X) = b$ (hence $t(\ast, X) \leq b$).

**Proof.** Let $B = \{f_\alpha: \alpha < b\}$ be a scale in $\omega^*$; $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta < b$. Denote for each $\alpha < b$, $R_\alpha = (V_\omega - U_{f_\alpha})^\omega$. One easily defines by transfinite induction subfamilies $S_\alpha \subset R_\alpha$ such that for each $\alpha < \beta$, the family $S_\alpha' = \bigcup \{S_\gamma: \gamma \leq \alpha\}$ is a mad subfamily of $R_\alpha' = \bigcup \{R_\gamma: \gamma \leq \alpha\}$ and $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. As soon as this is done, denote $S = \bigcup \{S_\alpha: \alpha < b\}$, choose a mad subfamily $Q_0$ of the family of all sequences in $V_\omega$ converging to $\ast$ and put $Q = S \cup Q_0$. Then $Q$ is a mad subfamily of $(V_\omega)^\omega$. Indeed, if $A \in (V_\omega)^\omega$ then either $A$ has infinite intersection with some element of $Q_0$ or $A \subset V_\omega - U_f$ for some $f \in \omega^*$. In the last case there is an $\alpha < b$ such that $f_\alpha \geq^* f$ and hence the set $A' = A - U_{f_\alpha}$ is infinite and it must have infinite intersection with some element of $S_\alpha'$. Also note that

(1) for each $\alpha < b$ and each $A \in Q$, either (i) the set $A \cap U_{f_\alpha}$ is finite, or (ii) the set $A - U_{f_\alpha}$ is finite.

Indeed, $A \in S_\beta$ for unique $\beta$. If $\alpha \geq \beta$ then $U_{f_\alpha} - U_{f_\beta}$ is finite, and since $A \subset X - U_{f_\beta}$, (i) holds. If $\alpha < \beta$ then (ii) holds, for otherwise $A$ would have infinite intersection with some element of $S_\alpha'$.

Put $X = V_\omega \cup S$. Points of $V_\omega$ are isolated. A basic neighborhood of a point $A \in S$ has the form $V_{A,K} = \{A\} \cup (A \setminus K)$ where $K$ is an arbitrary finite subset of $A$. A basic neighborhood of $\ast$ takes the form $O_{\alpha,K} = (U_{f_\alpha} \setminus K) \cup \{A \in Q: A \cap U_{f_\alpha} \text{ is infinite}\}$ where $\alpha < b$ and $K$ is finite. By (1), the sets $O_{\alpha,K}$ are clopen. So are the sets $V_{A,K}$. □

The proof of the next proposition uses the technique from [4] (and in fact from [6]). The space constructed here is a quotient space of Example 2.3.5 in [4].

**Proposition 1.5** ($\omega_1 = b$). $V_\omega$ can be embedded as a dense subspace into a zero-dimensional, compact space $X$ which is first countable at all points except $\ast$ and such that $\chi(\ast, X) = b$.

**Proof.** Let $B = \{f_\alpha: \alpha < b\}$ be a scale in $\omega^*$; $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta < b$. Put $X = V_\omega \cup B$. Points of $V_\omega$ are isolated in $X$. Basic neighborhoods of points from $B$ take the form

$$G(f_\alpha, f_\beta, K) = \{f_\gamma: \alpha < \gamma \leq \beta\} \cup ((L_{f_\alpha} - L_{f_\beta}) - K)$$
where \( L_f = V_\omega - U_f \) (\( f \in \omega^\omega \)) and \( K \) is a finite subset of \( V_\omega \). A basic neighborhood of \( * \) takes the form \( O_{\alpha,K} = \{ * \} \cup (U_{f_\beta} - K) \cup \{ f_\beta: \beta > \alpha \} \). Then \( B \cup \{ * \} \) is homeomorphic to the compact space \( \omega_1 + 1 \) with the order topology, and any sequence of points of \( V_\omega \) has a limit point in \( B \cup \{ * \} \). \( \square \)

**Question 1.** Is the converse to Propositions 1.4 and 1.5 true? (That is, does the possibility to embed the fan into a pseudocompact or countably compact space like specified in the propositions imply the corresponding set-theoretic assumptions?)

**Question 2.** Can "pseudocompact" be replaced by "countably compact" in Proposition 1.4?

### 2. Closed embeddings

**Proposition 2.1.** \( V_\omega \) can be embedded as a closed subspace into a countably tight Hausdorff pseudocompact space \( X \) which is regular at all points except \( * \).

**Proof.** We put \( X = V_\omega \cup Y \cup Z \) where:

1. \( Y = \{ b_{m,n,k}: m, n, k \in \omega \} \) consists of isolated points,
2. \( Z \) is a family of countable subsets of \( Y \) such that \( Z' = Z \cup \{ I_{m,n}: m, n \in \omega \} \) (where \( I_{m,n} = \{ b_{m,n,k}: k \in \omega \} \) is a mad family,
3. a basic neighborhood of a point \( z \in Z \) is of the form \( \{ z \} \cup (z \setminus K) \) where \( K \) is an arbitrary finite subset of \( z \),
4. a basic neighborhood of a point \( r_{m,n} \in V_\omega \) is of the form \( \{ a_{m,n} \} \cup (I_{m,n} \setminus K) \) where \( K \) is an arbitrary finite subset of \( I_{m,n} \),
5. a basic neighborhood of the point \( * \) is of the form

\[
U_{f,g} = \{ * \} \cup \{ a_{m,n}: m \in \omega, n \geq f(m) \} \\
\quad \cup \{ b_{m,n,k}: m \in \omega, n \geq f(m), k \geq g(m,n) \}
\]

where \( f \in \omega^\omega \) and \( g \in \omega^\omega \).

Then \( X \) is pseudocompact since \( Y \) is dense in \( X \) and \( Z' \) is mad; \( X \setminus \{ * \} \) is zero-dimensional and hence regular. \( \square \)

**Proposition 2.2 (\( b = \emptyset \)).** \( V_\omega \) can be embedded as a closed subspace into a zero-dimensional (hence regular) pseudocompact space \( X \) which is first-countable at all points except \( * \) and such that \( \chi(*,X) = b \) (and hence \( t(*,X) \geq b \)).

**Proof.** We construct \( X \) in the form

\[
X = X_\omega_1 = \bigcup \{ X_\alpha: \alpha < \omega_1 \}; \quad X = X_0 \cup Y \cup Z; \\
Y = Y_\omega_1 = \bigcup \{ Y_\alpha: \alpha < \omega_1 \}; \quad Z = Z_\omega_1 = \bigcup \{ Z_\alpha: \alpha < \omega_1 \}; \\
X_\alpha = Y_\alpha \cup Z_\alpha \text{ for } \alpha \leq \omega_1; \quad X_0 = Y_0 = V_\omega.
\]
\( Z_\alpha = \emptyset \) if \( \alpha = 0 \) or \( \alpha \) is a limit ordinal; \( Y \cap Z = \emptyset \); \( X_\alpha = \bigcup \{ X_\gamma : \gamma < \alpha \} \) if \( \alpha \) is a limit ordinal; \( X_\alpha \subset X_\beta, Y_\alpha \subset Y_\beta \) and \( Z_\alpha \subset Z_\beta \) whenever \( \alpha < \beta \). For \( \alpha = \beta + 1 \) we denote \( \bar{X}_\alpha = X_\alpha - X_{\alpha-1}, Y_\alpha = Y_\alpha - Y_{\alpha-1}, Z_\alpha = Z_\alpha - Z_{\alpha-1} \). We define the spaces \( X_\alpha \) by induction. Together with them we construct the following:

- bases \( B_\alpha \) of \( X_\alpha \) consisting of clopen sets,
- retractions \( \phi_\alpha^\beta : Y_\beta \to Y_\alpha \) (\( \alpha < \beta \)),
- mappings \( \psi_{\alpha,\beta} : B_\alpha \to B_\beta \) (\( n \in \omega, \alpha < \beta \)),
- almost disjoint families \( F_\alpha \) of countable subsets of \( Y_\alpha \),
- index sets \( A_\alpha \).

We demand that the following conditions should be fulfilled:
1. \( Y_\alpha \) is dense in \( X_\alpha \) (hence \( Y \) is dense in \( X \));
2. \( X_\alpha \) is bounded in \( X_{\alpha+1} \) (hence \( X \) is pseudocompact—see [8]);
3. \( \varphi_\alpha^\gamma = \varphi_\beta^\gamma \circ \varphi_\alpha^\beta \) whenever \( \alpha < \beta < \gamma < \omega \);
4. \( \psi_{\alpha,\alpha,\gamma} = \psi_{\beta,\alpha,\gamma} \circ \psi_{\alpha,\beta} \) whenever \( \alpha < \beta < \gamma < \omega \);
5. if \( \alpha < \beta \) and \( A \in F_\beta \) then \( \varphi_\alpha^\beta(A) \) is either a one-point set, or an element of \( F_\alpha \), or a sequence having a limit in \( X_\beta \);
6. if \( \alpha \leq \beta, U \in B_\alpha, A \in F_\beta, \) and \( n \in \omega \) then either the set \( A \cap \psi_{\alpha,\alpha,\beta}(U) \) is finite, or the set \( A - \psi_{\alpha,\alpha,\beta}(U) \) is finite;
7. if \( \alpha < \beta \) and \( x \in X_\alpha \cup U \) for some \( U \in B_\beta \) then \( U = \psi_{\alpha,\beta}(V) \) for some \( V \in B_\alpha \) and \( n \in \omega \).

To start, we take \( F_0 = S \) from the proof of Proposition 1.4 and put
\[
B_0 = \{ U_{f,\alpha} - K : \alpha <\beta, K \subset V_\alpha^* \text{ is finite} \} \cup \{ \{x\} : x \in V_\alpha^* \}.
\]

Non-limit step. Let \( \alpha = \alpha' + 1 \) and everything has been constructed for all \( \gamma < \alpha \). For each \( A \in F_{\alpha'} \), denote \( F_A \subset {}^\omega A \) the set of all functions from \( A \) to \( \omega \) which are either constant or do not take any value more than once. Note that for any function \( g \in {}^\omega A \), there exists an \( f \in F_A \) such that the set \( \{ x : f(x) = g(x) \} \) is infinite. Denote \( A_\alpha = \bigcup\{ F_A : A \in F_{\alpha'} \} \) and put \( \bar{Y}_\alpha = Y_{\alpha'} \times A_\alpha \times \omega \) with the product topology, where \( Y_{\alpha'} \) carries the topology defined on the previous step while \( A_\alpha \) and \( \omega \) carry the discrete topology. The projection \( \varphi_\alpha^\omega : Y_\alpha \to Y_{\alpha'} \) is defined as the identity on \( Y_{\alpha'} \) and as the projection onto the first factor on \( \bar{Y}_\alpha \). Put \( Z_\alpha = A_\alpha \). Let \( z \in Z_\alpha \). Then \( z \) is a function from the set \( A_\alpha \in F_{\alpha'} \) to \( \omega \). For every finite subset \( K \subset A \), denote
\[
O_K(z) = \{ z \} \cup \{ (y, z, z(y)) \in \bar{Y}_\alpha : y \in A - K \}.
\]

The sets \( O_K(z) \) form a countable local base of \( X_\alpha \) at \( z \). The points of \( \bar{Y}_\alpha \) are isolated in \( X_\alpha \). For \( V \in B_{\alpha'} \), denote
\[
\tilde{\psi}_{n,\alpha',\alpha}(V) = V \cup \{ (y, a, m) \in \bar{Y}_\alpha : y \in V \text{ and } m \geq n \} \quad \text{and} \quad \psi_{n,\alpha',\alpha} = \tilde{\psi}_{n,\alpha',\alpha} \cup \{ z \in Z_\alpha : O_K(z) - \{ z \} \subset \tilde{\psi}_{n,\alpha',\alpha}(V) \text{ for some } K \}.
\]

Note that by (6) the condition "\( O_K(z) - \{ z \} \subset \tilde{\psi}_{n,\alpha',\alpha}(V) \) for some \( K \)" is equivalent to the condition "\( O_K(z) \cap \tilde{\psi}_{n,\alpha',\alpha}(V) \) is infinite".
To define $\mathcal{F}_\alpha$, denote $\mathcal{G}_\alpha$ the set of all countable, discrete (in $X_\alpha$) subsets of $Y_\alpha$, and $\mathcal{H}_\alpha$ the subset of $\mathcal{G}_\alpha$ consisting of those elements $A$ for which either

(a) $|\varphi^\alpha_n(A)| = 1$, or
(b) $\varphi^\alpha_n(A) \subseteq F$ for some $F \in \mathcal{F}_\alpha$, or
(c) $\varphi^\alpha_n(A)$ is a sequence having a limit in $X_\alpha$.

Then every element of $\mathcal{G}_\alpha$ contains some element of $\mathcal{H}_\alpha$. We chose a mad subfamily $\mathcal{F}_\alpha$ of $\mathcal{H}_\alpha$ which therefore is a mad subfamily of $\mathcal{G}_\alpha$.

Limit step. Let $\alpha < \omega_1$ be a limit ordinal and everything have been constructed for all $\gamma < \alpha$. We put

$$X_\alpha = \bigcup\{X_\gamma : \gamma < \alpha\}, \quad Y_\alpha = \bigcup\{Y_\gamma : \gamma < \alpha\} \quad \text{and} \quad Z_\alpha = \bigcup\{Z_\gamma : \gamma < \alpha\}.$$

For each $x \in X_\alpha$ we denote $\text{por}(x)$ the first ordinal $\gamma < \alpha$ for which $x \in X_\gamma$. For $\lambda < \alpha$ and $y \in Y_\alpha$ we denote

$$\varphi^\alpha_\lambda(y) = \begin{cases} \varphi^\alpha_{\lambda+}(y) & \text{if } \text{por}(y) > \lambda \text{ and } y \in Y_\alpha, \\
y & \text{otherwise.} \end{cases}$$

For $\lambda < \alpha$, $n \in \omega$ and $U \in B_\lambda$ denote

$$\psi^{\alpha,n}_{\lambda,\alpha}(U) = \bigcup \{\psi^{\alpha,n}_{\lambda,\mu}(U) : \lambda < \mu < \alpha\}.$$

Put $B_\alpha = \{\psi^{\alpha,n}_{\lambda,\alpha}(U) : n \in \omega, \lambda < \alpha, U \in B_\lambda\}$. We set $A_\alpha = Z_\alpha = \emptyset$.

To define $\mathcal{F}_\alpha$, we denote $\mathcal{G}_\alpha$ the set of all countable, discrete (in $X_\alpha$) subsets of $Y_\alpha$. Note that for each $A \in \mathcal{G}_\alpha$ we have $\sup\{\text{por}(y) : y \in A\} = \alpha$. We denote $\mathcal{H}_\alpha$ the subfamily of $\mathcal{G}_\alpha$ consisting of those elements $A$ of $\mathcal{G}_\alpha$ for which either (a), or (b), or (c) holds (see nonlimit step) for each $\alpha' < \alpha$. Then every element of $\mathcal{G}_\alpha$ contains some element of $\mathcal{H}_\alpha$. So, we can choose a mad subfamily $\mathcal{F}_\alpha$ of $\mathcal{H}_\alpha$ which therefore is a mad subfamily of $\mathcal{G}_\alpha$. \qed

This construction is a modification of a construction from [7,8]—we take convergent sequences instead of ultrafilters; this is possible due to the simple structure of the bases $B_\alpha$. Another way to construct such an embedding is a two-step construction similar to those from Section 1.

3. $d$-retraction

**Definition.** A subspace $X_0 \subset X$ is called a $d$-retract of $X$ if there are subspaces $Y \subset X$ and $Y_0 \subset X_0 \cap Y$ such that $Y_0$ is dense in $Y$, $Y$ is dense in $X$ and there exists a retraction $d : Y \to Y_0$. We say that $d$ is a $d$-retraction of $X$ onto $X_0$ through $Y$ and $Y_0$, or, simply, a $d$-retraction.

Every dense subspace $X_0 \subset X$ is a $d$-retract of $X$, indeed, in this case one can take $Y_0 = Y = X_0$. Therefore, every Tychonoff space is a $d$-retract of a compact space—this sharply contrasts the fact that a “real” retraction, being a continuous surjection, preserves compactness-type properties. Note that in the constructions involved in Propositions 2.1 and 2.2, $V_\omega$ is embedded into a pseudocompact space as a $d$-retract (one just takes
$Y_0 = V^*_\ast$ and $Y$ from corresponding construction). Here, on the other hand, is a negative result which is an easy generalization of Proposition 1.3:

**Proposition 3.1.** If $V_\omega$ is embedded as a $d$-retract into a Hausdorff pseudocompact space $X$ which is regular at $\ast$ then $t(\ast, X) \geq b$.

**Proof.** Put $A = \{x \in X \setminus V_\omega : \exists$ neighborhoods $U$ of $\ast$ in $V_\omega$ and $V$ of $x$ in $X$ such that $U \cap r(V \cap Y) = \emptyset\}$ (here $r$ is a $d$-retraction of $X$ onto $T$ through $Y$ and $V^*_\ast$). Then the proof of Propositions 1.2 and 1.3 can be repeated almost literally. □

**References**