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Algebraic Equations with Exponential Terms

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The decomposition method is applied to algebraic equations containing exponential terms. The n term approximation ϕ_n is rapidly damped as n increases, yielding an oscillating convergence of superior accuracy (<0.01 % by $n=5$ and <0.0001 % by $n=9$ for $k=2$). © 1985 Academic Press, Inc.

INTRODUCTION

The decomposition method [1] of the first author has been shown to provide accurate approximation solutions to a very large class of differential equations (linear, nonlinear, deterministic, stochastic, or systems of coupled equations) and to be potentially useful for partial differential equations as well. Since the methods are operator methods, they can deal also with algebraic and trigonometric operators.

Consider as an example the equation $x=k+e^{-x}$, $k>0$. The solution is

$$x=k+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^{n-1}}{n!}e^{-nk}.$$

If we write $x=\sum_{n=0}^{\infty}x_n$ and $Nx=\sum_{n=0}^{\infty}A_n$, where the A_n represents the Adomian polynomials [2], we have

$$\sum_{n=0}^{\infty}x_n=k+\sum_{n=0}^{\infty}A_n$$

$$x_0=k$$

$$x_1=A_0=e^{-x_0}$$

$$x_2=A_1=e^{-x_0}(-x_1)$$

$$x_3=A_2=e^{-x_0}(-x_2+\tfrac{1}{2}x_1^2)$$

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$$x_4 = A_3 = e^{-x_0}(-x_3 + x_1 x_2 - \tfrac{1}{6}x_1^3)$$

$$x_5 = A_4 = e^{-x_0}(-x_4 + \tfrac{1}{2}x_2^2 + x_1 x_3 - \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{24}x_1^4)$$

$$x_6 = A_5 = e^{-x_0}(-x_5 + x_2 x_3 + x_1 x_4 - \tfrac{1}{2}x_1 x_2^2 - \tfrac{1}{2}x_1^2 x_3 + \tfrac{1}{6}x_1^3 x_2 - \tfrac{1}{120}x_1^5)$$

$$x_7 = A_6 = e^{-x_0}(-x_6 + \tfrac{1}{2}x_3^2 + x_2 x_4 + x_1 x_5 - \tfrac{1}{6}x_2^3 - x_1 x_2 x_3 - \tfrac{1}{2}x_1^2 x_4 + \tfrac{1}{4}x_1^2 x_2^2 + \tfrac{1}{6}x_1^3 x_3 - \tfrac{1}{24}x_1^4 x_2 + \tfrac{1}{720}x_1^6)$$

$$x_8 = A_7 = e^{-x_0}(-x_7 + x_3 x_4 + x_2 x_5 + x_1 x_6 - \tfrac{1}{2}x_2^2 x_3 - \tfrac{1}{2}x_1 x_3^2 - x_1 x_2 x_4 - \tfrac{1}{2}x_1^2 x_5 + \tfrac{1}{6}x_1 x_2^3 + \tfrac{1}{2}x_1^2 x_2 x_3 + \tfrac{1}{6}x_1^3 x_4 - \tfrac{1}{12}x_1^3 x_2^2 - \tfrac{1}{24}x_1^4 x_3 + \tfrac{1}{120}x_1^5 x_2 - \tfrac{1}{5040}x_1^7)$$

etc. Each term is successively calculated:

$$x_0 = k$$

$$x_1 = e^{-k}$$

$$x_2 = -e^{-2k}$$

$$x_3 = \tfrac{3}{2}e^{-3k}$$

$$x_4 = (-\tfrac{8}{3})\, e^{-4k}$$

$$x_5 = \tfrac{125}{24}e^{-5k}$$

$$x_6 = -\tfrac{6^5}{6!}e^{-6k}$$

etc. We notice the signs alternate for the x_n—positive for x_1, x_3, x_5,..., and minus for x_2, x_4, x_6,.... Thus for x_n we have a coefficient $(-1)^{n+1}$. The corresponding exponential is e^{-nk}. The numerical coefficient (other than $(-1)^{n+1}$) for e^{-nk} is given as

$n=1$	1	or	$1^0/1$
$n=2$	1	or	$2^1/1\cdot 2$
$n=3$	3/2	or	$3\cdot 3/2\cdot 3 = 3^2/3!$
$n=4$	8/3	or	$8\cdot 2\cdot 4/2\cdot 3\cdot 4 = 4^3/4!$
$n=5$	125/24	or	$125\cdot 5/24\cdot 5 = 5^4/5!$
$n=6$	1296/120	or	$1296\cdot 6/120\cdot 6 = 7776/720 = 6^5/6!$

Thus we can write $x_n = (-1)^{n+1}(n^{n-1}/n!)\, e^{-nk}$ and consequently, we have an algorithm making unnecessary the computation of more A_n polynomials.

A graphical verification is easily obtained by plotting e^{-x} versus x, raising the values by k (let us choose $k=2$) and looking for the intersection of the resulting curve with the line $y=x$. For $k=2$, we find the solution $x=2.120028239$.

The decomposition method solution is

$$x=k+\sum_{n=1}^{\infty}(-1)^{n+1}(n^{n-1}/n!)\,e^{-nk}$$

$$x_0=k=2$$

$$x_1=e^{-2}=1.3533528\times 10^{-1}$$

$$x_2=-e^{-4}.$$

The corresponding values of the exponential are

$$e^{-2}=1.3533528\times 10^{-1}$$
$$e^{-4}=1.8315639\times 10^{-2}$$
$$e^{-6}=2.4787522\times 10^{-3}$$
$$e^{-8}=3.3546263\times 10^{-4}$$
$$e^{-10}=4.539993\times 10^{-5}$$
$$e^{-12}=6.1442124\times 10^{-6}$$

The results for the x_n are

n	x_n
0	2.0
1	1.3533528×10^{-1}
2	-1.8315639×10^{-2}
3	3.7181283×10^{-3}
4	-8.9456701×10^{-4}
5	2.3645797×10^{-4}
6	-6.6357493×10^{-5}

The n term approximation $\phi_n=x_0+x_1+\cdots+x_n$. Values of ϕ_n are given in the following table:

n	ϕ_n	% error $\Psi_n = [(\phi_n - x)/x]\,100$
1	2.00...	-5.6
2	2.13533528	$+0.722$
3	2.117019641	-0.142
4	2.1207378	$+0.0330$
5	2.1198432	-0.00873
6	2.1200797	$+0.00243$
7	2.1200133	-0.000705

The accuracy is remarkable. A seven-term approximation has an error of 0.0007 %. The small amplitude oscillating (and rapidly damped) convergence is interesting to note and a subject of further study. If we plot Ψ_n versus Ψ_n, we see oscillations of decreasing amplitude. The envelope of the oscillations decreases asymptotically to zero. The calculated value of $\Psi_8 = 0.00021091328$; $\Psi_9 = -0.0001392257\%$. Thus with a nine-term approximation the error is less than (1/10000)%. The oscillating convergence means the solution is between ϕ_8 and ϕ_9. Thus, $\phi_8 < x < \phi_9$ or

$$2.1200327 < x < 2.1200269$$

The true solution is $x = 2.120028239$.

Finally it is interesting to consider the following: We determined x as

$$x = k + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-1}}{n!} e^{-nk}.$$

Substituting this into the original equation, we must have

$$e^{-[k + \sum_{n=1}^{\infty} (-1)^{n+1} (n^{n-1}/n!) e^{-nk}]} = \sum_{n=1}^{\infty} (-1)^{n+1} (n^{n-1}/n!)\, e^{-nk}$$

or

$$e^{-k} e^{-\sum (-1)^{n+1} (n^{n-1}/n!) e^{-nk}} = \sum_{n=1}^{\infty} (-1)^{n+1} (n^{n-1}/n!)\, e^{-nk}$$

which says

$$e^{-k} e^{-[e^{-k} - e^{-2k} + \dots]} = e^{-k} - e^{-2k} + \dots .$$

Multiply by e^k

$$e^{-[\ ]} = 1 - [\ ]$$

or

$$e^{-x} = 1 - x + \dots,$$

an approximate check.

References

1. G. Adomian, "Stochastic Systems," Academic Press, New York, 1983; "Nonlinear Stochastic Operator Equations," expected publication 1986.
2. R. Rach, The Adomian Polynomials, *J. Math. Anal. Appl.* **102**, No. 2, (1984), 415–419.