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Hierarchies of tree series transformations ☆

Zoltán Fülöp[a,*], Zsolt Gazdag[a], Heiko Vogler[b]

[a] *Department of Computer Science, University of Szeged, Árpád tér 2., H-6720 Szeged, Hungary*

[b] *Faculty of Computer Science, Dresden University of Technology, Dresden D-01062, Germany*

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Abstract

We study bottom-up and top-down tree series transducers over a semiring A and denote the tree series transformation classes computed by them by $BOT_{t-ts}(A)$ and $TOP_{t-ts}(A)$, respectively. We present the inclusion diagram of the classes $p\text{-}BOT^n_{t-ts}(A)$, $p\text{-}TOP^n_{t-ts}(A)$, $p\text{-}BOT^{n+1}_{t-ts}(A)$, and $p\text{-}TOP^{n+1}_{t-ts}(A)$ and prove its correctness, where A is a commutative izz-semiring (izz=idempotent, zero-divisor free, and zero-sum free) and the prefix p stands for polynomial. This inclusion diagram implies the properness of the following four hierarchies:

$$p\text{-}TOP_{t-ts}(A) \subsetneq p\text{-}TOP^2_{t-ts}(A) \subsetneq p\text{-}TOP^3_{t-ts}(A) \subsetneq \cdots,$$
$$p\text{-}BOT_{t-ts}(A) \subsetneq p\text{-}BOT^2_{t-ts}(A) \subsetneq p\text{-}BOT^3_{t-ts}(A) \subsetneq \cdots,$$
$$p\text{-}TOP_{t-ts}(A) \subsetneq p\text{-}BOT^2_{t-ts}(A) \subsetneq p\text{-}TOP^3_{t-ts}(A) \subsetneq p\text{-}BOT^4_{t-ts}(A) \subsetneq \cdots,$$
$$p\text{-}BOT_{t-ts}(A) \subsetneq p\text{-}TOP^2_{t-ts}(A) \subsetneq p\text{-}BOT^3_{t-ts}(A) \subsetneq p\text{-}TOP^4_{t-ts}(A) \subsetneq \cdots,$$

where the first hierarchy generalizes the famous top-down tree transformation hierarchy of Engelfriet (Math. Systems Theory 15 (1982) 92–125). As the second main result we prove that the first two hierarchies are proper even for arbitrary (i.e., not necessarily commutative) izz-semirings.

Keywords: Tree automata; Tree transducers; Tree series transducers

1. Introduction

Bottom-up tree transducers and top-down tree transducers have been introduced in [25] and [23,24], respectively. Roughly speaking, a tree transducer M is a finite state

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* Corresponding author. Tel.: +36-62-311-184; fax: +36-62-312-292.

E-mail addresses: fulop@inf.u-szeged.hu (Z. Fülöp), gazdag@inf.u-szeged.hu (Z. Gazdag), vogler@tcs.inf.tu-dresden.de (H. Vogler).

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machine which takes trees as input and produces trees as output. That is, M computes a function (called: tree transformation) $\tau_M : T_\Sigma \to \mathscr{P}(T_\Delta)$ where T_Σ and T_Δ are the sets of trees over the input ranked alphabet Σ and the output ranked alphabet Δ, respectively. Thus, tree transducers can be considered as a generalization of the usual sequential machines. The classes of tree transformations computed by bottom-up and top-down tree transducers are denoted by BOT_{tt} and TOP_{tt}, respectively.

Since their introduction, various contributions have been made to the theory of tree transducers as, e.g., (de-)composition of classes of tree transformations [1,6,7], hierarchy results [1,8,12,13], and constructions of semigroups generated by tree transformation classes [14] with composition. Survey articles and books which collect (at least parts of) the theory of tree transducers, are [5,15,17,18,21].

In this paper we consider hierarchy results. A tree transformation hierarchy is a sequence $C_1 \subseteq C_2 \subseteq \cdots$ such that, for every $n \geqslant 1$, C_n is a tree transformation class. If $C_n \subset C_{n+1}$ for every $n \geqslant 1$, then the hierarchy is proper. Two obvious examples of tree transformation hierarchies are the uniform top-down tree transformation hierarchy

$$TOP_{tt} \subseteq TOP_{tt}^2 \subseteq TOP_{tt}^3 \subseteq \cdots$$

and the uniform bottom-up tree transformation hierarchy

$$BOT_{tt} \subseteq BOT_{tt}^2 \subseteq BOT_{tt}^3 \subseteq \cdots$$

where C^n denotes the n-fold composition $\overbrace{C \circ \cdots \circ C}^{n}$ of the tree transformation class C.

Now we recall other tree transformation hierarchies which are relevant to this paper. The first such hierarchy result follows from Theorem 13 of [1] where the inclusions $TOP_{tt}^n \subseteq BOT_{tt}^{n+1}$ and $BOT_{tt}^n \subseteq TOP_{tt}^{n+1}$ were proved for every $n \geqslant 1$. These hierarchies are the alternating top-down tree transformation hierarchy

$$TOP_{tt} \subseteq BOT_{tt}^2 \subseteq TOP_{tt}^3 \subseteq BOT_{tt}^4 \subseteq \cdots$$

and the alternating bottom-up tree transformation hierarchy

$$BOT_{tt} \subseteq TOP_{tt}^2 \subseteq BOT_{tt}^3 \subseteq TOP_{tt}^4 \subseteq \cdots \; .$$

The next hierarchy result which we mention is based on Theorem 3.14 of [8]. It says that the uniform top-down tree transformation hierarchy is proper. The last relevant hierarchy result is Corollary 8.13 (iii) of [17] which states that the uniform bottom-up tree transformation hierarchy is proper.

In [22], these hierarchy results were combined into an inclusion diagram (cf. Fig. 1). An inclusion diagram of tree transformation classes is the Hasse-diagram of these classes with respect to the inclusion $\subseteq$ as partial order. That means, every inclusion shown by the diagram is proper (or strict) and the unrelated classes are incomparable with respect to the inclusion. However, the correctness of the diagram was not proved in [22]. Since the aim of the present paper is the generalization of this diagram to tree series transformation classes, we will prove its correctness.

Now we turn to (polynomial) tree series transducers. The investigation of tree series transducers was started in [20], where a restricted class of top-down tree transducers, called nondeterministically simple, was generalized. The generalization from tree

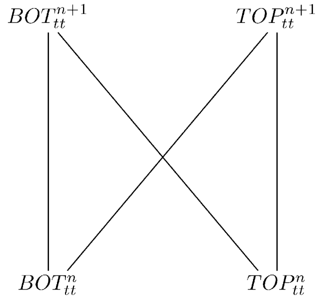

Fig. 1. The inclusion diagram of BOT_{tt}^n, TOP_{tt}^n, BOT_{tt}^{n+1}, and TOP_{tt}^{n+1} for every $n \geqslant 1$.

transducers to tree series transducers was carried out in a more systematic way in [9,16], where the full classes of both bottom-up and top-down tree transducers were generalized and the concept of tree series transducers was introduced.

Let us look at this concept in more detail. A tree series over a ranked alphabet Δ and semiring A, introduced in [4], is a mapping $\varphi : T_\Delta \to A$, cf. also [2,3,19]. A tree series φ is polynomial if the set $\{s \in T_\Delta \mid \varphi(s) \neq 0\}$ is finite. Then, as special case, a tree language $L \subseteq T_\Delta$ can be thought of as a tree series over the boolean semiring. A tree series transducer M computes a mapping $\tau_M : T_\Sigma \to A\langle\langle T_\Delta \rangle\rangle$, where $A\langle\langle T_\Delta \rangle\rangle$ denotes the class of all tree series over Δ and A. As a special case, if M is polynomial, then $\tau_M(s)$ is polynomial for every $s \in T_\Sigma$. The classes of tree series transformations computed by bottom-up and by top-down tree series transducers over a semiring A are denoted by $BOT_{t-ts}(A)$ and $TOP_{t-ts}(A)$, respectively. The corresponding classes for the polynomial tree series transducers are denoted by $p\text{-}BOT_{t-ts}(A)$ and $p\text{-}TOP_{t-ts}(A)$, respectively.

In this paper we generalize the inclusion diagram in Fig. 1 to a commutative izz-semiring A and to the classes $p\text{-}BOT_{t-ts}^n(A)$, $p\text{-}TOP_{t-ts}^n(A)$, $p\text{-}BOT_{t-ts}^{n+1}(A)$, and $p\text{-}TOP_{t-ts}^{n+1}(A)$ of polynomial tree series transformations. An izz-semiring A is an idempotent, zero-divisor free, and zero-sum free semiring. This leads to the inclusion diagram of the above classes which is obtained from the diagram of Fig. 1 by replacing the tree transformation classes by the corresponding polynomial tree series transformation classes (e.g. the class BOT_{tt}^n is replaced by $p\text{-}BOT_{t-ts}^n(A)$), cf. Fig. 2. From this inclusion diagram it follows that the uniform polynomial top-down tree series transformation hierarchy

$$p\text{-}TOP_{t-ts}(A) \subseteq p\text{-}TOP_{t-ts}^2(A) \subseteq p\text{-}TOP_{t-ts}^3(A) \subseteq \cdots,$$

the uniform polynomial bottom-up tree series transformation hierarchy

$$p\text{-}BOT_{t-ts}(A) \subseteq p\text{-}BOT_{t-ts}^2(A) \subseteq p\text{-}BOT_{t-ts}^3(A) \subseteq \cdots,$$

the alternating polynomial top-down tree series transformation hierarchy

$$p\text{-}TOP_{t-ts}(A) \subseteq p\text{-}BOT_{t-ts}^2(A) \subseteq p\text{-}TOP_{t-ts}^3(A) \subseteq p\text{-}BOT_{t-ts}^4(A) \subseteq \cdots,$$

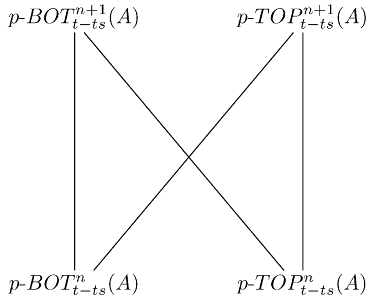

Fig. 2. The inclusion diagram of $p\text{-}BOT^n_{t-ts}(A)$, $p\text{-}TOP^n_{t-ts}(A)$, $p\text{-}BOT^{n+1}_{t-ts}(A)$, and $p\text{-}TOP^{n+1}_{t-ts}(A)$ for every $n \geqslant 1$ and commutative izz-semiring A.

and the alternating polynomial bottom-up tree series transformation hierarchy

$$p\text{-}BOT_{t-ts}(A) \subseteq p\text{-}TOP^2_{t-ts}(A) \subseteq p\text{-}BOT^3_{t-ts}(A) \subseteq p\text{-}TOP^4_{t-ts}(A) \subseteq \cdots$$

are proper. This is the first main result of our paper.

As the second main result we prove that, for every (not necessarily commutative) izz-semiring A, the uniform polynomial top-down tree series transformation hierarchy and the uniform polynomial bottom-up tree series transformation hierarchy are proper.

The contents of the paper is as follows. In Section 2 we collect the necessary basic definitions, notations, and propositions. In Section 3, we define the concept of a tree series transducer and those of its restricted versions which are necessary to develop the results of the paper. In Section 4, we prove the correctness of the inclusion diagram of tree transformation classes shown in Fig. 1. In Section 5, we prove the two inclusions $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$ from which the alternating polynomial top-down and bottom-up tree series transformation hierarchies follow. In Section 6, we obtain the inclusion diagram of the polynomial tree series transformation classes $p\text{-}BOT^n_{t-ts}(A)$, $p\text{-}TOP^n_{t-ts}(A)$, $p\text{-}BOT^{n+1}_{t-ts}(A)$, and $p\text{-}TOP^{n+1}_{t-ts}(A)$ for every commutative izz-semiring A. Also we prove here our second main result.

2. Preliminaries

2.1. Sets, strings, and trees

The power set of a set H is denoted by $\mathscr{P}(H)$ and $\emptyset$ denotes the empty set. $\mathbb{N}$ is the set of all nonnegative integers and, for every $n \in \mathbb{N}$, we let $[n] = \{1, \ldots, n\}$.

Let $\varrho \subseteq F \times G$ be a relation. The fact that $(a, b) \in \varrho$ for some $a \in F$ and $b \in G$ is also denoted by $a\varrho b$. The composition of ϱ and $\tau \subseteq G \times H$ is $\varrho \circ \tau = \{(a, c) \in F \times H \mid$

$(\exists b \in G): a\rho b$ and $b\tau c\}$. The concept of composition extends to classes of relations: for two classes C_1 and C_2, we define $C_1 \circ C_2 = \{\varrho \circ \tau \mid \varrho \in C_1$ and $\tau \in C_2\}$. For $n \geqslant 1$, the n-fold composition of a class C by itself is denoted by C^n.

If H is an alphabet, then H^* denotes the set of strings over H; the empty string is denoted by ε. For a string $w \in H^*$, $|w|$ denotes its length and, for a symbol $a \in H$, $|w|_a$ denotes the number of occurrences of a in w.

Let Σ be a ranked alphabet. For every $k \geqslant 0$, we denote by $\Sigma^{(k)}$ the set of all symbols of Σ which have rank k. Moreover, let H be a set disjoint with Σ. The set of (*finite*, *labelled and ordered*) *trees over* Σ *indexed by* H, denoted by $T_\Sigma(H)$, is the smallest subset T of $(\Sigma \cup H \cup \{(,)\} \cup \{,\})^*$, such that (i) $H \subseteq T$ and (ii) if $\sigma \in \Sigma^{(k)}$ with $k \geqslant 0$ and $s_1, \ldots, s_k \in T$, then $\sigma(s_1, \ldots, s_k) \in T$. In case $k = 0$, we identify $\sigma(\,)$ with σ. Moreover, $T_\Sigma(\emptyset)$ is denoted by T_Σ. It should be clear that $T_\Sigma = \emptyset$ if and only if $\Sigma^{(0)} = \emptyset$. Since we are not interested in this particular case, we assume that $\Sigma^{(0)} \neq \emptyset$ for every ranked alphabet Σ appearing as input or output ranked alphabet of some tree transducer in this paper.

We will need the set $X = \{x_1, x_2, \ldots\}$ of *variable symbols*. For every $k \geqslant 0$, we define $X_k = \{x_1, \ldots, x_k\}$, thus $X_0 = \emptyset$. We use the variables to occur in trees, so we will frequently consider the sets $T_\Sigma(X)$, $T_\Sigma(X_k)$, etc. of trees where Σ is a ranked alphabet.

Let Y be a finite subset of X and let $t \in T_\Sigma(Y)$. The tree t is called *linear in* Y (*nondeleting in* Y) if, for every $x_i \in Y$, $|t|_{x_i} \leqslant 1$ ($|t|_{x_i} \geqslant 1$, respectively) holds.

We distinguish a subset $\widehat{T_\Sigma}(X_k)$ of $T_\Sigma(X_k)$ as follows. Let a tree $t \in T_\Sigma(X_k)$ be in $\widehat{T_\Sigma}(X_k)$ if for every $1 \leqslant i \leqslant k$, $|t|_{x_i} = 1$ and, reading the leaves of t from left to right, the variables occur in the order $x_1 < x_2 < \cdots < x_k$. Note that elements of $\widehat{T_\Sigma}(X_k)$ are linear and nondeleting in X_k.

The *tree substitution* is defined as follows. Let $t \in T_\Sigma(X_k)$ for some ranked alphabet Σ and $k \geqslant 0$. Moreover, let $t_1, \ldots, t_k$ be also trees over (maybe other) ranked alphabets. Then $t[t_1, \ldots, t_k]$ stands for the tree which is obtained from t by substituting, for every $1 \leqslant i \leqslant k$, the tree t_i for every occurrence of x_i.

Let $t \in T_\Sigma(H)$. The *linearization of* t *with respect to* H, denoted by $lin_H(t)$, is defined as the unique pair (t', w) where $t' \in \widehat{T_\Sigma}(X_k)$ and $w = a_1 \ldots a_k \in H^*$ such that $t = t'[a_1, \ldots, a_k]$.

If Q is a unary ranked alphabet, i.e., the rank of every symbol in Q is 1, and Y is a finite subset of X, then $Q(Y)$ stands for the set $\{q(x_i) \mid q \in Q$ and $x_i \in Y\}$.

In the rest of the paper Σ, Δ, and Γ denote ranked alphabets.

By a *tree language* we mean a subset L of T_Σ. The class of *recognizable tree languages* is denoted by *REC*. A *tree transformation* is a function $\tau: T_\Sigma \to \mathscr{P}(T_\Delta)$. The classes of all *tree transformations computed by bottom-up tree transducers* (*by top-down tree transducers*) is denoted by BOT_{tt} (TOP_{tt}, respectively). The *linear* subclasses of BOT_{tt} and TOP_{tt} are denoted by $l\text{-}BOT_{tt}$ and $l\text{-}TOP_{tt}$, respectively. The class of homomorphism tree transformations, which is a subclass of both BOT_{tt} and TOP_{tt}, is denoted by HOM_{tt}. For more terminology and details about tree languages, tree transducers, and the composition theory of tree transformations the reader is advised to consult [1,6,14,15,17,18]. Note, in [17] the expressions root-to-frontier and frontier-to-root are used for top-down and for bottom-up, respectively.

2.2. Semirings

A *semiring* is an algebraic structure $\underline{A}=(A,\oplus,\odot,0,1)$ with two operations sum $\oplus$ and product $\odot$ such that $(A,\oplus,0)$ is a commutative monoid, $(A,\odot,1)$ is a monoid, and the following laws hold: for every $a,b,c\in A$, $(a\oplus b)\odot c=(a\odot c)\oplus(b\odot c)$, $a\odot(b\oplus c)=(a\odot b)\oplus(a\odot c)$, and $a\odot 0=0\odot a=0$. Whenever the operations ($\oplus$ and $\odot$) and the neutral elements (0 and 1) are clear from the context, then we denote the semiring $(A,\oplus,\odot,0,1)$ just by A. We give some examples of semirings which, besides, will be used also later.

- The *boolean semiring* is $\mathbb{B}=(\{0,1\},\vee,\wedge,0,1)$ with disjunction and conjunction as sum and product, respectively.
- The *semiring of natural numbers* is $Nat=(\mathbb{N},+,\cdot,0,1)$ with the obvious sum and product operations.
- The *tropical semiring* is $Trop=(\mathbb{N}\cup\{\infty\},\min,+,\infty,0)$ with min and $+$ as sum and product operations, respectively, where $\min\{a,\infty\}=\min\{\infty,a\}=a$ and $\infty+a=a+\infty=\infty$.
- Let Δ be an alphabet. The *semiring of finite formal languages* (*over* Δ) is $Lang_f(\Delta)=(\mathscr{P}_f(\Delta^*),\cup,\cdot,\emptyset,\{\varepsilon\})$ where $\mathscr{P}_f(\Delta^*)$ is the set of finite subsets of Δ^* and $\cdot$ is the usual concatenation of string languages.

Next we define some restrictions on semirings.

- A semiring A is *commutative* if, for every $a,b\in A$, the equation $a\odot b=b\odot a$ holds.
- A is *idempotent* if for every $a\in A$, $a\oplus a=a$ holds.
- A is *zero-divisor free* if, for every $a,b\in A$, $a\odot b=0$ implies $a=0$ or $b=0$.
- A is *zero-sum free* if, for every $a,b\in A$, $a\oplus b=0$ implies $a=0$ and $b=0$.
- A is *complete* if it is possible to define the sum for every family $(a_i\mid i\in I)$ of elements of A, where I is an index set, such that the following three conditions are satisfied:
  (i) $\sum_{i\in\emptyset} a_i=0$, $\sum_{i\in\{j\}} a_i=a_j$, $\sum_{i\in\{j,k\}} a_i=a_j\oplus a_k$ for $j\neq k$.
  (ii) $\sum_{j\in J}(\sum_{i\in I_j} a_i)=\sum_{i\in I} a_i$, if $\bigcup_{j\in J} I_j=I$ and $I_l\cap I_k=\emptyset$ for $l\neq k$.
  (iii) $\sum_{i\in I}(c\odot a_i)=c\odot(\sum_{i\in I} a_i)$, $\sum_{i\in I}(a_i\odot c)=(\sum_{i\in I} a_i)\odot c$ for every $c\in A$.

In the following table we summarize the properties of the semirings $\mathbb{B}$, Nat, $Trop$, and $Lang_f(\Delta)$.

	Commutative	Idempotent	Zero-divisor free	Zero-sum free	Complete
$\mathbb{B}$	Yes	Yes	Yes	Yes	Yes
Nat	Yes	No	Yes	Yes	No
$Trop$	Yes	Yes	Yes	Yes	Yes
$Lang_f(\Delta)$	No	Yes	Yes	Yes	No

We will frequently refer to semirings which are idempotent, zero-divisor free and zero-sum free. In order to avoid too long sentences, we call such semirings *izz-semirings*. For example, $\mathbb{B}$, $Trop$, and $Lang_f(\Delta)$ are izz-semirings.

We write the product $a \odot b$ of elements $a, b \in A$ in the form ab and, in order to omit parentheses, we fix that the semiring multiplication has a higher binding priority than the semiring addition. Also, in the sequel A denotes an arbitrary semiring.

For more details and a survey on the relevance of semirings and formal power series to formal languages and automata cf. [19].

2.3. Tree series

A *tree series* (*over* Δ *and* A) is a mapping $\varphi : T_\Delta \to A$. For every $t \in T_\Delta$, the element $\varphi(t) \in A$ is called the *coefficient* of t and it is also denoted by (φ, t). The *set of all tree series over* Δ *and* A is denoted by $A\langle\langle T_\Delta \rangle\rangle$.

Let $\varphi \in A\langle\langle T_\Delta \rangle\rangle$ be a tree series. We call φ *boolean* if, for every $t \in T_\Delta$, $(\varphi, t) \in \{0, 1\}$ holds. If there is an $a \in A$ such that for every $t \in T_\Delta$, we have $(\varphi, t) = a$, then φ is a *constant* and also denoted by $\tilde{a}$. The constants $\tilde{0}$ and $\tilde{1}$ are boolean. The *support of the tree series* φ is defined as the set $supp(\varphi) = \{t \in T_\Delta \mid (\varphi, t) \neq 0\}$. Moreover, φ is *polynomial* if $supp(\varphi)$ is finite and it is called a *singleton* if $supp(\varphi)$ is a singleton. Hence every singleton is polynomial.

Let $a \in A$ and $\varphi \in A\langle\langle T_\Delta \rangle\rangle$. We define the tree series $a\varphi \in A\langle\langle T_\Delta \rangle\rangle$ by letting, for every $t \in T_\Delta$, $((a\varphi), t) = a(\varphi, t)$.

Now we define the addition in $A\langle\langle T_\Delta \rangle\rangle$ in the following way. Let I be an index set and $(\varphi_i \in A\langle\langle T_\Delta \rangle\rangle \mid i \in I)$ a family of tree series. This family is called *locally finite* if for every $t \in T_\Delta$, the set $\{i \in I \mid (\varphi_i, t) \neq 0\}$ is finite. Now, if A is complete or the family of tree series is locally finite, then we can define the sum $\sum_{i \in I} \varphi_i \in A\langle\langle T_\Delta \rangle\rangle$ by $((\sum_{i \in I} \varphi_i), t) = \sum_{i \in I} (\varphi_i, t)$ for every $t \in T_\Delta$. Note that, if the family of tree series is not locally finite, then, by the completeness of A, the infinite sum $\sum_{i \in I} (\varphi_i, t)$ is defined.

It is easy to see that, for every tree series $\varphi : T_\Delta \to A$, the equation $\varphi = \sum_{t \in T_\Delta} (\varphi, t)t$ holds, because the family $((\varphi, t)t \mid t \in T_\Delta)$ of singletons is locally finite.

In fact, $(A\langle\langle T_\Delta \rangle\rangle, \oplus, \tilde{0})$ is acommutative monoid (cf. [19, p. 615]) and if A is complete, then this monoid is also complete. If A is complete, then the properties (i)-(ii) of the definition of completeness of a semiring A carry over to a family $(\varphi_i \in A\langle\langle T_\Delta \rangle\rangle \mid i \in I)$ of tree series as follows:

(i) $\sum_{i \in \emptyset} \varphi_i = \tilde{0}$, $\sum_{i \in \{j\}} \varphi_i = \varphi_j$, $\sum_{i \in \{j,k\}} \varphi_i = \varphi_j \oplus \varphi_k$ for $j \neq k$.

(ii) $\sum_{j \in J} (\sum_{i \in I_j} \varphi_i) = \sum_{i \in I} \varphi_i$, if $\bigcup_{j \in J} I_j = I$ and $I_l \cap I_k = \emptyset$ for $l \neq k$.

We will need the following property as well. Let A be complete or I finite, $a \in A$, $\varphi \in A\langle\langle T_\Delta \rangle\rangle$ and $(a_i \mid i \in I)$ is a family of elements of A. Then

(iii) $\sum_{i \in I} (a\varphi_i) = a(\sum_{i \in I} \varphi_i)$ and $\sum_{i \in I} (a_i \varphi) = (\sum_{i \in I} a_i)\varphi$.

The property can be proved easily from the definitions.

A *tree-to-tree series transformation* (for short: t-ts transformation) is a mapping $\tau : T_\Sigma \to A\langle\langle T_\Delta \rangle\rangle$. τ is boolean (polynomial) if for every $s \in T_\Sigma$, the tree series $\tau(s)$ is boolean (polynomial).

For two tree series transformations $\tau_1 : T_\Sigma \to A\langle\langle T_\Delta \rangle\rangle$ and $\tau_2 : T_\Delta \to A\langle\langle T_\Gamma \rangle\rangle$ such that A is complete or τ_1 is polynomial, we define the *composition* $\tau_1 \tilde{\circ} \tau_2 : T_\Sigma \to A\langle\langle T_\Gamma \rangle\rangle$ of τ_1 and τ_2 as follows. For every $s \in T_\Sigma$, let $\tau_1 \tilde{\circ} \tau_2(s) = \sum_{t \in T_\Delta} (\tau_1(s), t)\tau_2(t)$. Note that the composition of two t-ts transformations τ_1 and τ_2 is only defined if both are over the same semiring A. Also note that $\tilde{\circ}$ is not a composition of relations, which is denoted by $\circ$, cf. Section 2.1.

Let C_1 and C_2 be two t-ts transformation classes over the same semiring A. If A is complete or C_1 consists of polynomial t-ts transformations, then the *composition of* C_1 *and* C_2, denoted by $C_1 \tilde{\circ} C_2$, is the class $\{\tau_1 \tilde{\circ} \tau_2 \mid \tau_i \in C_i, 1 \leqslant i \leqslant 2\}$ of t-ts transformations. For $n \geqslant 1$, the n-fold composition of a class C by itself is denoted by C^n.

We will need a relation which "splits" a tree series over Δ and A into tree pieces and thus can be used to relate tree transformations and t-ts transformations. This is $pick_{A,\Delta} = \{(\varphi, t) \mid \varphi \in A\langle\langle T_\Delta \rangle\rangle$ and $t \in supp(\varphi)\} \subseteq A\langle\langle T_\Delta \rangle\rangle \times T_\Delta$. Note that here (φ, t) stands for a pair and not for the coefficient of t in φ. The class of relations $pick_{A,\Delta}$ for a ranked alphabet Δ is denoted by $PICK_A$.

If C is a class of t-ts transformations of type $T_\Sigma \to A\langle\langle T_\Delta \rangle\rangle$, then $C \circ pick_{A,\Delta}$ is a class of tree transformations of type $T_\Sigma \times T_\Delta$. Hence if C is a class of t-ts transformations over A, then $C \circ PICK_A$ is a class of tree transformations.

2.4. Hierarchies

The main subject of this paper are hierarchies. An *hierarchy* is a family $(C_n \mid n \geqslant 1)$, where, for every $n \geqslant 1$, C_n is a tree transformation class or a t-ts transformation class such that $C_n \subseteq C_{n+1}$. If $C_n \subset C_{n+1}$ for every $n \geqslant 1$, then the hierarchy is *proper*. We will also write an hierarchy in the form $C_1 \subseteq C_2 \subseteq \cdots$ or $C_1 \subset C_2 \subset \cdots$. If the C_n are tree transformation (t-ts transformation) classes, then we call the hierarchy a tree transformation (t-ts transformation) hierarchy. A well-known example of a tree transformation hierarchy is the hierarchy $(TOP^n \mid n \geqslant 1)$, which was shown to be proper in [8].

2.5. Substitution of tree series

Now we define the substitution of tree series.

Definition 2.1. Let $l \geqslant 0$, $\varphi \in A\langle\langle T_\Delta(X_l) \rangle\rangle$, and $\psi_1, \ldots, \psi_l \in A\langle\langle T_\Delta(H) \rangle\rangle$ such that A is complete or $\psi_1, \ldots, \psi_l$ are polynomial. The *substitution of* $(\psi_1, \ldots, \psi_l)$ *into* φ is the tree series $\varphi \leftarrow (\psi_1, \ldots, \psi_l)$ in $A\langle\langle T_\Delta(H) \rangle\rangle$, such that for every $u \in T_\Delta(H)$:

$$(\varphi \leftarrow (\psi_1, \ldots, \psi_l), u) = \sum_{\substack{t \in T_\Delta(X_l) \\ t_1, \ldots, t_l \in T_\Delta(H) \\ u = t[t_1, \ldots, t_l]}} (\varphi, t)(\psi_1, t_1) \ldots (\psi_l, t_l).$$

We note that this is an extension of Definition 2.5 of [9] and Definition 3.1 of [16] because in those papers A was assumed to be complete. However, if $\psi_1, \ldots, \psi_l$ are polynomial, then the completeness of A is not needed, because the sum in the right hand side of the defining equation is finite. On the contrary, if some ψ_i is not polynomial, then completeness is needed. For instance, let $l=1$, $u=\alpha\in\Delta^{(0)}$, $(\varphi,u)\neq 0$, and $supp(\psi_1)$ is infinite, and then the sum is infinite.

Next we give some properties of the tree series substitution which we will use in the forthcoming sections.

Proposition 2.2 (Engelfriet et al. [9, Proposition 2.8]). *Let A be a commutative semiring, $\varphi\in A\langle\langle T_\Delta(X_l)\rangle\rangle$, $\psi_1,\dots,\psi_l\in A\langle\langle T_\Delta(H)\rangle\rangle$, and $a,a_1,\dots,a_l\in A$. If A is complete or $\psi_1,\dots,\psi_l$ are polynomial, then*

$$a\varphi\leftarrow(a_1\psi_1,\dots,a_l\psi_l)=aa_1\dots a_l(\varphi\leftarrow(\psi_1,\dots,\psi_l)).$$

The next property is that $\leftarrow$-substitution distributes over the sums in the substitution.

Proposition 2.3 (Engelfriet et al. [9, Proposition 2.9]). *Let $F=(\varphi_i\in A\langle\langle T_\Delta(X_l)\rangle\rangle\mid i\in I)$ and,*
for every $1\leqslant j\leqslant l$, let $G_j=(\psi_{jk}\in A\langle\langle T_\Delta(H)\rangle\rangle\mid k\in I_j)$ be families of tree series, where $I,I_1,\dots,I_l$ are index sets. If (a) *A is complete or* (b) *$F,G_1,\dots,G_l$ are locally finite and for every $1\leqslant j\leqslant l$ and $1\leqslant k\leqslant I_j$ the tree series ψ_{jk} is polynomial, then*

$$\left(\sum_{i\in I}\varphi_i\right)\leftarrow\left(\sum_{k\in I_1}\psi_{1k},\dots,\sum_{k\in I_l}\psi_{lk}\right)=\sum_{i\in I,k_1\in I_1,\dots,k_l\in I_l}\varphi_i\leftarrow(\psi_{1k_1},\dots,\psi_{lk_l}).$$

Note that in the right-hand side of the above equation the priority of $\leftarrow$ is higher than that of the sum. We will assume this also in the rest of the paper.

The tree series substitution $\leftarrow$ is not associative even for polynomial and boolean tree series, cf. [11, p. 352]. On the other hand, the following associativity-like law for singletons was proved in [9].

Proposition 2.4 (Engelfriet et al. [9, Proposition 2.10]). *Let A be commutative, $a,a_1,\dots,a_k,b_1,\dots,b_l\in A$, $t\in T_\Delta(X_k)$, $u_1,\dots,u_k\in T_\Delta(X_l)$ and $v_1,\dots,v_l\in T_\Delta(H)$.*

Assume that there is a partition $(\{j_{i1},\dots,j_{il_i}\}\mid 1\leqslant i\leqslant k)$ of the set $[l]$ such that, for every $1\leqslant i\leqslant k$, $u_i\in T_\Delta(\{x_{j_{i1}},\dots,x_{j_{il_i}}\})$. Moreover, for every $1\leqslant i\leqslant k$, let $u_i'\in T_\Delta(X_{l_i})$ be such that $u_i=u_i'[x_1\leftarrow x_{j_{i1}},\dots,x_{l_i}\leftarrow x_{j_{il_i}}]$. Then

$$\begin{aligned}&(at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow(b_1v_1,\dots,b_lv_l)\\&\quad=at\leftarrow(a_1u_1'\leftarrow(b_{j_{11}}v_{j_{11}},\dots,b_{j_{1l_1}}v_{j_{1l_1}}),\dots,a_ku_k'\leftarrow(b_{j_{k1}}v_{j_{k1}},\dots,b_{j_{kl_k}}v_{j_{kl_k}})).\end{aligned}$$

In this paper, we need Proposition 2.4 in a more general form in which the singleton tree series $b_1v_1,\dots,b_lv_l$ are substituted by the polynomial ones $\psi_1,\dots\psi_l$. Next we prove this more general statement.

Proposition 2.5. *Let A be commutative, $a,a_1,\dots,a_k\in A$, $t\in T_\Delta(X_k)$, $u_1,\dots,u_k\in T_\Delta(X_l)$ and $\psi_1,\dots,\psi_l\in A\langle\langle T_\Delta(H)\rangle\rangle$ polynomial tree series. Let, for every $1\leqslant i\leqslant l$, $\psi_i=\sum_{v=1}^{m_i}b_{iv}v_{iv}$.*

Assume that there is a partition $(\{j_{i1},\dots,j_{il_i}\}\mid 1\leqslant i\leqslant k)$ of the set $[l]$ such that, for every $1\leqslant i\leqslant k$, $u_i\in T_\Delta(\{x_{j_{i1}},\dots,x_{j_{il_i}}\})$. Moreover, for every $1\leqslant i\leqslant k$, let $u'_i\in T_\Delta(X_{l_i})$ be such that $u_i=u'_i[x_1\leftarrow x_{j_{i1}},\dots,x_{l_i}\leftarrow x_{j_{il_i}}]$. Then

$$
\begin{aligned}
&(at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow(\psi_1,\dots,\psi_l)\\
&\quad= at\leftarrow(a_1u'_1\leftarrow(\psi_{j_{11}},\dots,\psi_{j_{1l_1}}),\dots,a_ku'_k\leftarrow(\psi_{j_{k1}},\dots,\psi_{j_{kl_k}})).
\end{aligned}
$$

(Note that the trees u_i with $1\leqslant i\leqslant k$ are not required to be nondeleting or linear in $\{x_{j_{i1}},\dots,x_{j_{il_i}}\}$, hence u'_i need not be nondeleting or linear in X_{l_i}. Moreover, note that the trees u'_i are not determined uniquely because the members $\{j_{i1},\dots,j_{il_i}\}$ of the partition are unordered. However, the statement is true for every u'_i which satisfies $u_i=u'_i[x_1\leftarrow x_{j_{i1}},\dots,x_{l_i}\leftarrow x_{j_{il_i}}]$.)

Proof.

$$
\begin{aligned}
&(at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow(\psi_1,\dots,\psi_l)\\
&= (at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow\left(\sum_{1\leqslant v\leqslant m_1} b_{1v}v_{1v},\dots,\sum_{1\leqslant v\leqslant m_l} b_{lv}v_{lv}\right)\\
&= \sum_{1\leqslant v_1\leqslant m_1,\dots,1\leqslant v_l\leqslant m_l}(at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow(b_{1v_1}v_{1v_1},\dots,b_{lv_l}v_{lv_l})\\
&\quad\text{(by Proposition 2.3)}\\
&= \sum_{1\leqslant v_1\leqslant m_1,\dots,1\leqslant v_l\leqslant m_l} at\leftarrow(a_1u'_1\leftarrow(b_{j_{11}v_1}v_{j_{11}v_1},\dots,b_{j_{1l_1}v_l}v_{j_{1l_1}v_l}),\dots,a_ku'_k\\
&\quad\leftarrow(b_{j_{k1}v_1}v_{j_{k1}v_1},\dots,b_{j_{kl_k}v_l}v_{j_{kl_k}v_l}))\\
&\quad\text{(by Proposition 2.4)}\\
&= at\leftarrow\left(a_1u'_1\leftarrow\left(\sum_{1\leqslant v\leqslant m_{j_{11}}} b_{j_{11}v}v_{j_{11}v},\dots,\sum_{1\leqslant v\leqslant m_{j_{1l_1}}} b_{j_{1l_1}v}v_{j_{1l_1}v}\right),\dots,\right.\\
&\quad\left. a_ku'_k\leftarrow\left(\sum_{1\leqslant v\leqslant m_{j_{k1}}} b_{j_{k1}v}v_{j_{k1}v},\dots,\sum_{1\leqslant v\leqslant m_{j_{kl_k}}} b_{j_{kl_k}v}v_{j_{kl_k}v}\right)\right)\\
&\quad\text{(by applying Proposition 2.3 twice)}\\
&= at\leftarrow(a_1u'_1\leftarrow(\psi_{j_{11}},\dots,\psi_{j_{1l_1}}),\dots,a_ku'_k\leftarrow(\psi_{j_{k1}},\dots,\psi_{j_{kl_k}}))\qquad\square
\end{aligned}
$$

Corollary 2.6. *Let A be commutative, $t\in T_\Delta(X_k)$, $u_1,\dots,u_k\in T_\Delta(X_l)$ and $\psi_1,\dots,\psi_l\in A\langle\langle T_\Delta(H)\rangle\rangle$ polynomial tree series. Assume that $u_1,\dots,u_k$ are linear in X_l and for every $1\leqslant j\leqslant l$, there is exactly one $1\leqslant i\leqslant k$ such that x_j occurs in u_i. Moreover, let $a,a_1,\dots,a_k\in A$ and, for every $1\leqslant i\leqslant l$, let $\psi_i=\sum_{v=1}^{m_i} b_{iv}v_{iv}$. Then*

$$
\begin{aligned}
&(at\leftarrow(a_1u_1,\dots,a_ku_k))\leftarrow(\psi_1,\dots,\psi_l)\\
&\quad= at\leftarrow(a_1u'_1\leftarrow(\psi_{j_{11}},\dots,\psi_{j_{1l_1}}),\dots,a_ku'_k\leftarrow(\psi_{j_{k1}},\dots,\psi_{j_{kl_k}})),
\end{aligned}
$$

where $lin_{X_l}(u_i)=(u'_i,x_{j_{i1}}\dots x_{j_{il_i}})$.

Proof. Since $(\{j_{i1},\ldots,j_{il_i}\} \mid 1 \leqslant i \leqslant k)$ is a partition of $[l]$ and, for every $1 \leqslant i \leqslant k$, the inclusions $u_i \in T_\Delta(\{x_{j_{i1}},\ldots,x_{j_{il_i}}\})$ and $u_i' \in T_\Delta(X_{l_i})$ hold and $u_i = u_i'[x_1 \leftarrow x_{j_{i1}},\ldots,x_{l_i} \leftarrow x_{j_{il_i}}]$, the statement follows from Proposition 2.5. $\square$

3. Tree series transducers

In this section, we define the concept of tree series transducer. Tree series transducers were introduced in [9] over a complete semiring A. Now we give a more general definition in which the completeness can be replaced by another independent restriction, called polynomial restriction. In the following, let Q be a unary ranked alphabet. Recall that Σ, Δ, and Γ are ranked alphabets and A is a semiring.

Definition 3.1. A *tree representation* (*over* Q, Σ, Δ, *and* A) is a family $\mu = (\mu_k \mid k \geqslant 0)$ of mappings

$$\mu_k : \Sigma^{(k)} \to (A\langle\langle T_\Delta(X)\rangle\rangle)^{Q\times(Q(X_k))^*}$$

such that only for finitely many indices $(q,w) \in Q \times (Q(X_k))^*$, $\mu_k(\sigma)_{q,w} \neq \tilde{0}$. Moreover, for every $(q,w) \in Q \times (Q(X_k))^*$, the membership $\mu_k(\sigma)_{q,w} \in A\langle\langle T_\Delta(X_l)\rangle\rangle$ holds, where $l = |w|$.

A tree representation is

- *polynomial* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, and $w \in (Q(X_k))^*$, the tree series $\mu_k(\sigma)_{q,w}$ is polynomial,
- *bottom-up* if the following additional condition holds. For every $\sigma \in \Sigma^{(k)}$, $q \in Q$, and $w \in (Q(X_k))^*$, if $w \neq q_1(x_1)\ldots q_k(x_k)$ for every $q_1,\ldots,q_k \in Q$, then $\mu_k(\sigma)_{q,w} = \tilde{0}$, and
- *top-down* if the following additional condition holds. For every $\sigma \in \Sigma^{(k)}$, $q \in Q$, and $w \in (Q(X_k))^*$ we have $supp(\mu_k(\sigma)_{q,w}) \subseteq \widehat{T}_\Delta(X_l)$, where $l = |w|$.

Definition 3.2. A *tree series transducer* (*over* A) is a tuple $M = (Q,\Sigma,\Delta,A,Q_d,\mu)$ where Q is a unary ranked alphabet (of states), Σ and Δ are ranked alphabets (of input symbols and output symbols, resp.), A is a semiring, $Q_d \subseteq Q$ (the set of designated states), and μ is a tree representation over Q, Σ, Δ, and A such that A is complete or μ is polynomial.

If μ is polynomial, then we call M *polynomial*.

With every tree series transducer M, we associate a family $(\tau_{M,q} \mid q \in Q)$ of t-ts transformations $\tau_{M,q} : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ and the t-ts transformation $\tau_M : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ computed by M. These are defined as follows.

Definition 3.3. Let $M = (Q,\Sigma,\Delta,A,Q_d,\mu)$ be a tree series transducer.

1. For every $q \in Q$, the t-ts transformation $\tau_{M,q} : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ is defined by induction on its argument as follows:
   - for every $\alpha \in \Sigma^{(0)}$, $\tau_{M,q}(\alpha) = \mu_0(\alpha)_{q,\varepsilon}$ and

- for every $k \geqslant 1$, $\sigma \in \Sigma^{(k)}$, and $s_1, \dots, s_k \in T_\Sigma$,

$$\tau_{M,q}(\sigma(s_1,\dots,s_k)) = \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l}))$$

2. The *t-ts transformation computed by* M is the mapping $\tau_M : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ which is defined by $\tau_M(s) = \sum_{q\in Q_d} \tau_{M,q}(s)$ for every $s \in T_\Sigma$.

This definition is clear in the case that A is complete (cf. Definition 2.1). In the case that A is not complete but μ is polynomial, the definition is correct if the tree series $\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})$ are polynomial. In the next proposition, we prove that the definition is correct also in this latter case.

Proposition 3.4. *For every tree series transducer $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ such that μ is polynomial, state $q \in Q$, and input tree $s \in T_\Sigma$, the tree series $\tau_{M,q}(s)$ is a polynomial tree series.*

Proof. We prove by induction on s. Let $s = \sigma(s_1,\dots,s_k)$. By Definition 3.3,

$$\tau_{M,q}(\sigma(s_1,\dots,s_k)) = \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})).$$

Now $\mu_k(\sigma)_{q,w}$ is polynomial because M is polynomial, and $\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})$ are also polynomial by the induction hypothesis. Hence, by Proposition 2.7 of [9], $\mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l}))$ is polynomial. Finally the sum of finitely many polynomial tree series is also polynomial. □

Let us show now that this definition coincides with Definition 3.5 of [9] in the case that A is complete. There the t-ts transformation which is computed by some tree series transducers was defined on the basis of a Σ-algebra and the corresponding homomorphism from T_Σ to this algebra. Let us recall these two concepts briefly.

Definition 3.5. Let A be complete and μ a tree representation. For every $k \geqslant 0$ and $\sigma \in \Sigma^{(k)}$, μ induces the mapping

$$\overline{\mu_k(\sigma)} : (A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1} \times \cdots \times (A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1} \to (A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1}$$

with k arguments. The mapping $\overline{\mu_k(\sigma)}$ is defined for every $P_1,\dots,P_k \in (A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1}$ and $q \in Q$ as

$$\overline{\mu_k(\sigma)}(P_1,\dots,P_k)_q = \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow ((P_{i_1})_{q_1},\dots,(P_{i_l})_{q_l}).$$

(Note that, for $k = 0$, $(Q(X_k))^* = \{\varepsilon\}$.)

Next we observe that the family of mappings $(\overline{\mu_k(\sigma)} \mid k \geqslant 0, \sigma \in \Sigma^{(k)})$ defines a Σ-algebra and we make explicit the initial homomorphism from T_Σ to that algebra.

Observation 3.6. The system $((A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1}, (\overline{\mu_k(\sigma)} \mid k \geqslant 0, \sigma \in \Sigma^{(k)}))$ is a Σ-algebra. Thus there is a unique homomorphism

$$h_\mu : T_\Sigma \to (A\langle\langle T_\Delta\rangle\rangle)^{Q\times 1}$$

defined as follows: for every $\sigma \in \Sigma^{(k)}$, $s_1, \ldots, s_k \in T_\Sigma$,

$$h_\mu(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}(h_\mu(s_1), \ldots, h_\mu(s_k)).$$

(Note that, for $\sigma \in \Sigma^{(0)}$, $h_\mu(\sigma) = \overline{\mu_0(\sigma)}() = \mu_0(\sigma)$.)

Now we can prove that for every tree series transducer M over a complete semiring A, τ_M of Definition 3.3 is the same as τ_M of Definition 3.5 of [9]. Note that in [9] τ_M was defined by $\tau_M(s) = \sum_{q \in Q_d} h_\mu(s)_q$. Hence, it is sufficient to show that for every input tree $s \in T_\Sigma$ and state $q \in Q$, $h_\mu(s)_q = \tau_{M,q}(s)$ holds.

Proposition 3.7. *For every tree series transducer $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ such that A is complete, and for every $q \in Q$ and $s \in T_\Sigma$, the equality $h_\mu(s)_q = \tau_{M,q}(s)$ holds.*

Proof. Let $s = \sigma(s_1, \ldots, s_k)$ for $k \geqslant 0$. Then

$$\begin{aligned}
&h_\mu(\sigma(s_1, \ldots, s_k))_q \\
&\quad = \overline{\mu_k(\sigma)}(h_\mu(s_1), \ldots, h_\mu(s_k))_q \\
&\qquad \text{(by Observation 3.6)} \\
&\quad = \sum_{w = q_1(x_{i_1}) \ldots q_l(x_{i_l}) \in (Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow (h_\mu(s_{i_1})_{q_1}, \ldots, h_\mu(s_{i_l})_{q_l}) \\
&\qquad \text{(by Definition 3.5)} \\
&\quad = \sum_{w = q_1(x_{i_1}) \ldots q_l(x_{i_l}) \in (Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}), \ldots, \tau_{M,q_l}(s_{i_l})) \\
&\qquad \text{(by induction hypothesis)} \\
&\quad = \tau_{M,q}(\sigma(s_1, \ldots, s_k)) \\
&\qquad \text{(by Definition 3.3).} \qquad \square
\end{aligned}$$

Now we define bottom-up and top-down tree series transducers.

Definition 3.8. A tree series transducer $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ is *bottom-up* if μ is a bottom-up tree representation. In this case the designated states are called final states.

Definition 3.9. A tree series transducer $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ is *top-down* if μ is a top-down tree representation. In this case the designated states are called initial states.

Next we give an example of a bottom-up and of a top-down tree series transducer and the tree series transformation computed by them.

Example 3.10. Let $M=(Q,\Sigma,\Delta,Nat,Q_d,\mu)$, where $Q=Q_d=\{q\}$, $\Sigma=\{\gamma^{(1)},\alpha^{(0)}\}$ and $\Delta=\{\sigma_1^{(2)},\sigma_2^{(2)},\alpha_1^{(0)},\alpha_2^{(0)}\}$. Note that, since $Q=Q_d=\{q\}$, for every input tree $s\in T_\Sigma$, $\tau_M(s)=\tau_{M,q}(s)$.

(a) We define M to be a bottom-up tree series transducer by giving the tree representation μ in the following way:

$$\begin{aligned}\mu_0(\alpha)_{q,\varepsilon} &= \alpha_1+2\alpha_2,\\ \mu_1(\gamma)_{q,q(x_1)} &= \sigma_1(x_1,x_1)+2\sigma_2(x_1,x_1).\end{aligned}$$

Intuitively, the cost of writing a symbol of index 1 to the output part is 1 while that of writing a symbol of rank 2 is 2. We demonstrate this by computing the value of $\tau_{M,q}$ on the input tree $\gamma(\alpha)$.

$$\begin{aligned}\tau_{M,q}(\gamma(\alpha)) &= \mu_1(\gamma)_{q,q(x_1)} \leftarrow (\tau_{M,q}(\alpha))\\ &= (\sigma_1(x_1,x_1)+2\sigma_2(x_1,x_1)) \leftarrow (\alpha_1+2\alpha_2)\\ &= (\sigma_1(x_1,x_1)\leftarrow\alpha_1)+(\sigma_1(x_1,x_1)\leftarrow 2\alpha_2)+(2\sigma_2(x_1,x_1)\leftarrow\alpha_1)\\ &\quad +(2\sigma_2(x_1,x_1)\leftarrow 2\alpha_2)\\ &\quad \text{(by Proposition 2.3)}\\ &= \sigma_1(\alpha_1,\alpha_1)+2\sigma_1(\alpha_2,\alpha_2)+2\sigma_2(\alpha_1,\alpha_1)+4\sigma_2(\alpha_2,\alpha_2).\end{aligned}$$

(b) Now we define the tree representation μ in the way that M becomes a top-down tree series transducer.

$$\begin{aligned}\mu_0(\alpha)_{q,\varepsilon} &= \alpha_1+2\alpha_2,\\ \mu_1(\gamma)_{q,q(x_1)q(x_1)} &= \sigma_1(x_1,x_2)+2\sigma_2(x_1,x_2).\end{aligned}$$

The coefficients 1 and 2 in the definition of μ have the same meaning as in the bottom-up case. Again, we compute $\tau_{M,q}(\gamma(\alpha))$.

$$\begin{aligned}\tau_{M,q}(\gamma(\alpha)) &= \mu_1(\gamma)_{q,q(x_1)q(x_1)} \leftarrow (\tau_{M,q}(\alpha),\tau_{M,q}(\alpha))\\ &= (\sigma_1(x_1,x_2)+2\sigma_2(x_1,x_2)) \leftarrow (\alpha_1+2\alpha_2,\alpha_1+2\alpha_2)\\ &= (\sigma_1(x_1,x_2)\leftarrow(\alpha_1,\alpha_1))+(\sigma_1(x_1,x_2)\leftarrow(\alpha_1,2\alpha_2))\\ &\quad +\cdots+(2\sigma_2(x_1,x_2)\leftarrow(2\alpha_2,2\alpha_2))\\ &\quad \text{(by Proposition 2.3)}\\ &= \sigma_1(\alpha_1,\alpha_1)+2\sigma_1(\alpha_1,\alpha_2)+2\sigma_1(\alpha_2,\alpha_1)+4\sigma_1(\alpha_2,\alpha_2)\\ &\quad +2\sigma_2(\alpha_1,\alpha_1)+4\sigma_2(\alpha_1,\alpha_2)+4\sigma_2(\alpha_2,\alpha_1)+8\sigma_2(\alpha_2,\alpha_2).\end{aligned}$$

In both cases (a) and (b), M is polynomial. The difference between the two results is due to the difference between the bottom-up and the top-down ways of computation. In fact, a bottom-up tree (series) transducer first computes the translation t' of a subtree s' of the input tree s and then makes copies of the resulting t' in the output. On the contrary, a top-down tree (series) transducer first makes some copies $s'_1, \dots, s'_n$ of s' and then translates them into output subtrees $t'_1, \dots, t'_n$.

Next we define some restricted versions of bottom-up and of top-down tree series transducers. These are the same restrictions which are imposed on tree series transducers in [9] and on transducers (except for 1) in [1,6,17]. We begin with the bottom-up case.

Definition 3.11. Let $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ be a bottom-up tree series transducer:

1. M is *boolean* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w = q_1(x_1) \dots q_k(x_k) \in (Q(X_k))^*$ the tree series $\mu_k(\sigma)_{q,w}$ is boolean.
2. M is *deterministic* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$ and sequence $w = q_1(x_1) \dots q_k(x_k) \in (Q(X_k))^*$, there is at most one $q \in Q$ such that $\mu_k(\sigma)_{q,w} \neq \tilde{0}$, and if $\mu_k(\sigma)_{q,w} \neq \tilde{0}$ then $\mu_k(\sigma)_{q,w}$ is a singleton.
3. M is *total* if $Q_d = Q$ and for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$ and sequence $w = q_1(x_1) \dots q_k(x_k) \in (Q(X_k))^*$, there is at least one $q \in Q$ such that $\mu_k(\sigma)_{q,w} \neq \tilde{0}$.
4. M is a *bottom-up homomorphism tree series transducer* if it is total deterministic and the set Q of states is a singleton.
5. M is *linear* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w = q_1(x_1) \dots q_k(x_k) \in (Q(X_k))^*$ and $t \in supp(\mu_k(\sigma)_{q,w})$, the tree t is linear in X_k.
6. M is a *bottom-up finite state relabeling tree series transducer* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w = q_1(x_1) \dots q_k(x_k) \in (Q(X_k))^*$ and $t \in supp(\mu_k(\sigma)_{q,w})$ the tree t has the form $\delta(x_1, \dots, x_k)$ for some $\delta \in \Delta^{(k)}$. □

Note that all bottom-up tree series transducers over $\mathbb{B}$ are boolean. Moreover, deterministic bottom-up tree series transducers are polynomial, hence bottom-up homomorphism tree series transducers are polynomial. Also bottom-up finite state relabeling tree series transducers are polynomial.

The *class of t-ts transformations computed by bottom-up tree series transducers over a semiring A* is denoted by $BOT_{t-ts}(A)$. The classes of t-ts transformations which correspond to the syntactic subclasses 1–6 of Definition 3.11 and to the polynomial case are denoted by x-$BOT_{t-ts}(A)$ where $x \in \{b, d, t, h, l, r, p\}$, resp. If more than one syntactic restriction holds, then $BOT_{t-ts}(A)$ is prefixed by the string of corresponding letters, e.g. dtl-$BOT_{t-ts}(A)$ denotes the class of t-ts transformations induced by deterministic, total, and linear bottom-up tree series transducers over A.

Now we introduce the corresponding restrictions for top-down tree series transducers.

Definition 3.12. Let $M = (Q, \Sigma, \Delta, A, Q_d, \mu)$ be a top-down tree series transducer:

1. M is *boolean* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w \in (Q(X_k))^*$ the tree series $\mu_k(\sigma)_{q,w}$ is boolean.
2. M is *deterministic* if the following conditions hold. Q_d is a singleton. For every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, there is at most one $w \in (Q(X_k))^*$ such that $\mu_k(\sigma)_{q,w} \neq \tilde{0}$, and if $\mu_k(\sigma)_{q,w} \neq \tilde{0}$ then $\mu_k(\sigma)_{q,w}$ is a singleton.
3. M is *total* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, there is at least one $w \in (Q(X_k))^*$ such that $\mu_k(\sigma)_{q,w} \neq \tilde{0}$, and $Q_d \neq \emptyset$.

4. M is a *top-down homomorphism tree series transducer* if it is total deterministic and the set Q of states is a singleton.
5. M is *linear* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w \in (Q(X_k))^*$ such that $\mu_k(\sigma)_{q,w} \neq \tilde{0}$, the string w is linear in X_k.
6. M is a *top-down finite state relabeling tree series transducer* if for every $k \geqslant 0$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, $w \in (Q(X_k))^*$, if $\mu_k(\sigma)_{q,w} \neq \tilde{0}$, then $w = q_1(x_1) \ldots q_k(x_k)$ and every $t \in supp(\mu_k(\sigma)_{q,w})$ has the form $\delta(x_1, \ldots, x_k)$ for some $\delta \in \Delta^{(k)}$.

Again like in the bottom-up case, all top-down tree series transducers over $\mathbb{B}$ are boolean. Moreover, deterministic top-down tree series transducers are polynomial, hence top-down homomorphism tree series transducers are polynomial. Also, top-down finite state relabeling tree series transducers are polynomial.

The *class of t-ts transformations computed by top-down tree series transducers over* A is denoted by $TOP_{t-ts}(A)$. The classes of t-ts transformations which correspond to the syntactic subclasses 1–6 of Definition 3.12 and to the polynomial case are denoted by x-$TOP_{t-ts}(A)$ where $x \in \{b, d, t, h, l, r, p\}$, resp. For combinations of syntactic restrictions we use a notation similar to that of the bottom-up case.

We use two more notations concerning t-ts transformations. In [9, Proposition 3.11], it was shown that bottom-up and top-down finite state relabeling t-ts transformations coincide, that is, for every semiring A, we have r-$BOT_{t-ts}(A) = r$-$TOP_{t-ts}(A)$. Thus, we can denote both classes by the symbol $QREL_{t-ts}(A)$.

Moreover, in [9, Corollary 4.15] it was shown that bottom-up and top-down boolean homomorphism t-ts transformations coincide, that is, for every semiring A, the equation bh-$BOT_{t-ts}(A) = bh$-$TOP_{t-ts}(A)$ holds. Thus, we denote both classes by the same symbol b-$HOM_{t-ts}(A)$.

We finish this section by showing a property which we will need later.

Proposition 3.13. *For every boolean homomorphism tree series transducer* $M = (\{*\}, \Sigma, \Delta, \{*\}, A, \mu)$ *and input tree* $s \in T_\Sigma$, *the tree series* $\tau_{M,*}(s)$ *is a singleton with coefficient* 1.

Proof. The proof is similar to that of Proposition 3.4. $\square$

4. The tree transformation hierarchies

In this section, we recall and summarize the hierarchy results concerning tree transformation classes which we will generalize to t-ts transformations over a semiring A with certain properties in the next two sections. In fact we not only summarize but we combine these hierarchy results into one inclusion diagram.

An inclusion diagram is a Hasse diagram of the considered tree transformation classes with respect to the partial order $\subseteq$. Hence, all inclusions shown by an inclusion diagram are strict and the unrelated classes in the diagram are incomparable with respect to $\subseteq$.

For more details and an intensive discussion of inclusion diagrams the reader is referred to Section 2.2 of [15] (also cf. [14]).

The hierarchy results transpire from different works, mainly from [1,6,8,17]. The inclusion diagram was shown in [22]; however, he did not give a proof. Therefore, we reconstruct the proof. For the notation TOP_{tt}, BOT_{tt} and the nth power of a tree transformation class see Section 2.1.

Proposition 4.1. *For every* $n \geqslant 1$, *the diagram in Fig.* 1 *is the inclusion diagram of the tree transformation classes* BOT_{tt}^n, TOP_{tt}^n, BOT_{tt}^{n+1}, *and* TOP_{tt}^{n+1}.

Proof. It is obvious that, for every $n \geqslant 1$, $TOP_{tt}^n \subseteq TOP_{tt}^{n+1}$ and $BOT_{tt}^n \subseteq BOT_{tt}^{n+1}$. Moreover, it was shown in [1, Theorem 13] that, for every $n \geqslant 1$, $TOP_{tt}^n \subseteq BOT_{tt}^{n+1}$ and $BOT_{tt}^n \subseteq TOP_{tt}^{n+1}$.

Then, in [8, Theorem 3.14], it was shown that the top-down tree transformation hierarchy is proper, i.e., that $TOP_{tt}^n \subset TOP_{tt}^{n+1}$ for every $n \geqslant 1$. In fact, he proved a stronger statement namely that the hierarchy of tree transformation languages for top-down tree transducers is proper, however, we do not consider tree transformation languages in this paper.

In order to show the properness of the bottom-up tree transformation hierarchy, we need some preparations. Based on the results of [8], it was obtained in Corollary 8.12 of Chapter IV of [17] that for every $n \geqslant 1$, $TOP_{tt}^n(REC) \subset BOT_{tt}^{n+1}(REC) \subset TOP_{tt}^{n+1}(REC)$.

On the other hand, the inclusion $BOT_{tt}(REC) \subset TOP_{tt}(REC)$ also holds, which can be seen as follows. By (4,6) of Section 6 in [6], $BOT_{tt} = l\text{-}TOP_{tt} \circ HOM_{tt}$, by [24], $l\text{-}TOP_{tt}(REC) = REC$, hence $BOT_{tt}(REC) = HOM_{tt}(REC)$. Moreover, by Theorem 3.2.5 of [10], $HOM_{tt}(REC) \subset TOP_{tt}(REC)$, which verifies that $BOT_{tt}(REC) \subset TOP_{tt}(REC)$.

Hence the following stronger statement is also true:

$$\text{For every } n \geqslant 1,\ BOT_{tt}^n(REC) \subset TOP_{tt}^n(REC) \subset BOT_{tt}^{n+1}(REC). \qquad (*)$$

From the above it easily follows that the bottom-up tree transformation hierarchy is also proper, i.e., that $BOT_{tt}^n \subset BOT_{tt}^{n+1}$ for every $n \geqslant 1$. (The proof is by contradiction as follows. Assume $BOT_{tt}^n = BOT_{tt}^{n+1}$ for some $n \geqslant 1$, then $BOT_{tt}^n(REC) = BOT_{tt}^{n+1}(REC)$, which contradicts to $(*)$.)

Now to show that the diagram in Fig. 1 is an inclusion diagram, it is sufficient to prove that the classes BOT_{tt}^n and TOP_{tt}^n are incomparable with respect to inclusion for every $n \geqslant 1$. We can prove again by contradiction.

(a) Assume that $TOP_{tt}^n \subseteq BOT_{tt}^n$ for some $n \geqslant 1$. Then, certainly, $TOP_{tt}^n(REC) \subseteq BOT_{tt}^n(REC)$ which contradicts to $(*)$.

(b) Now assume that $BOT_{tt}^n \subseteq TOP_{tt}^n$ for some $n \geqslant 1$. Then we can infer as follows:

$$\begin{aligned} & BOT_{tt}^n \subseteq TOP_{tt}^n \\ \Rightarrow\ & (l\text{-}TOP_{tt} \circ HOM_{tt})^n \subseteq TOP_{tt}^n \\ & \quad (\text{since, by } (4,6) \text{ of Section } 6 \text{ in } [6],\ BOT_{tt} = l\text{-}TOP_{tt} \circ HOM_{tt}) \end{aligned}$$

$\Rightarrow (HOM_{tt} \circ l\text{-}TOP_{tt})^n \circ HOM_{tt} \subseteq HOM_{tt} \circ TOP_{tt}^n$
(composition with HOM_{tt} from the left and using the associativity of $\circ$)
$\Rightarrow TOP_{tt}^n \circ HOM_{tt} \subseteq HOM_{tt} \circ TOP_{tt}^n$
(since, by (3) of Section 6 in [6], $TOP_{tt} = HOM_{tt} \circ l\text{-}TOP_{tt}$)
$\Rightarrow TOP_{tt}^n \circ HOM_{tt} \subseteq TOP_{tt}^n$
(since, by Lemma 3.11 of Chapter IV of [17], $HOM_{tt} \circ TOP_{tt} \subseteq TOP_{tt}$)
$\Rightarrow TOP_{tt}^n \circ HOM_{tt} \circ l\text{-}TOP_{tt} \subseteq TOP_{tt}^n \circ l\text{-}TOP_{tt}$
$\Rightarrow TOP_{tt}^{n+1} \subseteq (HOM_{tt} \circ l\text{-}TOP_{tt})^n \circ l\text{-}TOP_{tt}$
(since, by (3) of Section 6 in [6], $TOP_{tt} = HOM_{tt} \circ l\text{-}TOP_{tt}$)
$\Rightarrow TOP_{tt}^{n+1} \subseteq (HOM_{tt} \circ l\text{-}BOT_{tt})^n \circ l\text{-}BOT_{tt}$
(since, by (3) of Section 6 in [6], $l\text{-}TOP_{tt} \subseteq l\text{-}BOT_{tt}$)
$\Rightarrow TOP_{tt}^{n+1} \subseteq HOM_{tt} \circ (l\text{-}BOT_{tt} \circ HOM_{tt})^{n-1} \circ l\text{-}BOT_{tt} \circ l\text{-}BOT_{tt}$
(by the associativity of $\circ$)
$\Rightarrow TOP_{tt}^{n+1} \subseteq HOM_{tt} \circ BOT_{tt}^{n-1} \circ l\text{-}BOT_{tt} \circ l\text{-}BOT_{tt}$
(since, by (4) on p. 229 of [6], $BOT_{tt} = l\text{-}BOT_{tt} \circ HOM_{tt}$)
$\Rightarrow TOP_{tt}^{n+1} \subseteq HOM_{tt} \circ BOT_{tt}^{n-1} \circ l\text{-}BOT_{tt}$
(since, by Theorem 4.5(2) of [6], $l\text{-}BOT_{tt} = l\text{-}BOT_{tt} \circ l\text{-}BOT_{tt}$)
$\Rightarrow TOP_{tt}^{n+1} \subseteq BOT_{tt}^{n+1}$

Obviously, this again contradicts to $(*)$. $\square$

We observe that the diagram in Fig. 1 contains four fundamental tree transformation hierarchies. Now we list them and give names to them in order to be able to speak about them more easily:

$TOP_{tt} \subseteq TOP_{tt}^2 \subseteq TOP_{tt}^3 \subseteq \cdots$ is the *uniform top-down tree transformation hierarchy*,
$TOP_{tt} \subseteq BOT_{tt}^2 \subseteq TOP_{tt}^3 \subseteq \cdots$ is the *alternating top-down tree transformation hierarchy*,
$BOT_{tt} \subseteq BOT_{tt}^2 \subseteq BOT_{tt}^3 \subseteq \cdots$ is the *uniform bottom-up tree transformation hierarchy*,
$BOT_{tt} \subseteq TOP_{tt}^2 \subseteq BOT_{tt}^3 \subseteq \cdots$ is the *alternating bottom-up tree transformation hierarchy*.

We saw that each of them is proper.

In the next sections we will generalize (or: lift up) the inclusion diagram in Fig. 1 to the level of polynomial t-ts transformations over a commutative izz-semiring A. We call the so-obtained hierarchies the uniform polynomial top-down t-ts transformation hierarchy over A, alternating polynomial top-down t-ts transformation hierarchy over A, and so on. We also show that the uniform polynomial top-down t-ts transformation hierarchy and the uniform polynomial bottom-up t-ts transformation hierarchy are proper over an (not necessarily commutative) izz-semiring A.

5. The alternating bottom-up and top-down t-ts transformation hierarchies

In this section, we generalize the alternating bottom-up and top-down tree transformation hierarchies to polynomial t-ts transformation hierarchies over a commutative semiring A.

First we introduce a notation. Let $C(A)$ be a class of bottom-up or top-down t-ts transformation classes over a semiring A and $n \geqslant 1$ an integer. From now on the n-fold composition $C(A)^n$ of $C(A)$ by itself (cf. Section 2.3) will be written as $C^n(A)$.

The generalization amounts to prove that for every commutative semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$. Note that this is the generalization of the inclusions $TOP^n_{tt} \subseteq BOT^{n+1}_{tt}$ and $BOT^n_{tt} \subseteq TOP^{n+1}_{tt}$, which were proved in [1, Theorem 13].

We begin with showing the first inclusion.

Theorem 5.1. *For every commutative semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$.*

Proof. We prove by induction on n. We only prove the induction step because its proof includes the proof of the induction base $n = 1$.

$$
\begin{aligned}
& p\text{-}TOP^n_{t-ts}(A) \\
&\quad = p\text{-}TOP^{n-1}_{t-ts}(A) \,\tilde{\circ}\, p\text{-}TOP_{t-ts}(A) \\
&\quad \subseteq p\text{-}TOP^{n-1}_{t-ts}(A) \,\tilde{\circ}\, b\text{-}HOM_{t-ts}(A) \,\tilde{\circ}\, l\,p\text{-}TOP_{t-ts}(A) \\
&\qquad \text{(by Lemma 5.9 of [9])} \\
&\quad \subseteq p\text{-}BOT^n_{t-ts}(A) \,\tilde{\circ}\, b\text{-}HOM_{t-ts}(A) \,\tilde{\circ}\, l\,p\text{-}TOP_{t-ts}(A) \\
&\qquad \text{(by I.H)} \\
&\quad \subseteq p\text{-}BOT^n_{t-ts}(A) \,\tilde{\circ}\, l\,p\text{-}TOP_{t-ts}(A) \\
&\qquad \text{(by Corollary 5.5(2) of [9], note that the lemma holds for an arbitrary} \\
&\qquad \text{commutative semiring } A \text{ which is not necessarily } \omega\text{-continuous as was} \\
&\qquad \text{required in [9])} \\
&\quad \subseteq p\text{-}BOT^n_{t-ts}(A) \,\tilde{\circ}\, l\,p\text{-}BOT_{t-ts}(A) \\
&\qquad \text{(by Theorem 5.14(2) of [16], with the above note)} \\
&\quad \subseteq p\text{-}BOT^{n+1}_{t-ts}(A).
\end{aligned}
$$

In fact, Theorem 5.14 (2) of [16] only states that $l\text{-}TOP_{t-ts}(A)\subseteq l\text{-}BOT_{t-ts}(A)$; however, it is not hard to check that the property polynomial is preserved under the construction shown in that theorem. □

We note that Theorem 5.1 also holds for tree series transducers which are not polynomial, assuming that the semiring A is complete.

Next we generalize the inclusion $BOT_{tt}^{n}\subseteq TOP_{tt}^{n+1}$ to bottom-up and top-down t-ts transformation classes over a commutative semiring A. We need a rather long preparation due to the following facts. We need and therefore prove that, for every commutative semiring A, the inclusion $b\text{-}HOM_{t-ts}(A)\,\tilde{\circ}\,p\text{-}TOP_{t-ts}(A)\subseteq p\text{-}TOP_{t-ts}(A)$ holds. In fact we already obtained a similar composition result in [9, Lemma 5.17] where we proved that for every commutative semiring A, the inclusion $bdt\text{-}TOP_{t-ts}(A)\,\tilde{\circ}\,d\text{-}TOP_{t-ts}(A)\subseteq d\text{-}TOP_{t-ts}(A)$ holds. The composition result we need now is, on the one hand, more special because in the first component we restrict to a boolean homomorphism, which is a special boolean, total and deterministic top-down tree series transducer. On the other hand it is more general because in the second component of the composition we allow to appear a polynomial top-down tree series transducer rather than a deterministic one. Nevertheless, we follow the line of the proof of Lemma 5.17 [9].

First we define the composition of a boolean homomorphism tree series transducer and a polynomial top-down tree series transducer. In order to do so, we need the following auxiliary concept.

Definition 5.2. Let Q_1 and Q_2 be unary ranked alphabets (e.g. state sets of top-down tree series transducers), $k\geqslant 0$, $w\in(Q_1(X_k))^*$ with $|w|=l$ and let $u\in(Q_2(X_l))^*$. Then $u\langle w\rangle$ is the word in $(Q(X_k))^*$, where $Q=Q_2\times Q_1$, which is obtained from u by replacing, for every $q\in Q_2$ and $x_i\in X_l$, the expression $q(x_i)$ by $\langle q,p\rangle(x_j)$ where $p(x_j)$ is the ith letter in w. (In more detail, if $w=p_1(x_{i_1})\ldots p_l(x_{i_l})$, where $p_1,\ldots,p_l\in Q_1$ and $x_{i_1},\ldots,x_{i_l}\in X_k$, and $u=q_1(x_{j_1})\ldots q_m(x_{j_m})$, where $q_1,\ldots,q_m\in Q_2$ and $x_{j_1},\ldots,x_{j_m}\in X_l$, then $u\langle w\rangle=\langle q_1,p_{j_1}\rangle(x_{i_{j_1}})\ldots\langle q_m,p_{j_m}\rangle(x_{i_{j_m}})$. In case $k=0$, $(Q_1(X_k))^*=\{\varepsilon\}$ hence $w=u=u\langle w\rangle=\varepsilon$. If Q_1 is a singleton, i.e. $Q_1=\{*\}$, then we identify $u\langle w\rangle$ with $q_1(x_{i_{j_1}})\ldots q_m(x_{i_{j_m}})$.)

For instance, if $w=p_1(x_2)p_2(x_1)p_3(x_2)p_4(x_1)$ and $u=q_1(x_3)q_2(x_1)q_3(x_1)q_4(x_4)$, then

$$u\langle w\rangle=\langle q_1,p_3\rangle(x_2)\langle q_2,p_1\rangle(x_2)\langle q_3,p_1\rangle(x_2)\langle q_4,p_4\rangle(x_1).$$

As another example, if $Q_1=\{*\}$ and $w=*(x_2)*(x_1)*(x_2)*(x_1)$, then $u\langle w\rangle=q_1(x_2)q_2(x_2)q_3(x_2)q_4(x_1)$.

Definition 5.3. Let $M_1=(\{*\},\Sigma,\Delta,A,*,\mu_1)$ be a boolean top-down homomorphism tree series transducer and $M_2=(Q,\Delta,\Gamma,A,Q_d,\mu_2)$ a polynomial top-down tree series transducer. The *composition of* M_1 *and* M_2 is the top-down tree series transducer $M=(Q,\Sigma,\Gamma,A,Q_d,\mu)$ defined as follows.

Let $mx=\max\{l\mid$ there are $k\geqslant 0$, $\sigma\in\Sigma^{(k)}$, $t\in\widehat{T_\Delta}(X_l)$ such that $t\in supp((\mu_1)_k(\sigma)_{*,w})$ where $w\in(\{*\}(X_k))^*\}$. We extend μ_2 to μ_2' by letting $\mu_2'=((\mu_2')_0,(\mu_2)_k\mid k\geqslant 1)$, where

$(\mu_2')_0 : (\Delta^{(0)} \cup X_{mx}) \to A\langle\langle T_{\Gamma \cup Q(X_{mx})}\rangle\rangle^{Q \times \{\varepsilon\}}$. The mapping $(\mu_2')_0$ is defined such that, for every $\alpha \in \Delta^{(0)}$, $(\mu_2')_0(\alpha) = (\mu_2)_0(\alpha)$ and, for every $x_i \in X_{mx}$ and $q \in Q$, $(\mu_2')_0(x_i)_{q,\varepsilon} = q(x_i)$.

In this extension the elements of X_{mx} and of $Q(X_{mx})$ are considered as 0-ary symbols, therefore μ_2' is a top-down tree representation over Q, $\Delta \cup X_{mx}$, $\Gamma \cup Q(X_{mx})$, and A. Let $M_2' = (Q, \Delta \cup X_{mx}, \Gamma \cup Q(X_{mx}), A, Q_d, \mu_2')$. Note that M_2' is polynomial.

Now we can construct M as follows. The mapping μ is defined in the following way:

For every $\sigma \in \Sigma^{(k)}$ with $k \geqslant 0$,
  for every $w \in (\{*\}(X_k))^*$ with $l = |w|$,
    if $(\mu_1)_k(\sigma)_{*,w} = t$ for a $t \in \widehat{T_\Delta}(X_l)$,
    then { for every $q \in Q$,
      if $\tau_{M_2',q}(t) = a_1\hat{t}_1 \oplus \cdots \oplus a_r\hat{t}_r$ for some $a_1, \ldots, a_r \in A - \{0\}$
      and $\hat{t}_1, \ldots, \hat{t}_r \in T_\Gamma(Q(X_l))$, (by Proposition 3.4, $\tau_{M_2',q}(t)$ is polynomial)
      then define, for every $1 \leqslant j \leqslant r$,
        $\mu_k(\sigma)_{q,v_j} = \sum_{1 \leqslant i \leqslant r, v_i = v_j} a_i t_i$,
        where, for every $1 \leqslant \kappa \leqslant r$, $lin_{Q(X_l)}(\hat{t}_\kappa) = (t_\kappa, u_\kappa)$, $t_\kappa \in \widehat{T_\Gamma}(X_{m_\kappa})$,
        $u_\kappa \in (Q(X_l))^*$, $|u_\kappa| = m_\kappa$ and $v_\kappa = u_\kappa\langle w\rangle$ }.

Moreover, for every $\sigma \in \Sigma^{(k)}$ with $k \geqslant 0$, $q \in Q$ and $v \in (Q(X_k))^*$ not defined by the above conditions, let $\mu_k(\sigma)_{q,v} = \tilde{0}$.

Proposition 5.4. *The composition of a boolean top-down homomorphism tree series transducer $M_1 = (\{*\}, \Sigma, \Delta, A, *, \mu_1)$ and polynomial top-down tree series transducer $M_2 = (Q, \Delta, \Gamma, A, Q_d, \mu_2)$ is polynomial.*

Proof. Let $M = (Q, \Sigma, \Gamma, A, Q_d, \mu)$ be the composition of M_1 and M_2. By Definition 5.3, for every $\sigma \in \Sigma^{(k)}$ with $k \geqslant 0$, $q \in Q$ and $v \in (Q(X_k))^*$, $\mu_k(\sigma)_{q,v}$ is $\tilde{0}$ or polynomial. □

Now the most difficult part of the preparation follows. We prove that the composition of a boolean homomorphism tree series transducer M_1 and a polynomial top-down tree series transducer M_2 induces the composition of the t-ts transformations τ_{M_1} and τ_{M_2}.

Lemma 5.5 (cf. Englefriet et al. [9, Lemma 5.17]). *Let A be commutative. Let $M_1 = (\{*\}, \Sigma, \Delta, A, *, \mu_1)$ be a boolean top-down homomorphism tree series transducer, $M_2 = (Q, \Delta, \Gamma, A, Q_d, \mu_2)$ a polynomial top-down tree series transducer, and $M = (Q, \Sigma, \Gamma, A, Q_d, \mu)$ the composition of M_1 and M_2. Then $\tau_M = \tau_{M_1} \tilde{\circ}\, \tau_{M_2}$.*

Proof.

$$\begin{aligned}
&\tau_M(s)\\
&= \sum_{q \in Q_d} \tau_{M,q}(s)\\
&=^+ \sum_{q \in Q_d} \tau_{M_1,*} \tilde{\circ}\, \tau_{M_2,q}(s)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q\in Q_d} \Big(\sum_{t\in T_\Delta} (\tau_{M_1,*}(s),t)\tau_{M_2,q}(t) \Big) \\
&= \sum_{t\in T_\Delta} (\tau_{M_1,*}(s),t) \Big(\sum_{q\in Q_d} \tau_{M_2,q}(t) \Big) \\
&= \sum_{t\in T_\Delta} (\tau_{M_1,*}(s),t)\tau_{M_2}(t) \\
&= \tau_{M_1,*} \,\tilde{\circ}\, \tau_{M_2}(s) \\
&= \tau_{M_1} \,\tilde{\circ}\, \tau_{M_2}(s).
\end{aligned}
$$

Hence, it is sufficient to show that the equation at + holds, i.e., for every $s\in T_\Sigma$ and $q\in Q$ the equation $\tau_{M,q}(s) = \tau_{M_1,*} \,\tilde{\circ}\, \tau_{M_2,q}(s)$ holds. We prove this by induction on s. Let $s=\sigma(s_1,\dots,s_k)$:

$$
\begin{aligned}
&\tau_{M,q}(s) \\
&= \sum_{v=q_1(x_{e_1})\dots q_m(x_{e_m})} \mu_k(\sigma)_{q,v} \leftarrow (\tau_{M,q_1}(s_{e_1}),\dots,\tau_{M,q_m}(s_{e_m})) \\
&= \sum_{\substack{v=q_1(x_{e_1})\dots q_m(x_{e_m}) \\ \mu_k(\sigma)_{q,v}\neq\tilde{0}}} \mu_k(\sigma)_{q,v} \leftarrow (\tau_{M,q_1}(s_{e_1}),\dots,\tau_{M,q_m}(s_{e_m})) \\
&= \sum_{\substack{v=q_{j,1}(x_{i_{\kappa_{j,1}}})\dots q_{j,m_j}(x_{i_{\kappa_{j,m_j}}}) \\ v\in\{u_1\langle w\rangle,\dots,u_r\langle w\rangle\}}} \mu_k(\sigma)_{q,v} \leftarrow (\tau_{M,q_{j,1}}(s_{i_{\kappa_{j,1}}}),\dots,\tau_{M,q_{j,m_j}}(s_{i_{\kappa_{j,m_j}}}))
\end{aligned}
$$

(where $w = *(x_{i_1})\dots *(x_{i_l}) \in (\{*\}X_k)^*$ such that $(\mu_1)_k(\sigma)_{*,w} = t$ for $a\ t \in \widehat{T_\Delta}(X_l)$, $\tau_{M_2',p}(t) = a_1\hat{t}_1 \oplus\cdots\oplus a_r\hat{t}_r$ for some $a_1,\dots,a_r \in A-\{0\}$ and $\hat{t}_1,\dots,\hat{t}_r \in T_\Gamma(Q(X_l))$, for every $1\leqslant j\leqslant r$, $lin_{Q(X_l)}(\hat{t}_j) = (t_j,u_j)$, $t_j \in \widehat{T_\Gamma}(X_{m_j})$, $u_j \in (Q(X_l))^*$ with $u_j = q_{j,1}(x_{\kappa_{j,1}})\dots q_{j,m_j}(x_{\kappa_{j,m_j}})$)

$$
\begin{aligned}
&= \sum_{\substack{v=q_{j,1}(x_{i_{\kappa_{j,1}}})\dots q_{j,m_j}(x_{i_{\kappa_{j,m_j}}}) \\ v\in\{u_1\langle w\rangle,\dots,u_r\langle w\rangle\}}} \left(\sum_{\substack{1\leqslant j\leqslant r \\ v=u_j\langle w\rangle}} a_j t_j \right) \leftarrow (\tau_{M,q_{j,1}}(s_{i_{\kappa_{j,1}}}),\dots,\tau_{M,q_{j,m_j}}(s_{i_{\kappa_{j,m_j}}})) \\
&= \sum_{\substack{v=q_{j,1}(x_{i_{\kappa_{j,1}}})\dots q_{j,m_j}(x_{i_{\kappa_{j,m_j}}}) \\ v\in\{u_1\langle w\rangle,\dots,u_r\langle w\rangle\}}} \left(\sum_{\substack{1\leqslant j\leqslant r \\ v=u_j\langle w\rangle}} a_j t_j \leftarrow (\tau_{M,q_{j,1}}(s_{i_{\kappa_{j,1}}}),\dots,\tau_{M,q_{j,m_j}}(s_{i_{\kappa_{j,m_j}}})) \right)
\end{aligned}
$$

(by Proposition 2.3)

$$
= \sum_{\substack{1\leqslant j\leqslant r \\ u_j\langle w\rangle=q_{j,1}(x_{i_{\kappa_{j,1}}})\dots q_{j,m_j}(x_{i_{\kappa_{j,m_j}}})}} a_j t_j \leftarrow (\tau_{M,q_{j,1}}(s_{i_{\kappa_{j,1}}}),\dots,\tau_{M,q_{j,m_j}}(s_{i_{\kappa_{j,m_j}}}))
$$

$$\stackrel{\text{I.H.}}{=} \sum_{\substack{1 \leqslant j \leqslant r \\ u_j\langle w\rangle = q_{j,1}(x_{i_{\kappa_{j,1}}})\dots q_{j,m_j}(x_{i_{\kappa_{j,m_j}}})}} a_j t_j \leftarrow (\tau_{M_1,*} \,\tilde{\circ}\, \tau_{M_2,q_{j,1}}(s_{i_{\kappa_{j,1}}}), \dots, \tau_{M_1,*} \,\tilde{\circ}\, \tau_{M_2,q_{j,m_j}}(s_{i_{\kappa_{j,m_j}}}))$$

$$\stackrel{\dagger}{=} \tau_{M_2,q}((\mu_1)_k(\sigma)_{*,w} \leftarrow (\tau_{M_1,*}(s_{i_1}), \dots, \tau_{M_1,*}(s_{i_l})))$$

$$= \tau_{M_2,q}(\tau_{M_1,*}(\sigma(s_1, \dots, s_k)))$$

$$= \tau_{M_1,*} \,\tilde{\circ}\, \tau_{M_2,q}(\sigma(s_1, \dots, s_k)).$$

Next we prove the equation †. Actually, we prove a more general statement which justifies †. In fact, $\tau_{M_1,*}(s_{i_1}), \dots, \tau_{M_1,*}(s_{i_l})$ are singletons with coefficient 1 (cf. Proposition 3.13). Also note that, since M_1 is a boolean homomorphism, the tree series $(\mu_1)_k(\sigma)_{*,w}$ is also a singleton with coefficient 1. Since the general statement is independent of the above computation, we can freely reuse the notations which have been used up to now.

Statement: For every $l \geqslant 0$, $t \in T_\Delta(X_l), s_1, \dots, s_l \in T_\Delta$ and $q \in Q$, the equation

$$\tau_{M_2,q}(t[s_1, \dots, s_l]) = \sum_{j=1}^{r} a_j t_j \leftarrow (\tau_{M_2,q_{j,1}}(s_{\kappa_{j,1}}), \dots, \tau_{M_2,q_{j,m_j}}(s_{\kappa_{j,m_j}}))$$

holds, where $\tau_{M_2',q}(t) = a_1\hat{t}_1 \oplus \cdots \oplus a_r\hat{t}_r$ for some $a_1, \dots, a_r \in A - \{0\}$ and $\hat{t}_1, \dots, \hat{t}_r \in T_\Gamma(Q \ (X_l))$, and, for every $1 \leqslant j \leqslant r$, $lin_{Q(X_l)}(\hat{t}_j) = (t_j, q_{j,1}(x_{\kappa_{j,1}}) \dots q_{j,m_j}(x_{\kappa_{j,m_j}}))$, $t_j \in \widehat{T_\Gamma}(X_{m_j})$.

We prove by induction on the structure of t. If $t = x_j$, then $\tau_{M_2',q}(t) = q(x_j)$ and thus both sides of the equation are equal to $\tau_{M_2,q}(s_j)$. Now let $t = \delta(v_1, \dots, v_n)$ with $n \geqslant 0$. Then

$$\tau_{M_2,q}(t[s_1, \dots, s_l])$$
$$= \tau_{M_2,q}(\delta(v_1[s_1, \dots, s_l], \dots, v_n[s_1, \dots, s_l]))$$
$$= \sum_{w = p_1(x_{i_1})\dots p_d(x_{i_d}) \in (Q(X_n))^*} (\mu_2)_n(\delta)_{q,w} \leftarrow (\dots, \tau_{M_2,p_j}(v_{i_j}[s_1, \dots, s_l]), \dots)$$

(for the sake of brevity, we consider only the jth element, $1 \leqslant j \leqslant d$)

$$\stackrel{\text{I.H.}}{=} \sum_{w = p_1(x_{i_1})\dots p_d(x_{i_d}) \in (Q(X_n))^*} (\mu_2)_n(\delta)_{q,w} \leftarrow \left(\dots, \sum^{(j)} a_j u_j \leftarrow (\tau_{M_2,r_{j,1}}(s_{c_{j,1}}), \dots, \tau_{M_2,r_{j,m_j}}(s_{c_{j,m_j}})), \dots \right),$$

where $\tau_{M_2',p_j}(v_{i_j}) = \sum^{(j)} a_j\hat{u}_j$, $lin_{Q(X_l)}(\hat{u}_j) = (u_j, r_{j,1}(x_{c_{j,1}}) \dots r_{j,m_j}(x_{c_{j,m_j}}))$,

note that $u_j \in \widehat{T_\Gamma}(X_{m_j})$ and $\sum^{(j)} \langle\text{tree series}\rangle_j$ denotes a finite sum of tree series of which the number depends on j and which we do not detail in order to avoid unreadable indexes

$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*}$$
$$\sum^{(1)} \cdots \sum^{(d)} (\mu_2)_n(\delta)_{q,w} \leftarrow (\dots, a_j u_j \leftarrow (\tau_{M_2,r_{j,1}}(s_{c_{j,1}}),\dots,\tau_{M_2,r_{j,m_j}}(s_{c_{j,m_j}})),\dots),$$
(by Proposition2.3)
$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*}$$
$$\sum^{(1)} \cdots \sum^{(d)} ((\mu_2)_n(\delta)_{q,w} \leftarrow (\dots, a_j u_j[x_1 \leftarrow x_{z_{j-1}+1},\dots,x_{m_j} \leftarrow x_{z_j}],\dots))$$
$$\leftarrow (\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}}))$$
(by Corollary 2.6, where $z_j = m_1 + \cdots + m_j$, $1 \leqslant j \leqslant d$,
note that by Proposition 3.4, $\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}})$
are polynomials)
$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)}$$
$$\left((\textstyle\sum bu) \leftarrow (\dots, a_j u_j[x_1 \leftarrow x_{z_{j-1}+1},\dots,x_{m_j} \leftarrow x_{z_j}],\dots)\right)$$
$$\leftarrow (\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}}))$$
where $\sum bu$ abbreviates the detailed explanation of$(\mu_2)_n(\delta)_{q,w}$
$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)}$$
$$\textstyle\sum(bu \leftarrow (\dots, a_j u_j[x_1 \leftarrow x_{z_{j-1}+1},\dots,x_{m_j} \leftarrow x_{z_j}],\dots))$$
$$\leftarrow (\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}}))$$
(by Proposition 2.3 and the commutativity and associativity of
the sum of tree series)
$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)}$$
$$\textstyle\sum(ba_1 \dots a_d u[\dots, u_j[x_1 \leftarrow x_{z_{j-1}+1},\dots,x_{m_j} \leftarrow x_{z_j}],\dots])$$
$$\leftarrow (\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}}))$$
(by Proposition 2.2)
$$= \sum_{w=p_1(x_{i_1})\dots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)} \textstyle\sum \hat{a}t \leftarrow (\tau_{M_2,r_{1,1}}(s_{c_{1,1}}),\dots,\tau_{M_2,r_{d,m_d}}(s_{c_{d,m_d}}))$$
where $\hat{a} = ba_1 \dots a_d$ and $t = u[\dots, u_j[x_1 \leftarrow x_{z_{j-1}+1},\dots,x_{m_j} \leftarrow x_{z_j}],\dots]$.

This is equal to the right-hand side of *Statement* because

$$\tau_{M_2',q}(t)$$
$$= \tau_{M_2',q}(\delta(v_1,\dots,v_n))$$

$$= \sum_{w=p_1(x_{i_1})\ldots p_d(x_{i_d})\in(Q(X_n))^*} (\mu_2)_n(\delta)_{q,w} \leftarrow (\ldots, \tau_{M_2', p_j}(v_{i_j}), \ldots)$$

$$= \sum_{w=p_1(x_{i_1})\ldots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)} \sum ba_1 \ldots a_d u[\ldots, \hat{u}_j, \ldots]$$

$$= \sum_{w=p_1(x_{i_1})\ldots p_d(x_{i_d})\in(Q(X_n))^*} \sum^{(1)} \cdots \sum^{(d)}$$

$$\sum ba_1 \ldots a_d u[\ldots, u_j[x_1 \leftarrow r_{j,1}(x_{c_{j,1}}), \ldots, x_{m_j} \leftarrow r_{j,m_j}(x_{c_{j,m_j}})], \ldots]$$

and

$$lin_{Q(X_l)}(u[\ldots, \hat{u}_j, \ldots])$$
$$= (u[\ldots, u_j[x_1 \leftarrow x_{z_{j-1}+1}, \ldots, x_{m_j} \leftarrow x_{z_j}], \ldots], r_{1,1}(x_{c_{1,1}}) \ldots r_{d,m_d}(x_{c_{d,m_d}})). \qquad \square$$

Corollary 5.6. *For every commutative semiring A, $b\text{-}HOM_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}TOP_{t-ts}(A) \subseteq p\text{-}TOP_{t-ts}(A)$.*

Proof. It immediately follows from Proposition 5.4 and Lemma 5.5. $\square$

Now we can generalize the inclusion $BOT_{tt}^n \subseteq TOP_{tt}^{n+1}$ in the following way.

Theorem 5.7. *For every commutative semiring A and $n \geqslant 1$, $p\text{-}BOT_{t-ts}^n(A) \subseteq p\text{-}TOP_{t-ts}^{n+1}(A)$.*

Proof. Again we prove by induction on n and only prove the induction step because its proof includes the proof of the induction base $n=1$.

$$\begin{aligned}
& p\text{-}BOT_{t-ts}^n(A) \\
= \; & p\text{-}BOT_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}BOT_{t-ts}^{n-1}(A) \\
\subseteq \; & p\text{-}BOT_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}TOP_{t-ts}^{n}(A) \\
& \text{(by I.H)} \\
\subseteq \; & QREL_{t-ts}(A) \mathbin{\tilde{\circ}} b\text{-}HOM_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}TOP_{t-ts}^{n}(A) \\
& \text{(by Lemma 5.6(1) of [9])} \\
\subseteq \; & QREL_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}TOP_{t-ts}^{n}(A) \\
& \text{(by Corollary 5.6)} \\
\subseteq \; & p\text{-}TOP_{t-ts}(A) \mathbin{\tilde{\circ}} p\text{-}TOP_{t-ts}^{n}(A) \\
& \text{(by Proposition 3.11 of [9], in fact this property} \\
& \text{is true for every semiring)} \\
= \; & p\text{-}TOP_{t-ts}^{n+1}(A). \qquad \square
\end{aligned}$$

Theorem 5.8. *For every commutative izz-semiring A, the following inclusions hold*:

$$p\text{-}TOP_{t-ts}(A) \subseteq p\text{-}BOT^2_{t-ts}(A) \subseteq p\text{-}TOP^3_{t-ts}(A) \subseteq p\text{-}BOT^4_{t-ts}(A) \subset \cdots$$

and

$$p\text{-}BOT_{t-ts}(A) \subseteq p\text{-}TOP^2_{t-ts}(A) \subseteq p\text{-}BOT^3_{t-ts}(A) \subseteq p\text{-}TOP^4_{t-ts}(A) \subset \cdots .$$

Proof. The statement follows from Theorems 5.1 and 5.7. □

These inclusions form the alternating polynomial top-down and bottom-up tree series transformation hierarchies.

6. Lifting up the inclusion diagram of tree transformation classes

In this section, we lift up the whole inclusion diagram of tree transformation classes in Fig. 1 to the level of polynomial t-ts transformations. In fact, we show that, for every commutative izz-semiring A and $n \geqslant 1$, the diagram in Fig. 2 is the inclusion diagram of the t-ts transformation classes $p\text{-}BOT^n_{tt}(A)$, $p\text{-}TOP^n_{tt}(A)$, $p\text{-}BOT^{n+1}_{tt}(A)$ and $p\text{-}TOP^{n+1}_{tt}(A)$.

We have shown in Section 5 that for every commutative semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$. Moreover, for every semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$.

Therefore, in order to show that the diagram in Fig. 2 is an inclusion diagram, it is sufficient to prove that, for every izz-semiring A and $n \geqslant 1$, the classes $p\text{-}BOT^n_{t-ts}(A)$ and $p\text{-}TOP^n_{t-ts}(A)$ are incomparable with respect to inclusion. We will prove the incomparability by lifting up the incomparability between BOT^n_{tt} and TOP^n_{tt} shown in Proposition 4.1 to polynomial t-ts transformation classes over an izz-semiring A. For this purpose, we prove a lift lemma (Lemma 6.19) which can be used to lift up the incomparability relation between tree transformation classes to the incomparability relation between polynomial t-ts transformation classes over an izz-semiring A.

From the fact that the diagram in Fig. 2 is an inclusion diagram for a commutative izz-semiring A it follows that the four fundamental hierarchies, i.e., the uniform polynomial top-down t-ts transformation hierarchy, the alternating polynomial top-down t-ts transformation hierarchy, etc. are proper for every commutative izz-semiring A.

We also show that, out of the four, the uniform polynomial top-down t-ts transformation hierarchy and the uniform polynomial bottom-up t-ts transformation hierarchy are also proper for every izz-semiring A. (Hence, the commutativity of A is not needed.) We will do this in the way that we make a second lift lemma and a third lift lemma. The second one is Lemma 6.22 which can lift up a strict inclusion between tree transformation classes to the same strict inclusion between polynomial t-ts transformation classes over $\mathbb{B}$. The third lift lemma is Lemma 6.23 which can lift up a strict inclusion between t-ts transformation classes over $\mathbb{B}$ to the same strict inclusion between polynomial t-ts transformation classes over an izz-semiring A.

Now we start to elaborate the lift lemmas, which needs a rather long preparation.

Lemma 6.1. *If A is idempotent and $M=(Q,\Sigma,\Delta,A,Q_d,\mu)$ is a boolean and polynomial tree series transducer, then for every $s\in T_\Sigma$, the tree series $\tau_M(s)$ is boolean.*

Proof. By induction on the structure of s we show that, for every $u\in T_\Delta$, $(\tau_M(s),u)\in\{0,1\}$. Let $s=\sigma(s_1,\dots,s_k)$, then

$$\begin{aligned}
&(\tau_M(s),u)\\
&=\Big(\sum_{q\in Q_d}\tau_{M,q}(\sigma(s_1,\dots,s_k)),u\Big)\\
&\quad\text{(by Definition 3.3(2))}\\
&=\Bigg(\sum_{q\in Q_d}\ \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*}\mu_k(\sigma)_{q,w}\leftarrow(\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})),u\Bigg)\\
&\quad\text{(by Definition 3.3(1))}\\
&=\sum_{q\in Q_d}\ \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*}(\mu_k(\sigma)_{q,w}\leftarrow(\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})),u)\\
&\quad\text{(by the definition of the sum of tree series)}\\
&=\sum_{q\in Q_d}\ \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*}\\
&\qquad\sum_{\substack{t\in T_\Delta(X_l),\\ t_1,\dots,t_l\in T_\Delta,\\ u=t[t_1,\dots,t_l]}}(\mu_k(\sigma)_{q,w},t)(\tau_{M,q_1}(s_{i_1}),t_1)\dots(\tau_{M,q_l}(s_{i_l}),t_l)\\
&\quad\text{(by Definition 2.1).}
\end{aligned}$$

Now M is boolean, hence $(\mu_l(\sigma)_{q,w},t)\in\{0,1\}$. By induction hypothesis, $(\tau_{M,q_1}(s_{i_1}),t_1)\in\{0,1\},\dots,(\tau_{M,q_l}(s_{i_l}),t_l)\in\{0,1\}$, hence $(\mu_l(\sigma)_{q,w},t)(\tau_{M,q_1}(s_{i_1}),t_1)\dots(\tau_{M,q_l}(s_{i_l}),t_l)\in\{0,1\}$. Since M is polynomial, only finitely many members of the form $(\mu_l(\sigma)_{q,w},t)(\tau_{M,q_1}(s_{i_1}),t_1)\dots(\tau_{M,q_l}(s_{i_l}),t_l)$ are 1, cf. Proposition 3.4. Thus, since A is idempotent, the result of the last sum in our computation is 0 or 1. □

In the proof of the lift lemmas we will need to "booleanize" tree series, t-ts transformations and tree series transducers over a semiring A. The booleanization of a tree series φ over A means to change the non zero coefficients of φ to 1 and it is based on the extension of a signum function *sgn* over A to tree series over A. Then, the booleanization of t-ts transformations is defined in terms of the booleanization of tree series.

Now we formally give the definition of the signum function *sgn* over a semiring A and generalize it to tree series and t-ts transformation over A.

Definition 6.2. 1. For a semiring A, the mapping $sgn_A : A \to A$ is defined such that, for every $a \in A$,

$$sgn_A(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

2. For every ranked alphabet Δ, we extend sgn_A to a mapping of type $A\langle\langle T_\Delta\rangle\rangle \to A\langle\langle T_\Delta\rangle\rangle$ by letting, for every $\varphi \in A\langle\langle T_\Delta\rangle\rangle$ and $t \in T_\Delta$, $(sgn_A(\varphi), t) = sgn_A((\varphi, t))$. Certainly $sgn_A(\varphi)$ is a boolean tree series. We call $sgn_A(\varphi)$ the *signum frame of* φ.

3. Finally, we extend sgn_A to t-ts transformations. For a t-ts transformation $\tau : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$, we define the signum frame of τ by $sgn_A(\tau) : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ by letting, for every $s \in T_\Sigma$, $sgn_A(\tau)(s) = sgn_A(\tau(s))$. We call $sgn_A(\tau)$ the *signum frame of a t-ts transformation* τ.

In what follows we write sgn for sgn_A provided A is clear from the context. Moreover, we omit the outside parenthesis in expressions of the form $sgn((\varphi, t))$ and write just $sgn(\varphi, t)$.

Proposition 6.3. *If A is idempotent and zero-sum free, then for every $a, b \in A$, $sgn(a \oplus b) = sgn(a) \oplus sgn(b)$ holds.*

Proof. The proof can be performed easily by a simple case analysis. □

In the following, we prove several properties of the sgn function over tree series and t-ts transformations. The first one is that sgn distributes over finite sums of tree series over an idempotent and zero-sum free semiring.

Proposition 6.4. *If A is idempotent and zero-sum free, then, for every family $(\varphi_i \mid i \in [n])$ over $A\langle\langle T_\Delta\rangle\rangle$, the equality $sgn(\sum_{i\in[n]} \varphi_i) = \sum_{i\in[n]} sgn(\varphi_i)$ holds.*

Proof. For every $t \in T_\Delta$,

$$\begin{aligned}
\Big(\sum_{i\in[n]} sgn(\varphi_i), t\Big) &= \sum_{i\in[n]} (sgn(\varphi_i), t) \\
&\quad \text{(by the definition of sum of tree series)} \\
&= \sum_{i\in[n]} sgn(\varphi_i, t) \\
&\quad \text{(by Definition 6.2)} \\
&= sgn\Big(\sum_{i\in[n]} (\varphi_i, t)\Big) \\
&\quad \text{(by Proposition 6.3)} \\
&= sgn\Big(\sum_{i\in[n]} \varphi_i, t\Big) \\
&\quad \text{(by the definition of sum of tree series)}
\end{aligned}$$

$$= (sgn\left(\sum_{i\in[n]} \varphi_i\right), t)$$
(by Definition 6.2). □

Proposition 6.5. *If A is zero-divisor free, then for every $a, b \in A$, $sgn(ab) = sgn(a)\, sgn(b)$ holds.*

Proof. Again, it can be performed by a simple case analysis. □

The next property is that *sgn* distributes over finite compositions of polynomial t-ts transformations provided the underlying semiring is an izz-semiring.

Lemma 6.6. *Assume the A is an izz-semiring. For every $n \geqslant 2$ and polynomial* t-ts *transformations $\tau_i : T_{\Sigma_i} \to A\langle\langle T_{\Sigma_{i+1}}\rangle\rangle$ with $1 \leqslant i \leqslant n$,*

$$sgn(\tau_1 \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_n) = sgn(\tau_1)\,\tilde{\circ} \cdots \tilde{\circ}\, sgn(\tau_n).$$

Proof. We prove by induction on n.

(i) Let $n = 2$, $\tau_1 : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ and $\tau_2 : T_\Delta \to A\langle\langle T_\Gamma\rangle\rangle$. We show that, for every $s \in T_\Sigma$ and $u \in T_\Gamma$, $(sgn(\tau_1 \,\tilde{\circ}\, \tau_2)(s), u) = (sgn(\tau_1)\,\tilde{\circ}\, sgn(\tau_2)(s), u)$.

$$(sgn(\tau_1 \,\tilde{\circ}\, \tau_2)(s), u)$$
$$= (sgn(\tau_1 \,\tilde{\circ}\, \tau_2(s)), u)$$
(by Definition 6.2(3))
$$= sgn(\tau_1 \,\tilde{\circ}\, \tau_2(s), u)$$
(by Definition 6.2(2))
$$= sgn\left(\sum_{t\in T_\Delta} (\tau_1(s), t)\tau_2(t), u\right)$$
(by the definition of the composition of t-ts transformations)
$$= sgn\left(\sum_{t\in T_\Delta} ((\tau_1(s), t)\tau_2(t), u)\right)$$
(by the definition of sum of tree series)
$$= sgn\left(\sum_{t\in T_\Delta} (\tau_1(s), t)(\tau_2(t), u)\right)$$
(by the definition of the product of an element of A and a tree series over A)
$$= \sum_{t\in T_\Delta} sgn((\tau_1(s), t)(\tau_2(t), u))$$
(Proposition 6.3 is extended to finitely many members note that τ_1 and τ_2 are polynomial)
$$= \sum_{t\in T_\Delta} sgn(\tau_1(s), t) sgn(\tau_2(t), u)$$
(by Proposition 6.5)

$$
\begin{aligned}
&= \sum_{t\in T_\Delta} (sgn(\tau_1(s)),t)(sgn(\tau_2(t)),u)\\
&= \sum_{t\in T_\Delta} (sgn(\tau_1)(s),t)(sgn(\tau_2)(t),u)\\
&\quad \text{(by Definition 6.2)}\\
&= \sum_{t\in T_\Delta} ((sgn(\tau_1)(s),t)sgn(\tau_2)(t),u)\\
&\quad \text{(by the definition of the product of an element of } A\\
&\quad \text{and a tree series over } A)\\
&= \Big(\sum_{t\in T_\Delta} (sgn(\tau_1)(s),t)sgn(\tau_2)(t),u\Big)\\
&\quad \text{(by the definition of sum of tree series)}\\
&= (sgn(\tau_1)\,\tilde{\circ}\,sgn(\tau_2)(s),u)\\
&\quad \text{(by the definition of the composition of t-ts transformations)}
\end{aligned}
$$

(ii) The induction step from n to $n+1$ looks as follows.

$$
\begin{aligned}
&sgn(\tau_1\,\tilde{\circ}\cdots\tilde{\circ}\,\tau_{n+1})\\
&\quad = sgn(\tau_1\,\tilde{\circ}\cdots\tilde{\circ}\,\tau_n)\,\tilde{\circ}\,sgn(\tau_{n+1}).\\
&\qquad \text{(by case } (i))\\
&\quad = sgn(\tau_1)\,\tilde{\circ}\cdots\tilde{\circ}\,sgn(\tau_{n+1})\\
&\qquad \text{(by induction hypothesis).} \qquad \Box
\end{aligned}
$$

Now we extend the signum function to tree series transducers over a semiring A as follows.

Definition 6.7. Let $M=(Q,\Sigma,\Delta,A,Q_d,\mu)$ be a tree series transducer. The *signum frame of M* is the tree series transducer $sgn(M)=(Q,\Sigma,\Delta,A,Q_d,v)$, where, for every $k\geqslant 0$, $\sigma\in\Sigma^{(k)}$, $q\in Q, w\in(Q(X_k))^*$, the equality $v_k(\sigma)_{q,w}=sgn(\mu_k(\sigma)_{q,w})$ holds.

The “operation” sgn over tree series transducers has the following properties. The tree series transducer $sgn(M)$ is boolean. Moreover, if M is bottom-up, (top-down, polynomial) then $sgn(M)$ is also bottom-up, (top-down, polynomial).

Now we prove that the t-ts transformation induced by the signum frame of a polynomial tree series transducer M over an izz-semiring is the signum frame of τ_M, i.e., of the t-ts transformation induced by M.

Lemma 6.8. *Let A be an izz-semiring. Then, for every polynomial tree series transducer $M=(Q,\Sigma,\Delta,A,Q_d,\mu)$, the equality $sgn(\tau_M)=\tau_{sgn(M)}$ holds.*

Proof. Let $sgn(M)=(Q,\Sigma,\Delta,A,Q_d,v)$. By Lemma 6.1, the tree series $\tau_{sgn(M)}$ is boolean, hence, it is enough to show that, for every $s\in T_\Sigma$ and $u\in T_\Delta$, $(\tau_{sgn(M)}(s),u)=$

$1 \Leftrightarrow (sgn(\tau_M)(s), u) = 1$.

$$
\begin{aligned}
&(\tau_{sgn(M)}(s), u) = 1 \\
&\quad \Leftrightarrow \Big(\sum_{q \in Q_d} \tau_{sgn(M),q}(s), u \Big) = 1 \\
&\qquad \text{(by Definition 3.3(2))} \\
&\quad \Leftrightarrow \sum_{q \in Q_d} (\tau_{sgn(M),q}(s), u) = 1 \\
&\qquad \text{(by the definition of sum of tree series)} \\
&\quad \Leftrightarrow^{\dagger} \sum_{q \in Q_d} (sgn(\tau_{M,q})(s), u) = 1 \\
&\qquad \text{(since, for every } q \in Q,\ \tau_{sgn(M),q} = sgn(\tau_{M,q}), \text{ see below)} \\
&\quad \Leftrightarrow \sum_{q \in Q_d} (sgn(\tau_{M,q}(s)), u) = 1 \\
&\qquad \text{(by Definition 6.2)} \\
&\quad \Leftrightarrow \Big(\sum_{q \in Q_d} sgn(\tau_{M,q}(s)), u \Big) = 1 \\
&\qquad \text{(by the definition of sum of tree series)} \\
&\quad \Leftrightarrow \Big(sgn \Big(\sum_{q \in Q_d} \tau_{M,q}(s) \Big), u \Big) = 1 \\
&\qquad \text{(by Proposition 6.4)} \\
&\quad \Leftrightarrow (sgn(\tau_M(s)), u) = 1 \\
&\quad \Leftrightarrow (sgn(\tau_M)(s), u) = 1 \\
&\qquad \text{(by Definition 6.2).}
\end{aligned}
$$

Now we prove the equivalence marked by †, i.e., that for every $q \in Q$, $\tau_{sgn(M),q} = sgn(\tau_{M,q})$ by induction on s. For this, let $s = \sigma(s_1, \ldots, s_l) \in T_\Sigma$ and $u \in T_\Delta$. Then

$$
\begin{aligned}
&(\tau_{sgn(M),q}(\sigma(s_1, \ldots, s_l)), u) = 1 \\
&\quad \Leftrightarrow \Bigg(\sum_{w = q_1(x_{i_1}) \ldots q_l(x_{i_l}) \in (Q(X_k))^*} v_k(\sigma)_{q,w} \leftarrow (\tau_{sgn(M),q_1}(s_{i_1}), \ldots, \tau_{sgn(M),q_l}(s_{i_l})), u \Bigg) \\
&\quad = 1 \\
&\qquad \text{(by Definition 3.3(1))} \\
&\quad \Leftrightarrow \sum_{w = q_1(x_{i_1}) \ldots q_l(x_{i_l}) \in (Q(X_k))^*} (v_k(\sigma)_{q,w} \leftarrow (\tau_{sgn(M),q_1}(s_{i_1}), \ldots, \tau_{sgn(M),q_l}(s_{i_l})), u) = 1 \\
&\qquad \text{(by the definition of sum of tree series)} \\
&\quad \Leftrightarrow \sum_{w = q_1(x_{i_1}) \ldots q_l(x_{i_l}) \in (Q(X_k))^*} \sum_{\substack{t \in T_\Delta(X_l) \\ t_1, \ldots, t_l \in T_\Delta, \\ u = t[t_1, \ldots, t_l]}} (v_k(\sigma)_{q,w}, t)(\tau_{sgn(M),q_1}(s_{i_1}), t_1) \ldots (\tau_{sgn(M),q_l}(s_{i_l}), t_l) = 1 \\
&\qquad \text{(by Definition 2.1)}
\end{aligned}
$$

$$\Leftrightarrow \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \sum_{\substack{t\in T_\Delta(X_l)\\ t_1,\dots,t_l\in T_\Delta,\\ u=t[t_1,\dots,t_l]}} (v_k(\sigma)_{q,w},t)(sgn(\tau_{M,q_1})(s_{i_1}),t_1)\dots(sgn(\tau_{M,q_l})(s_{i_l}),t_l)=1$$

(by induction hypothesis)

$$\Leftrightarrow \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \sum_{\substack{t\in T_\Delta(X_l)\\ t_1,\dots,t_l\in T_\Delta,\\ u=t[t_1,\dots,t_l]}} (v_k(\sigma)_{q,w},t)(sgn(\tau_{M,q_1}(s_{i_1})),t_1)\dots(sgn(\tau_{M,q_l}(s_{i_l})),t_l=1$$

(by Definition 6.2)

$\Leftrightarrow \exists w = q_1(x_{i_1})\dots q_l(x_{i_l}) \in (Q(X_k))^*, t\in T_\Delta(X_l), t_1,\dots,t_l\in T_\Delta$

with $t[t_1,\dots,t_l]=u$, such that

$(v_k(\sigma)_{q,w},t)(sgn(\tau_{M,q_1}(s_{i_1})),t_1)\dots(sgn(\tau_{M,q_l}(s_{i_l})),t_l)=1$

(since $v_k(\sigma)_{q,w}$, $sgn(\tau_{M,q_1}(s_{i_1})),\dots,sgn(\tau_{M,q_l}(s_{i_l}))$ are boolean, M is polynomial and A is idempotent)

$\Leftrightarrow \exists w = q_1(x_{i_1})\dots q_l(x_{i_l}) \in (Q(X_k))^*, t\in T_\Delta(X_l), t_1,\dots,t_l\in T_\Delta$ with

$t[t_1,\dots,t_l]=u$, such that

$(v_k(\sigma)_{q,w},t)=1$, $(sgn(\tau_{M,q_1}(s_{i_1})),t_1)=1,\dots,(sgn(\tau_{M,q_l}(s_{i_l})),t_l)=1$

(since $v_k(\sigma)_{q,w}$, $sgn(\tau_{M,q_1}(s_{i_1})),\dots,sgn(\tau_{M,q_l}(s_{i_l}))$ are boolean)

$\Leftrightarrow \exists w = q_1(x_{i_1})\dots q_l(x_{i_l}) \in (Q(X_k))^*, t\in T_\Delta(X_l), t_1,\dots,t_l\in T_\Delta$ with

$t[t_1,\dots,t_l]=u$, such that

$$(\mu_k(\sigma)_{q,w},t)\neq 0,\ (\tau_{M,q_1}(s_{i_1}),t_1)\neq 0,\dots,(\tau_{M,q_l}(s_{i_l}),t_l)\neq 0$$

(by Definitions 6.2 and 6.7)

$\Leftrightarrow \exists w = q_1(x_{i_1})\dots q_l(x_{i_l}) \in (Q(X_k))^*, t\in T_\Delta(X_l), t_1,\dots,t_l\in T_\Delta$ with

$t[t_1,\dots,t_l]=u$, such that

$(\mu_k(\sigma)_{q,w},t)(\tau_{M,q_1}(s_{i_1}),t_1)\dots(\tau_{M,q_l}(s_{i_l}),t_l)\neq 0$

(since A is zero-divisor free)

$$\Leftrightarrow \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \sum_{\substack{t\in T_\Delta(X_l)\\ t_1,\dots,t_l\in T_\Delta,\\ u=t[t_1,\dots,t_l]}} (\mu_k(\sigma)_{q,w},t)(\tau_{M,q_1}(s_{i_1}),t_1),\dots,(\tau_{M,q_l}(s_{i_l}),t_l))\neq 0$$

(since A is zero-sum free and M is polynomial)

$$\Leftrightarrow \sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} (\mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})),u)\neq 0$$

(by Definition 2.1)

$$\Leftrightarrow \left(\sum_{w=q_1(x_{i_1})\dots q_l(x_{i_l})\in(Q(X_k))^*} \mu_k(\sigma)_{q,w} \leftarrow (\tau_{M,q_1}(s_{i_1}),\dots,\tau_{M,q_l}(s_{i_l})),u\right)\neq 0$$

(by the definition of sum of tree series)

$\Leftrightarrow (\tau_{M,q}(\sigma(s_1,\ldots,s_l)),u) \neq 0$
(by Definition 3.3(1))
$\Leftrightarrow (sgn(\tau_{M,q})(\sigma(s_1,\ldots,s_l)),u) = 1$
(by Definition 6.2). □

Now we prove the next lemma which will be one of the keys to prove our main results. It states that if a boolean t-ts transformation τ appears as the composition of n (not necessarily boolean) t-ts transformations induced by polynomial bottom-up or top-down tree series transducers over an izz-semiring A, then there are further n boolean and polynomial tree series transducers over A such that the composition of the t-ts transformations induced by them is τ.

Lemma 6.9. *Let A be an izz-semiring, τ is a boolean* t-ts *transformation over A such that $\tau = \tau_{N_1} \tilde{\circ} \cdots \tilde{\circ} \tau_{N_n}$, where every N_i is either a bottom-up or a top-down polynomial tree series transducer over A. Then there are boolean and polynomial bottom-up or top-down tree series transducers $M_1,\ldots,M_n$ over A such that $\tau = \tau_{M_1} \tilde{\circ} \cdots \tilde{\circ} \tau_{M_n}$. Moreover, N_i and M_i have the same type, i.e., N_i is top-down if and only if M_i is top-down.*

Proof. We show that $\tau = \tau_{sgn(N_1)} \tilde{\circ} \cdots \tilde{\circ} \tau_{sgn(N_n)}$.

$\tau = sgn(\tau)$
(since τ is boolean)
$= sgn(\tau_{N_1} \tilde{\circ} \cdots \tilde{\circ} \tau_{N_n})$
$= sgn(\tau_{N_1}) \tilde{\circ} \cdots \tilde{\circ} sgn(\tau_{N_n})$
(by Lemma 6.6, note that, by Proposition 3.4, $\tau_{N_1},\ldots,\tau_{N_n}$ are polynomial)
$= \tau_{sgn(N_1)} \tilde{\circ} \cdots \tilde{\circ} \tau_{sgn(N_n)}$
(by Lemma 6.8).

Now $sgn(N_1),\ldots,sgn(N_n)$ are boolean and polynomial bottom-up or top-down tree series transducers, the properties bottom-up and top-down are also preserved under sgn. This proves our lemma. □

Now a next part of the preparation follows. In this part, we consider tree series and t-ts transformations over $\mathbb{B}$ as tree series and t-ts transformations over a semiring A by identifying the two elements 0 and 1 of $\mathbb{B}$ with the additive and the multiplicative unit elements 0_A and 1_A of A, respectively. This can be described by an embedding function em_A from $\mathbb{B}$ to A. First we define the function em, then we show some properties which we will need.

Definition 6.10. Let $(A,\oplus,\odot,0_A,1_A)$ be a semiring. 1. The *natural embedding of* $\mathbb{B}$ *into* A is the mapping $em_A : \{0,1\} \to A$ defined by $em_A(0) = 0_A$ and $em_A(1) = 1_A$.

2. For every ranked alphabet Δ, em_A extends to a mapping of type $\mathbb{B}\langle\langle T_\Delta\rangle\rangle \to A\langle\langle T_\Delta\rangle\rangle$ by letting, for every $\varphi \in \mathbb{B}\langle\langle T_\Delta\rangle\rangle$ and $t \in T_\Delta$, $(em_A(\varphi), t) = em_A((\varphi, t))$. We call $em_A(\varphi)$ the natural extension of φ over A.

3. The natural extension of a t-ts transformation $\tau : T_\Sigma \to \mathbb{B}\langle\langle T_\Delta\rangle\rangle$ over a semiring A is the t-ts transformation $em_A(\tau) : T_\Sigma \to A\langle\langle T_\Delta\rangle\rangle$ which satisfies the following condition: for every $s \in T_\Sigma$, $em_A(\tau)(s) = em_A(\tau(s))$.

Note that, for every $\varphi \in \mathbb{B}\langle\langle T_\Delta\rangle\rangle$ and semiring A, the tree series $em_A(\varphi)$ is boolean and $supp(\varphi) = supp(em_A(\varphi))$ holds.

In what follows we abbreviate em_A to em whenever A is clear from the context. Moreover, we omit the outside parentheses in expressions of the form $em((\varphi, t))$ and write just $em(\varphi, t)$.

Proposition 6.11. *Let em be the natural embedding into* $(A, \oplus, \odot, 0_A, 1_A)$ *and let* $b_1, b_2 \in \{0, 1\}$. *Then* $em(b_1 \wedge b_2) = em(b_1)em(b_2)$ *and if A is idempotent, then* $em(b_1 \vee b_2) = em(b_1) \oplus em(b_2)$.

Proof. It can be performed by a simple case analysis. $\square$

The first property we show is that *em* distributes over finite compositions of polynomial t-ts transformations provided the underlying semiring is idempotent.

Lemma 6.12. *If A is idempotent, then, for every* $n \geqslant 2$ *and polynomial* t-ts *transformations* $\tau_i : T_{\Sigma_i} \to \mathbb{B}\langle\langle T_{\Sigma_{i+1}}\rangle\rangle$, $1 \leqslant i \leqslant n$,

$$em(\tau_1 \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_n) = em(\tau_1) \,\tilde{\circ} \cdots \tilde{\circ}\, em(\tau_n).$$

Proof. We prove by induction on n.

(i) Let $n = 2$, $\tau_1 : T_\Sigma \to \mathbb{B}\langle\langle T_\Delta\rangle\rangle$ and $\tau_2 : T_\Delta \to \mathbb{B}\langle\langle T_\Gamma\rangle\rangle$. We prove that, for every $s \in T_\Sigma$ and $u \in T_\Gamma$, the equality $(em(\tau_1 \,\tilde{\circ}\, \tau_2)(s), u) = (em(\tau_1) \,\tilde{\circ}\, em(\tau_2)(s), u)$ holds.

$$\begin{aligned}
&(em(\tau_1 \,\tilde{\circ}\, \tau_2)(s), u)\\
&\quad = (em(\tau_1 \,\tilde{\circ}\, \tau_2(s)), u)\\
&\qquad \text{(by Definition 6.10(3))}\\
&\quad = em(\tau_1 \,\tilde{\circ}\, \tau_2(s), u)\\
&\qquad \text{(by Definition 6.10(2))}\\
&\quad = em\Bigl(\sum_{t \in T_\Delta} (\tau_1(s), t)\,(\tau_2(t), u)\Bigr)\\
&\qquad \text{(by standard arguments)}\\
&\quad = \sum_{t \in T_\Delta} em((\tau_1(s), t)\,(\tau_2(t), u))\\
&\qquad \text{(Proposition 6.11 is extended to a finite sum, note that } \tau_1 \text{ and } \tau_2\\
&\qquad \text{are polynomial)}
\end{aligned}$$

$$= \sum_{t\in T_\Delta} em(\tau_1(s),t)\, em(\tau_2(t),u)$$
(by Proposition 6.11)
$$= \sum_{t\in T_\Delta} (em(\tau_1(s)),t)\,(em(\tau_2(t)),u)$$
$$= \sum_{t\in T_\Delta} (em(\tau_1)(s),t)\,(em(\tau_2)(t),u)$$
(by Definition 6.10)
$$= (em(\tau_1)\,\tilde{\circ}\, em(\tau_2)(s),u)$$
(by standard arguments).

(ii) Induction step from n to $n+1$. It is left to the reader. □

Next we show that em distributes over finite sums of tree series over an idempotent semiring.

Proposition 6.13. *Let $(\varphi_i \mid i\in[n])$ be a family over $\mathbb{B}\langle\langle T_\Delta\rangle\rangle$ and em the natural embedding into an idempotent semiring A. Then $em(\sum_{i\in[n]}\varphi_i) = \sum_{i\in[n]} em(\varphi_i)$.*

Proof. For every $t\in T_\Delta$,

$$\Big(\sum_{i\in[n]} em(\varphi_i),t\Big) = \sum_{i\in[n]} (em(\varphi_i),t)$$
(by the definition of sum of tree series)
$$= \sum_{i\in[n]} em(\varphi_i,t)$$
(by Definition 6.10)
$$= em\Big(\sum_{i\in[n]} (\varphi_i,t)\Big)$$
(by Proposition 6.11)
$$= em\Big(\sum_{i\in[n]} \varphi_i,t\Big)$$
$$= \Big(em\Big(\sum_{i\in[n]} \varphi_i\Big),t\Big)$$
(by Definition 6.10). □

Next we show that em distributes over substitution of polynomial tree series over an idempotent semiring.

Proposition 6.14. *Let $\varphi\in\mathbb{B}\langle\langle T_\Delta(X_l)\rangle\rangle$ and $\psi_1,\dots,\psi_l\in\mathbb{B}\langle\langle T_\Delta\rangle\rangle$ be polynomial tree series and em the natural embedding into an idempotent semiring A. Then*

$$em(\varphi \leftarrow (\psi_1,\dots,\psi_l)) = em(\varphi) \leftarrow (em(\psi_1),\dots,em(\psi_l)).$$

Proof. For every $u \in T_\Delta$,

$$
\begin{aligned}
& em(\varphi \leftarrow (\psi_1,\dots,\psi_l),u) \\
&= em\left(\sum_{\substack{t\in T_\Delta(X_l)\\ t_1,\dots,t_l\in T_\Delta\\ u=t[t_1,\dots,t_l]}} (\varphi,t)(\psi_1,t_1)\dots(\psi_l,t_l)\right) \\
&\quad \text{(by Definition 2.1)} \\
&= em\left(\sum_{\substack{t\in supp(\varphi)\\ \forall 1\leqslant i\leqslant l: t_i\in supp(\psi_i)\\ u=t[t_1,\dots,t_l]}} (\varphi,t)(\psi_1,t_1)\dots(\psi_l,t_l)\right) \\
&= \sum_{\substack{t\in supp(\varphi)\\ \forall 1\leqslant i\leqslant l: t_i\in supp(\psi_i)\\ u=t[t_1,\dots,t_l]}} em((\varphi,t)(\psi_1,t_1)\dots(\psi_l,t_l)) \\
&\quad \text{(Proposition 6.11 is generalized to finitely many members,} \\
&\quad \text{note that } \varphi,\psi_1,\dots,\psi_l \text{ are polynomial)} \\
&= \sum_{\substack{t\in supp(\varphi)\\ \forall 1\leqslant i\leqslant l: t_i\in supp(\psi_i)\\ u=t[t_1,\dots,t_l]}} em(\varphi,t)em(\psi_1,t_1)\dots em(\psi_l,t_l) \\
&\quad \text{(by Proposition 6.11)} \\
&= \sum_{\substack{t\in supp(em(\varphi))\\ \forall 1\leqslant i\leqslant l: t_i\in supp(em(\psi_i))\\ u=t[t_1,\dots,t_l]}} em(\varphi,t)em(\psi_1,t_1)\dots em(\psi_l,t_l) \\
&\quad \text{(because } supp(\varphi) = supp(em(\varphi)) \text{ and } supp(\psi_i) = supp(em(\psi_i))\text{)} \\
&= \sum_{\substack{t\in supp(em(\varphi))\\ \forall 1\leqslant i\leqslant l: t_i\in supp(em(\psi_i))\\ u=t[t_1,\dots,t_l]}} (em(\varphi),t)(em(\psi_1),t_1)\dots(em(\psi_l),t_l) \\
&\quad \text{(by Definition 6.10)} \\
&= \sum_{\substack{t\in T_\Delta(X_l)\\ t_1,\dots,t_l\in T_\Delta\\ u=t[t_1,\dots,t_l]}} (em(\varphi),t)(em(\psi_1),t_1)\dots(em(\psi_l),t_l) \\
&= (em(\varphi) \leftarrow (em(\psi_1),\dots,em(\psi_l)),u). \qquad \square
\end{aligned}
$$

Now we extend the concept of embedding to tree series transducers over $\mathbb{B}$.

Definition 6.15. The tree series transducers $M=(Q,\Sigma,\Delta,\mathbb{B},Q_d,\mu)$ and $N=(Q,\Sigma,\Delta,A,Q_d,v)$ are *associated by embedding* if v is defined in the following way. For every $k\geqslant 0, \sigma\in\Sigma^{(k)}, q\in Q, w\in(Q(X_k))^*$, let $v_k(\sigma)_{q,w}=em(\mu_k(\sigma)_{q,w})$, where em is the natural embedding into A.

Note that if $M=(Q,\Sigma,\Delta,\mathbb{B},Q_d,\mu)$ and $N=(Q,\Sigma,\Delta,A,Q_d,v)$ are associated by embedding, then N is boolean. Moreover, M is bottom-up, (top-down, polynomial) if and only if N is bottom-up, (top-down, polynomial).

Now we prove that the embedding of a tree series transducer M over $\mathbb{B}$ into an idempotent semiring computes the embedding of the t-ts transformation τ_M.

Lemma 6.16. *Let A be an idempotent semiring and let the tree series transducers $M=(Q,\Sigma,\Delta,\mathbb{B},Q_d,\mu)$ and $N=(Q,\Sigma,\Delta,A,Q_d,v)$ be polynomial such that M and N are associated by embedding. Then, $\tau_N=em(\tau_M)$.*

Proof. Let $s\in T_\Sigma$. Then

$$\begin{aligned}
\tau_N(s) &= \sum_{q\in Q_d}\tau_{N,q}(s) \quad \text{(by Definition 3.3)}\\
&=^{\dagger} \sum_{q\in Q_d} em(\tau_{M,q}(s))\\
&= em\Bigg(\sum_{q\in Q_d}\tau_{M,q}(s)\Bigg) \quad \text{(by Proposition 6.13)}\\
&= em(\tau_M(s))\\
&= em(\tau_M)(s) \quad \text{(by standard arguments)}
\end{aligned}$$

Now we prove the equation marked by †, i.e., that for every $s\in T_\Sigma$ and $q\in Q$, $\tau_{N,q}(s)=em(\tau_{M,q}(s))$ holds. To this end, let $s=\sigma(s_1,\ldots,s_k)$. Then

$$\begin{aligned}
&\tau_{N,q}(\sigma(s_1,\ldots,s_k))\\
&= \sum_{w=q_1(x_{i_1})\ldots q_l(x_{i_l})\in(Q(X_k))^*} v_k(\sigma)_{q,w}\leftarrow(\tau_{N,q_1}(s_{i_1}),\ldots,\tau_{N,q_l}(s_{i_l}))\\
&\quad\text{(by Definition 3.3)}\\
&= \sum_{w=q_1(x_{i_1})\ldots q_l(x_{i_l})\in(Q(X_k))^*} em(\mu_k(\sigma)_{q,w})\leftarrow(em(\tau_{M,q_1}(s_{i_1})),\ldots,em(\tau_{M,q_l}(s_{i_l})))\\
&\quad\text{(by I.H. and the fact that M and N are associated by embedding)}\\
&= \sum_{w=q_1(x_{i_1})\ldots q_l(x_{i_l})\in(Q(X_k))^*} em(\mu_k(\sigma)_{q,w}\leftarrow(\tau_{M,q_1}(s_{i_1}),\ldots,\tau_{M,q_l}(s_{i_l})))\\
&\quad\text{(by Proposition 6.14, note that M is polynomial)}\\
&= em\Bigg(\sum_{w=q_1(x_{i_1})\ldots q_l(x_{i_l})\in(Q(X_k))^*} \mu_k(\sigma)_{q,w}\leftarrow(\tau_{M,q_1}(s_{i_1}),\ldots,\tau_{M,q_l}(s_{i_l}))\Bigg)\\
&\quad\text{(by Proposition 6.13, note that $\mu_k(\sigma)_{q,w}\neq\tilde{0}$ only for finitely many w}\\
&\quad\text{cf. Definition 3.1)}\\
&= em(\tau_{M,q}(\sigma(s_1,\ldots,s_k))). \qquad \square
\end{aligned}$$

Now we prepare to state our first lift lemma.

Lemma 6.17. *For every* $n \geqslant 2$ *and polynomial* t-ts *transformations* $\tau_i : T_{\Sigma_i} \to \mathbb{B}\langle\langle T_{\Sigma_{i+1}} \rangle\rangle$, $1 \leqslant i \leqslant n$,

$$(\tau_1 \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_n) \circ pick_{\mathbb{B},\Sigma_{n+1}} = (\tau_1 \circ pick_{\mathbb{B},\Sigma_2}) \circ \cdots \circ (\tau_n \circ pick_{\mathbb{B},\Sigma_{n+1}}).$$

Proof. We prove by induction on n:
(i) Let $n = 2$, $\tau_1 : T_\Sigma \to \mathbb{B}\langle\langle T_\Delta \rangle\rangle$ and $\tau_2 : T_\Delta \to \mathbb{B}\langle\langle T_\Gamma \rangle\rangle$. Then

$$\begin{aligned}
&(s,t) \in (\tau_1 \,\tilde{\circ}\, \tau_2) \circ pick_{\mathbb{B},\Gamma} \\
&\quad \Leftrightarrow (\exists s' \in T_\Delta) : (\tau_1(s), s') = 1 \text{ and } (\tau_2(s'), t) = 1 \\
&\quad \Leftrightarrow (\exists s' \in T_\Delta) : (s, s') \in \tau_1 \circ pick_{\mathbb{B},\Delta} \text{ and } (s', t) \in \tau_2 \circ pick_{\mathbb{B},\Gamma} \\
&\quad \Leftrightarrow (s,t) \in (\tau_1 \circ pick_{\mathbb{B},\Delta}) \circ (\tau_2 \circ pick_{\mathbb{B},\Gamma})
\end{aligned}$$

(ii) The proof of the induction step is left to the reader. □

Corollary 6.18. *Let* $m \geqslant 1$, C_i *be either* $p\text{-}TOP_{t-ts}$ *or* $p\text{-}BOT_{t-ts}$ *for every* $1 \leqslant i \leqslant m$. *Then*

$$\begin{aligned}
&(C_1(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(\mathbb{B})) \circ PICK_{\mathbb{B}} \\
&\quad = (C_1(\mathbb{B}) \circ PICK_{\mathbb{B}}) \circ \cdots \circ (C_m(\mathbb{B}) \circ PICK_{\mathbb{B}}).
\end{aligned}$$

Proof. The proof immediately follows from Lemma 6.17. □

Now we state and prove our first lift lemma, which lifts up an inequality from the level of tree transformations to the level of tree series transformations. In the statement of the lemma, by the type of a tree (series) transducer M we mean that M is either a bottom-up or a top-down tree (series) transducer.

Lemma 6.19 (Lift lemma 1). *Let A be an izz-semiring, let $m, n \geqslant 1$, C_i and D_j are BOT_{tt} or TOP_{tt} and $\overline{C_i}$ and $\overline{D_j}$ are $p\text{-}BOT_{t-ts}$ or $p\text{-}TOP_{t-ts}$ such that C_i has the same type as $\overline{C_i}$ and D_j has the same type as $\overline{D_j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.*

If

$$C_1 \circ \cdots \circ C_m - D_1 \circ \cdots \circ D_n \neq \emptyset,$$

then also

$$\overline{C_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{C_m}(A) - \overline{D_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(A) \neq \emptyset.$$

Proof. In fact, we prove that if

$$\overline{C_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{C_m}(A) \subseteq \overline{D_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(A)$$

then also

$$C_1 \circ \cdots \circ C_m \subseteq D_1 \circ \cdots \circ D_n.$$

We prove our statement in two main steps. In the first main step, we show that the inclusion

$$\overline{C_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{C_m}(\mathbb{B}) \subseteq \overline{D_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(\mathbb{B})$$

holds. To this end let us take polynomial bottom-up or top-down tree series transducers $M_1, \ldots, M_m$ over $\mathbb{B}$ and consider $\tau = \tau_{M_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{M_m}$. By Lemmas 6.12 and 6.16, $em(\tau) = \tau_{N_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{N_m}$, where $N_1, \ldots, N_m$ are the polynomial bottom-up or top-down tree series transducers over A which are associated with $M_1, \ldots, M_m$ by embedding, respectively. Thus, $em(\tau) \in \overline{C_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{C_m}(A)$ and thus, by our assumption, $em(\tau) \in \overline{D_1}(A) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(A)$. Now note that $em(\tau)$ is boolean. Then, by Lemma 6.9, there are boolean and polynomial bottom-up or top-down tree series transducers $\overline{N}_1, \ldots, \overline{N}_n$ over A such that $em(\tau) = \tau_{\overline{N}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{N}_n}$. Now let $\overline{M}_1, \ldots, \overline{M}_n$ be bottom-up or top-down tree series transducers over $\mathbb{B}$ such that $\overline{M}_1, \ldots, \overline{M}_n$ are associated with $\overline{N}_1, \ldots, \overline{N}_n$ by embedding, respectively. Then, again by Lemma 6.16, $\tau_{\overline{N}_i} = em(\tau_{\overline{M}_i})$ for every $1 \leqslant i \leqslant n$. Thus $em(\tau) = em(\tau_{\overline{M}_1}) \,\tilde{\circ} \cdots \tilde{\circ}\, em(\tau_{\overline{M}_n})$ and, by Lemma 6.12, $em(\tau) = em(\tau_{\overline{M}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{M}_n})$. From this $\tau = \tau_{\overline{M}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{M}_n}$ follows, which implies that $\tau \in \overline{D_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(\mathbb{B})$. This finishes the proof of the first step.

In the second step we prove that $C_1 \circ \cdots \circ C_m \subseteq D_1 \circ \cdots \circ D_n$ as follows:

$$\begin{aligned}
& C_1 \circ \cdots \circ C_m \\
&\quad = (\overline{C_1}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \circ \cdots \circ (\overline{C_m}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \\
&\qquad \text{(by Corollaries 4.7 and 4.14 of [9])} \\
&\quad = (\overline{C_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{C_m}(\mathbb{B})) \circ PICK_{\mathbb{B}} \\
&\qquad \text{(by Corollary 6.18)} \\
&\quad \subseteq (\overline{D_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(\mathbb{B})) \circ PICK_{\mathbb{B}} \\
&\qquad \text{(by the inclusion shown in the first step)} \\
&\quad = (\overline{D_1}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \circ \cdots \circ (\overline{D_n}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \\
&\qquad \text{(by Corollary 6.18)} \\
&\quad = D_1 \circ \cdots \circ D_m \\
&\qquad \text{(by Corollaries 4.7 and 4.14 of [9]).} \qquad \square
\end{aligned}$$

To prove that the diagram on Fig. 2 is an inclusion diagram, it remained to prove the following incomparability result.

Theorem 6.20. *For every izz-semiring A and $n \geqslant 1$, the classes $p\text{-}BOT^n_{t-ts}(A)$ and $p\text{-}TOP^n_{t-ts}(A)$ are incomparable with respect to inclusion.*

Proof. First we prove that $p\text{-}BOT^n_{t-ts}(A) - p\text{-}TOP^n_{t-ts}(A) \neq \emptyset$. By Proposition 4.1, $BOT^n_{tt} - TOP^n_{tt} \neq \emptyset$. Then by Lemma 6.19, $p\text{-}BOT^n_{t-ts}(A) - p\text{-}TOP^n_{t-ts}(A) \neq \emptyset$. In a similar way we can prove that $p\text{-}TOP^n_{t-ts}(A) - p\text{-}BOT^n_{t-ts}(A) \neq \emptyset$. $\square$

Now we can lift up the inclusion diagram of tree transformation classes in Fig. 1 to the polynomial t-ts transformation level in the following way.

Theorem 6.21. *For every $n \geqslant 1$ and commutative izz-semiring A, the diagram in Fig. 2 is the inclusion diagram of $p\text{-}BOT^n_{tt}(A)$, $p\text{-}TOP^n_{tt}(A)$, $p\text{-}BOT^{n+1}_{tt}(A)$ and $p\text{-}TOP^{n+1}_{tt}(A)$.*

Proof. For every semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$. By Theorems 5.1 and 5.7, for every commutative semiring A and $n \geqslant 1$, $p\text{-}TOP^n_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$ and $p\text{-}BOT^n_{t-ts}(A) \subseteq p\text{-}TOP^{n+1}_{t-ts}(A)$. Thus our theorem follows from Theorem 6.20. $\square$

Now we prove our second lift lemma which says that if a proper inclusion holds between two compositions of tree transformation classes, then the same strict inclusion holds between the corresponding compositions of polynomial t-ts transformation classes over $\mathbb{B}$.

Lemma 6.22 (Lift lemma 2). *Let $m, n \geqslant 1$, C_i and D_j are BOT_{tt} or TOP_{tt} and $\overline{C_i}$ and $\overline{D_j}$ are $p\text{-}BOT_{t-ts}$ or $p\text{-}TOP_{t-ts}$ such that C_i has the same type as $\overline{C_i}$ and D_j has the same type as $\overline{D_j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.*

If

$$C_1 \circ \cdots \circ C_m \subset D_1 \circ \cdots \circ D_n,$$

then

$$\overline{C_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{C_m}(\mathbb{B}) \subset \overline{D_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{D_n}(\mathbb{B}).$$

Proof. First we observe the following fact. If $C(\mathbb{B})$ is a class of t-ts transformations over $\mathbb{B}$, then there is a bijection between $C(\mathbb{B})$ and the tree transformation class $C(\mathbb{B}) \circ PICK_{\mathbb{B}}$. In fact, the mapping $f : C(\mathbb{B}) \to C(\mathbb{B}) \circ PICK_{\mathbb{B}}$ defined as follows is a bijection: for a t-ts transformation $\tau : T_\Sigma \to \mathbb{B}\langle\langle T_\Delta \rangle\rangle$, let $f(\tau) = \tau \circ pick_{\mathbb{B},\Delta}$.

Now since both $\overline{C_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{C_m}(\mathbb{B})$ and $\overline{D_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{D_n}(\mathbb{B})$ are classes of t-ts transformations over $\mathbb{B}$, it is sufficient to show that

$$(\overline{C_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{C_m}(\mathbb{B})) \circ PICK_{\mathbb{B}} \subset (\overline{D_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{D_n}(\mathbb{B})) \circ PICK_{\mathbb{B}}.$$

This can be seen as follows:

$$\begin{aligned} &(\overline{C_1}(\mathbb{B}) \tilde{\circ} \cdots \tilde{\circ} \overline{C_m}(\mathbb{B})) \circ PICK_{\mathbb{B}} \\ &= (\overline{C_1}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \circ \cdots \circ (\overline{C_m}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \\ &\quad \text{(by Corollary 6.18)} \end{aligned}$$

$= C_1 \circ \cdots \circ C_m$

(by Corollaries 4.7 and 4.11 of [9])

$\subset D_1 \circ \cdots \circ D_n$

(by our assumption)

$= (\overline{D_1}(\mathbb{B}) \circ PICK_{\mathbb{B}}) \circ \cdots \circ (\overline{D_n}(\mathbb{B}) \circ PICK_{\mathbb{B}})$

(by Corollaries 4.7 and 4.11 of [9])

$= (\overline{D_1}(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, \overline{D_n}(\mathbb{B})) \circ PICK_{\mathbb{B}}$

(by Corollary 6.18). □

Now we prove the third lift lemma, which says that if an inclusion holds for two compositions of polynomial t-ts transformation classes over an izz-semiring A *and* the proper version of the same inclusion holds for the same two compositions of the same t-ts transformation classes over $\mathbb{B}$, then the first mentioned inclusion over A is also proper.

Lemma 6.23 (Lift lemma 3). *Let A be an izz-semiring. Assume that the two inclusions*:
(1) $C_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(A) \subseteq D_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, D_n(A)$ *and*
(2) $C_1(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(\mathbb{B}) \subset D_1(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, D_n(\mathbb{B})$
hold, where C_i and D_j are either p-TOP_{t-ts} or p-BOT_{t-ts} for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then also

$$C_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(A) \subset D_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, D_n(A).$$

Proof. By our assumption (2), there are polynomial bottom-up or top-down tree series transducers $M_1, \ldots, M_n$ over $\mathbb{B}$ such that $\tau = \tau_{M_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{M_n}$ and $\tau \notin C_1(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(\mathbb{B})$. By Lemma 6.12, $em(\tau) = em(\tau_{M_1}) \,\tilde{\circ} \cdots \tilde{\circ}\, em(\tau_{M_n})$. Now consider the bottom-up or top-down tree series transducers $N_1, \ldots, N_n$ over A which are associated with $M_1, \ldots, M_n$ by embedding, respectively. By Lemma 6.16, $\tau_{N_i} = em(\tau_{M_i})$ for every $1 \leqslant i \leqslant n$. Note that the N_i are boolean. Thus, $em(\tau) = \tau_{N_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{N_n}$ hence we get $em(\tau) \in D_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, D_n(A)$. We show that $em(\tau) \notin C_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(A)$.

On the contrary, assume that $em(\tau) \in C_1(A) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(A)$, note that $em(\tau)$ is boolean. Then, by Lemma 6.9, there are boolean and polynomial bottom-up or top-down tree series transducers $\overline{N}_1, \ldots, \overline{N}_m$ over A such that $em(\tau) = \tau_{\overline{N}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{N}_m}$. Now let $\overline{M}_1, \ldots, \overline{M}_m$ be bottom-up or top-down tree series transducers over $\mathbb{B}$ such that $\overline{M}_1, \ldots, \overline{M}_m$ are associated with $\overline{N}_1, \ldots, \overline{N}_m$ by embedding. Then, again by Lemma 6.16, $\tau_{\overline{N}_i} = em(\tau_{\overline{M}_i})$ for every $1 \leqslant i \leqslant m$. Thus $em(\tau) = em(\tau_{\overline{M}_1}) \,\tilde{\circ} \cdots \tilde{\circ}\, em(\tau_{\overline{M}_m})$ and, by Lemma 6.12, $em(\tau) = em(\tau_{\overline{M}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{M}_m})$. From this $\tau = \tau_{\overline{M}_1} \,\tilde{\circ} \cdots \tilde{\circ}\, \tau_{\overline{M}_m}$ follows, which implies that $\tau \in C_1(\mathbb{B}) \,\tilde{\circ} \cdots \tilde{\circ}\, C_m(\mathbb{B})$. A contradiction, hence $em(\tau) \notin C_1(A) \,\tilde{\circ} \cdots \tilde{\circ} C_m(A)$. □

Now we can prove that the uniform top-down and the uniform bottom-up t-ts transformation hierarchies are proper for every (not necessarily commutative) izz-semiring A.

Theorem 6.24. *For every izz-semiring A and $n \geqslant 1$, $p\text{-}BOT^{n}_{t-ts}(A) \subset p\text{-}BOT^{n+1}_{t-ts}(A)$ and $p\text{-}TOP^{n}_{t-ts}(A) \subset p\text{-}TOP^{n+1}_{t-ts}(A)$.*

Proof. We prove only the bottom-up case because the top-down one can be handled in the same way. Obviously, for every semiring A, $p\text{-}BOT^{n}_{t-ts}(A) \subseteq p\text{-}BOT^{n+1}_{t-ts}(A)$. Moreover, by Proposition 4.1, $BOT^{n}_{tt} \subset BOT^{n+1}_{tt}$, which, by Lemma 6.22, implies that $p\text{-}BOT^{n}_{t-ts}(\mathbb{B}) \subset p\text{-}BOT^{n+1}_{t-ts}(\mathbb{B})$. Then it follows from Lemma 6.23 that, for every izz-semiring A and $n \geqslant 1$, $p\text{-}BOT^{n}_{t-ts}(A) \subset p\text{-}BOT^{n+1}_{t-ts}(A)$. □

Corollary 6.25. *The uniform polynomial bottom-up and the uniform polynomial top-down* t-ts *transformation hierarchies over every izz-semiring A are proper.*

We note that, for every ranked alphabet Δ, $Lang_f(\Delta)$ is a natural example of an izz-semiring which is not commutative (and not complete either). Another example of a semiring with these properties is the semiring consisting of the 2×2 0-matrix over $\mathbb{B}$ and all 2×2-matrices M over $\mathbb{B}$ such that $M_{1,1} = 1$.

For the sake of completeness, we state the following result about the alternating t-ts transformation hierarchies.

Corollary 6.26. *The alternating polynomial bottom-up and the alternating polynomial top-down* t-ts *transformation hierarchies over every commutative izz-semiring A are proper.*

Proof. It follows immediately from Theorem 6.21. □

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