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MATHEMATICS

SOME NEW SUBALGEBRAS OF $L^1(G)$

BY

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In order to obtain more information about the convolution structure of $L^1(G)$ (G a locally compact group) it is very useful to study ideals and subalgebras of $L^1(G)$. Among these the subalgebras which are dense in $L^1(G)$ and Banach algebras in their own right are of most interest. Up to now *Beurling Algebras* and *Segal Algebras* have been studied more extensively. By using a more general concept it is possible to consider Segal algebras as "normed left ideals" of $L^1(G)$. The same methods are applicable to normed left ideals of Beurling algebras, e.g. the intersection of a Segal algebra with a Beurling algebra. In this paper a new class of Banach subalgebras of $L^1(G)$ is introduced which is different from those mentioned above. All groups shall be *noncompact*!

It is the purpose of this note to give a first outline of the properties of this class of subalgebras. Detailed proofs will appear elsewhere.

Notations and definitions which are not explicitly stated are entirely taken from H. Reiter's book [2].

1. First of all we give some definitions

Definition 1: A real-valued function g on $\mathbb{R}^n$ ($n \geq 1$, $n \in \mathbb{N}$) is said to be a special *gage function*, if it is of the form:

(SG 1) $g(x) = h(|x|)$ for all $x \in \mathbb{R}^n$, where the function h has the properties;
(SG 2) $h(y) > 0$ for all $y \in \mathbb{R}^+$ and h is decreasing for $|y| \to \infty$;
(SG 3) $h(y) \leq B \cdot h(2y)$ for all $y \in \mathbb{R}^+$, where B is a constant, $1 \leq B < \infty$.

Definition 2: A real-valued, strictly positive, measurable and essentially bounded function g on G is said to be a *gage function*, if there exist

(G 1) i) a real number A with $1 \leq A < \infty$ and
ii) a system of (not necessary pairwise different) neighbourhoods $(U_x)_{x \in M}$ of the neutral element, with M the complement of a locally negligible set, such that the following holds:
(G 2) $x \notin U_x U_x$ for all $x \in M$,
(G 3) $\operatorname{ess\,sup}_{z \notin U_x} g(z) \leq A \cdot g(x)$ for all $x \in M$,
(G 4) g is locally essentially bounded away from 0.

Of course every special gage function is a gage function in this more general sense. Now we can define our algebra.

DEFINITION 3: Let g be a strictly positive, measurable function; then we put

$$\Lambda_g(G) := \{f \in L^1(G) | f(x) = f^0(x)g(x),\ f^0 \in L^\infty(G)\}$$
$$\Lambda_g{}^0(G) := \{f \in L^1(G) | f(x) = f^0(x)g(x),\ f^0 \in C^0(G)\}.$$

If g is a gage function and G a unimodular group, the elements of $\Lambda_g(G)$ form a subalgebra of $L^1(G)$ which is a Banach algebra with the norm

$$\|f\|_g = A(\|f\|_1 + \|f^0\|_\infty) \quad (\text{cf. } G\ 3).$$

We call two gage functions equivalent, if both of their quotients are locally essentially bounded. Of course, equivalent gage functions correspond to the same spaces $\Lambda_g(G)$ with equivalent norms. Since every gage function is equivalent to a continuous one, we assume from now on, without loss of generality, that *gage functions* are *continuous*. Then $\Lambda_g{}^0(G)$ is a space of continuous functions, the closure of $\mathscr{K}(G)$ in $\Lambda_g(G)$. Thus $\Lambda_g{}^0(G)$ is a Banach algebra with the norm above. $\Lambda_g{}^0(G)$ is an ideal of $\Lambda_g(G)$ if and only if g is essentially bounded away from 0, that is if $\Lambda_g{}^0(G) = L^1 \cap C^0(G)$.

The next important consequence of the definitions is the fact that w, $w(x) := 1/g(x)$, is (up to a constant) a weight function if g is a gage function; we say that w is *associated* with g. Thus $\Lambda_g(G)$ and $\Lambda_g{}^0(G)$ are invariant under the translations L_y, R_y, $y \in G$. Moreover we have $\|L_a f\|_g \leqslant w(a)\|f\|_g$ and $\|R_a f\|_g \leqslant \max\{1, w(a)\Delta(a^{-1})\}\|f\|_g$, $a \in G$. For $f \in \Lambda_g{}^0(G)$ we also have $\|L_y f - f\|_g < \varepsilon$ and $\|R_y f - f\|_g < \varepsilon$ if $y \in U = U(\varepsilon, f)$.

Hence $\Lambda_g{}^0(G)$ posesses twosided approximate units (which are not bounded). The algebras $\Lambda_g(G)$ $[\Lambda_g{}^0(G)]$, for abelian groups G, do not have the factorization property. Beurling algebras always have the factorization property, since they have bounded approximate units.

The closed left (right) ideals of $\Lambda_g{}^0(G)$ coincide with the closed left (right) translation invariant linear subspaces of $\Lambda_g{}^0(G)$. In general a Banach algebra $\Lambda_g(G)$, with g a continuous and strictly positive function, need not be translation invariant. As in the case of Beurling algebras, the involution $f \to f^*(x) = f(x^{-1})\Delta(x^{-1})$ is not in general applicable in $\Lambda_g(G)$; for the case of special gage functions (Definition 1) it is applicable.

If $\Lambda_{g_1}(G)$ is properly contained in $\Lambda_{g_2}(G)$, then $\Lambda_{g_1}(G)$ is not an ideal in $\Lambda_{g_2}(G)$. In particular, the algebras $\Lambda_g(G)$ cannot be ideals in $L^1(G)$ as in the case of Segal algebras.

2. EXAMPLES

For $G = \mathbf{R}^n$ some special functions $h(x)$ (cf. Definition 1) can be defined as follows:

a) Let $\alpha > 1$ and $\{c_i\}_{i=0}^\infty$ with $1 \leqslant c_i \leqslant c < \infty$ for $i = 0, 1, 2, \ldots$ be given.

We put

$$h_1(x) := \begin{cases} (\prod_{i=0}^{k} c_i)^{-1} & x \in [\alpha^{k-1}, \alpha^k[;\ k=1,2,\ldots \\ c_0^{-1} & x \in [0,1[. \end{cases}$$

b) Let $\{c_{\alpha_i}\}_{i=1}^n$, $\{\alpha_i\}_{i=1}^n$, $c_{\alpha_i}>0$, $\alpha_i>0$ be given and put

$$p(x) := \sum_{i=1}^{n} c_{\alpha_i} x^{\alpha_i},\ h_2(x) := 1/1+p(x).$$

c) Let ϱ be any positive function, decreasing on the right half line and put

$$k(x) := \int_0^x \varrho(\xi)\,d\xi,\ h_3(x) := 1/1+k(x).$$

Every special gage function on $\mathbf{R}^n$ is equivalent to a gage function defined by means of a function h_1 of the type of example a).

On every locally compact, compactly generated group there are gage functions such that $\Lambda_g(G) \neq L^1(G)$. They can be constructed by generalizing example a).

Of special interest is the case that $g \in L^1(G)$. In this case we have

$$(*)\quad g \star g(x) \leqslant A \cdot g(x) \text{ almost everywhere on } G.$$

On every group of polynomial growth (e.g. connected nilpotent Lie groups) there exist gage functions in $L^1(G)$. Whether there is, for instance, such a function on F_2, the free group with two generators, is an open question; the answer is likely to be negative. The above inequality throws some light on the connection between gage functions and weight functions, since on a discrete group any function g fulfilling (*) is of the form $g(x) = K \cdot 1/w(x)$, $K>0$, with w a weight function. A similar results holds for such functions g on $\mathbf{R}$ which are decreasing as $|x| \to \infty$. In spite of this fact there are weight functions such that $1/w \in L^1(\mathbf{R})$, but $1/w$ does not fulfill (*), e.g. $w(x) = e^{|x|}$. A classification of all functions fulfilling (*) is at this moment beyond our reach.

3. For a weight function w associated to g we call the corresponding Beurling algebra $L^1{}_w(G)$ *associated* to $\Lambda_g(G)$. Then the following relations hold:

i) $L^1{}_w(G) \star \Lambda_g(G) \subseteq \Lambda_g(G)$,

ii) $L^1{}_w(G) \star \Lambda_g{}^0(G) = \Lambda_g{}^0(G)$.

From there, for the case of an *abelian* group G we can get further information; in particular we can conclude, that $\Lambda_g(G)$ and $\Lambda_g{}^0(G)$ are of type F in the sense of DOMAR [1] if the associated Beurling $L^1{}_w(G)$ algebra is of type F.

If g is a special gage function on $\mathbf{R}^n$, then the associated Beurling algebra $L^1{}_w(G)$ is of type F. As a consequence we have: The only multiplicative linear functionals on $A_g^0(G)$ are of the form $f \mapsto \hat{f}(x_0)$, $x_0 \in \hat{G}$. Therefore the space of regular maximal ideals of $A_g^0(G)$ coincides with $\hat{G}$ and the Fourier transform of $f \in A_g^0(G)$ coincides with the Gelfand transform. Also $\lim_{n\to\infty} \|f \star f \star \ldots \star f\|_g{}^{1/n} = \|\hat{f}\|_\infty$ (n-fold convolution of f).

The functions $f \in A_g^0(G)$ with $\hat{f}$ having compact support are dense in $A_g^0(G)$ and there exist approximate units $\{u_\alpha\}_{\alpha \in I}$ with $\hat{u}_\alpha$ having compact support. The most important consequence is that the only closed ideal with empty cospectrum is all of $A_g^0(G)$. The author does not know whether $A_g^0(G)$ fullfills the condition of Wiener-Ditkin.

4. Let now H be a closed, normal subgroup of G and consider the mapping T_H of $L^1(G)$ onto $L^1(G/H)$ (cf. [2, Chap. 3, § 4.4.]). If $g \in C^0(G)$ and the restriction of the gage function g to H belongs to $L^1(H)$, then $\dot{g}(\dot{x}) = \int_H g(x\xi)d\xi$, $\dot{x} = \pi_H(x)$, is a continuous function of $\dot{x} \in G/H$ and therefore the space $A_{\dot{g}}(G/H)$ can be defined as in definition 3.

Thus T_H is a contraction from $A_g(G)$ *onto* $A_{\dot{g}}(G/H)$. Since T_H is an algebra homomorphism, $A_{\dot{g}}(G/H)$ is again a subalgebra of $L^1(G/H)$. Of course $T_H A_g^0(G) \subseteq A_{\dot{g}}^0(G/H)$. If G is of the form $H \times H_1$ it can be shown, that T_H maps $A_g^0(G)$ *onto* $A_{\dot{g}}^0(G/H)$.

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Added in proof:

Summary

In this note subalgebras of $L^1(G)$, G a unimodular, locally compact group, are defined by means of certain "gage functions". Several properties of these algebras, in part similar to those of Beurling algebras and Segal algebras, are discussed. Detailed proof will appear elsewhere.