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The spectrum of bicyclic antiautomorphisms of directed triple systems

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Abstract

A transitive triple, (a,b,c), is defined to be the set $\{(a,b),(b,c),(a,c)\}$ of ordered pairs. A directed triple system of order v, DTS(v), is a pair (D,β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D such that any ordered pair of distinct points of D is contained in precisely one transitive triple of β. An antiautomorphism of a directed triple system, (D,β), is a permutation of D which maps β to β^{-1}, where $\beta^{-1} = \{(c,b,a)|(a,b,c)\in\beta\}$. In this paper we complete the necessary and sufficient conditions for the existence of a directed triple system of order v admitting an antiautomorphism consisting of two cycles.

Keywords: Antiautomorphism; Bicyclic; Directed triple system

1. Introduction

A *Steiner triple system of order* v, STS(v), is a pair (S,β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [8] showed that there is an STS(v) if and only if $v \equiv 1$ or $3 \pmod 6$ or $v = 0$.

An *automorphism* of (S,β) is a permutation of S which maps β to itself. An automorphism, α, of (S,β) is called *cyclic* if the permutation defined by α consists of a single cycle of length v. Peltesohn [12] proved that an STS(v) having a cyclic automorphism exists if and only if $v \equiv 1$ or $3 \pmod 6$ and $v \neq 9$. An automorphism,

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α, of (S,β) is called *bicyclic* if the permutation defined by α consists of two cycles. Calahan-Zijlstra and Gardner [1] have shown that there exists an STS(v) admitting a bicyclic automorphism having cycles of length M and N, with $1 < M \leqslant N$, if and only if $M \equiv 1$ or $3 \pmod 6), M \neq 9$, $M|N$, and $M + N \equiv 1$ or $3 \pmod 6$.

A *transitive triple*, (a,b,c), is defined to be the set $\{(a,b),(b,c),(a,c)\}$ of ordered pairs. A *directed triple system of order* v, DTS(v), is a pair (D,β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D, called *triples*, such that any ordered pair of distinct points of D is contained in precisely one element of β. Hung and Mendelsohn [6] have shown that necessary and sufficient conditions for the existence of a DTS(v) are that $v \equiv 0$ or $1 \pmod 3$.

For a DTS$(v), (D,\beta)$, we define β^{-1} by $\beta^{-1} = \{(c,b,a)|(a,b,c)\in\beta\}$. Then (D,β^{-1}) is a DTS(v) and is called the *converse* of (D,β). A DTS(v) which is isomorphic to its converse is said to be *self-converse*. Kang et al. [7] have shown that a self-converse DTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$ and $v \neq 6$. An *automorphism* of (D,β) is a permutation of D which maps β to itself. A DTS(v) is called *cyclic* if there is an automorphism consisting of a single cycle of order v. Colbourn and Colbourn have shown that a cyclic DTS(v) exists if and only if $v \equiv 1, 4$, or $7 \pmod{12}$ [5]. An *antiautomorphism* of (D,β) is a permutation of D which maps β to β^{-1}. Clearly, a DTS(v) is self-converse if and only if it admits an antiautomorphism.

An automorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v - d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic automorphism have been given by Micale and Pennisi [10]. An automorphism, α, on a DTS(v) is called *f-bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v - f)/2$ and f fixed points. Micale and Pennisi [9] have given conditions for the existence of f-bicyclic directed triple systems.

An antiautomorphism, α, on a DTS(v) is called *d-cyclic* if the permutation defined by α consists of a single cycle of length d and $v-d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a d-cyclic antiautomorphism have been given by Carnes et al. [2]. We call an antiautomorphism, α, on a DTS(v) *f-bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v - f)/2$ and f fixed points. A *bicyclic* antiautomorphism of a DTS(v) is an antiautomorphism, α, which consists of two cycles of length M and N respectively, where $v = M + N$. Carnes et al. [4] have shown that a necessary condition for a DTS(v) to admit a bicyclic antiautomorphism with cycles of length M and N, $1 < M \leqslant N$, is that $M|N$. Carnes, Dye, and Reed gave necessary and sufficient conditions for the case where $N = M$ [3], and for the case where $N = 2M$ [4]. In this paper we consider the case where $N = kM, k > 2$.

2. Preliminaries

If K is the length of a cycle, $K \in \{M,N\}$, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (K-1)_i), i \in \{0,1\}$. Let $\Delta = \{0,1,2,\ldots,(K-1)\}$. We shall use all additions modulo K in the triples. For $a_i, b_j, c_k \in D, i,j,k \in \{0,1\}, (a_i,b_j,c_k)\in\beta$, let the *orbit* of (a_i,b_j,c_k) be

$\{((a+t)_i,(b+t)_j,(c+t)_k)\,|\,t\in\Delta, t\,\text{even}\}\cup\{((c+t)_k,(b+t)_j,(a+t)_i)\,|\,t\in\Delta, t\,\text{odd}\}$. Clearly the orbits of the elements of β yield a partition of β.

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of a DTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of each orbit occurs in $\bar{\beta}$. Also, we say that the *reverse* of the transitive triple (a,b,c) is the transitive triple (c,b,a).

Let (S,β') be an STS(v). Let $\beta=\{(a,b,c),(c,b,a)\,|\,\{a,b,c\}\in\beta'\}$. Then (S,β) is called the *corresponding* DTS(v), and the identity map on the point set is an antiautomorphism. This yields a self-converse DTS(v) for $v\equiv 1$ or 3 (mod 6). For $v\equiv 1$ (mod 6) the cyclic STS(v) from Peltesohn's constructions [12] has no orbit of length less than v, hence the corresponding DTS(v) admits a cyclic antiautomorphism.

In the constructions of the base triples, we use the following systems. An (A,n)-*system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\ldots,n$ that partition the set $\{1,2,\ldots,2n\}$, such that $b_r=a_r+r$ for $r=1,2,\ldots,n$. Skolem [13] showed that an (A,n)-system exists if and only if $n\equiv 0$ or 1 (mod 4). A (B,n)-*system* is a collection of ordered pairs (a_r,b_r) for $r=1,2,\ldots,n$ that partition the set $\{1,2,\ldots,2n-1,2n+1\}$, such that $b_r=a_r+r$ for $r=1,2,\ldots,n$. O'Keefe [11] showed that a (B,n)-system exists if and only if $n\equiv 2$ or 3 (mod 4). In either case, the triples used in the constructions are of the form $(0_i,(a_r+n)_i,(b_r+n)_i)$ for $r=1,2,\ldots,n$, where $i=0$ in the cycle of length M and $i=1$ in the cycle of length N.

3. Necessary conditions

The types of cyclic triples possible are:

(1) *Type* 1: (x_0,y_0,z_0) where x_0,y_0,z_0 are in the cycle of length M,
(2) *Type* 2: (x_1,y_1,z_1) where x_1,y_1,z_1 are in the cycle of length N,
(3) *Type* 3: (x_0,y_1,z_1) or (y_1,x_0,z_1) or (y_1,z_1,x_0) where x_0 is in the cycle of length M and y_1,z_1 are in the cycle of length N,
(4) *Type* 4: (x_0,y_0,z_1) or (x_0,z_1,y_0) or (z_1,x_0,y_0) where x_0,y_0 are in the cycle of length M and z_1 is in the cycle of length N.

Carnes et al. [4] have shown that if a DTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1<M<N$, and a type 4 triple occurs, then M is odd and $N=2M$. Since the cases for $N=2M$ are settled in [4], for the remainder, we consider the cases where $N=kM$, $k>2$, so that only triples of type 1, 2, or 3 are possible.

If a DTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1<M<N$, then the restriction of the permutation to the points of the cycle of length M is clearly a cyclic subsystem.

Lemma 1. *If a DTS(v) admits a bicyclic antiautomorphism with cycles of length M and N, where $1<M<N$, then $M\equiv 1,4,$ or 7 (mod 12).*

Proof. If M is even, then $M \equiv 4 \pmod{12}$ by Carnes et al. [2].

If M is odd, then α^2 is a cyclic automorphism of the form $(0_0, 2_0, \ldots, (M-1)_0, 1_0, 3_0, \ldots, (M-2)_0)$. But then $M \equiv 1$ or $7 \pmod{12}$ by Colbourn and Colbourn [5]. □

Lemma 2. *If there exists a DTS(v) which admits a bicyclic antiautomorphism with cycles of length M and N, where $1 < M < N$ and $N > 2M$, then $M \equiv 1, 4$, or $7 \pmod{12}$ and $N = kM$, where $k \equiv 2 \pmod 3$ if $M \equiv 4 \pmod{12}$ and $k \equiv 2 \pmod 6$ if $M \equiv 1$ or $7 \pmod{12}$.*

Proof. We need only show that $k \equiv 2 \pmod 3$ if $M \equiv 4 \pmod{12}$ and $k \equiv 2 \pmod 6$ if $M \equiv 1$ or $7 \pmod{12}$.

There are $[v(v-1)]/3$ triples and $[M(M-1)]/3$ type 1 triples. The number of other triples, type 2 and 3 triples, is then $[v(v-1)]/3 - [M(M-1)]/3 = [N(N+2M-1)]/3$. All of these have orbits of length N, so there are $[N+2M-1]/3 = [(k+2)M-1]/3$ orbits. Then $(k+2)M \equiv 1 \pmod 3$ and since $M \equiv 1 \pmod 3$, we have $(k+2) \equiv 1 \pmod 3$, so that $k \equiv 2 \pmod 3$.

Now, suppose $M \equiv 1$ or $7 \pmod{12}$. If N is odd, then $\alpha^N(x_i, y_j, z_l) = (z_l, y_j, x_i)$, so that we have an STS(v) where $v = M + N$. But $M + N$ is even, a contradiction. Therefore, if M is odd, N is even and so k is even. Since k is even and $k \equiv 2 \pmod 3$, we have $k \equiv 2 \pmod 6$. □

4. $M \equiv 1 \pmod{12}$

Lemma 3. *If $v = M + N, N = kM, M \equiv 1 \pmod{12}$ and $k \equiv 2 \pmod 6$, there exists a DTS(v) which admits a bicyclic antiautomorphism where M and N are the lengths of the cycles.*

Proof. Let $M = 12t + 1$.

For $k = 12r + 2$ and $r + t$ even, the base triples include the following and their reverses:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right),$$

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \qquad \text{for } s = 0, 1, \ldots, 3t-1,$$

$$\left(0_0, (6t+s+1)_1, \left(\frac{N}{2} + 6t - s\right)_1\right) \qquad \text{for } s = 0, 1, \ldots, 3t-1.$$

The remaining triples in the cycle of length N are formed using an $(A, 24rt + 2r + 2t)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $k=12r+2$ and $r+t$ odd, the base triples include the following and their reverses:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right),$$

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t-2,$$

$$\left(0_0, (6t+s+1)_1, \left(\frac{N}{2} + 6t - s\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t.$$

The remaining triples in the cycle of length N are formed using a $(B, 24rt + 2r + 2t)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $k = 12r + 8$ and $r + t$ even, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right).$$

Also included are the following triples and their reverses:

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t-1,$$

$$\left(0_0, (6t+s+1)_1, \left(\frac{N}{2} + 6t - s\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t-1.$$

The remaining triples in the cycle of length N are formed using an $(A, 24rt+2r+14t+1)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $k = 12r + 8$ and $r + t$ odd, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right).$$

Also included are the following triples and their reverses:

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t-2,$$

$$\left(0_0, (6t+s+1)_1, \left(\frac{N}{2} + 6t - s\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t.$$

The remaining triples in the cycle of length N are formed using a $(B, 24rt+2r+14t+1)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M). □

5. $M \equiv 4 \pmod{12}$

Lemma 4. *If $v = M + N, N = kM, M \equiv 4 \pmod{12}$ and $k \equiv 2 \pmod 3$, there exists a DTS(v) which admits a bicyclic antiautomorphism where M and N are the lengths of the cycles.*

Proof. Let $M = 12t + 4$ and $k = 6r + 2$.

For $t = 0$, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right), \left(0_0, 1_1, \left(\frac{N}{2} - 1\right)_1\right), \left(\left(\frac{N}{2} - 1\right)_1, (N-2)_1, 0_0\right),$$

$$\left(\left(\frac{N}{2} - 3\right)_1, 0_0, \left(\frac{N}{2} - 2\right)_1\right), \left(0_1, \left(\frac{N}{2} - 2\right)_1, \left(\frac{N}{2} - 1\right)_1\right).$$

For $r = 1$, also included are the following triples and their reverses:

$$(0_1, 6_1, 11_1), (0_1, 8_1, 12_1), (0_1, 10_1, 13_1), (0_1, 7_1, 9_1).$$

For $r \geqslant 2$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{N}{6} + s + \frac{2}{3}\right)_1, \left(\frac{N}{3} - s - \frac{5}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24} - \frac{7}{3},$$

$$\left(0_1, \left(\frac{N}{3} + s + \frac{1}{3}\right)_1, \left(\frac{N}{2} - s - 3\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24} - \frac{7}{3},$$

$$\left(0_1, \left(\frac{3N}{8} + s\right)_1, \left(\frac{11N}{24} - s - \frac{5}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24} - \frac{7}{3},$$

$$\left(0_1, \left(\frac{5N}{24} + s + \frac{1}{3}\right)_1, \left(\frac{7N}{24} - s - \frac{1}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24} - \frac{10}{3},$$

$$\left(0_1, \left(\frac{5N}{24} - \frac{2}{3}\right)_1, \left(\frac{3N}{8} - 1\right)_1\right), \left(0_1, \left(\frac{N}{4}\right)_1, \left(\frac{5N}{12} - \frac{4}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{N}{3} - \frac{2}{3}\right)_1, \left(\frac{5N}{12} - \frac{1}{3}\right)_1\right), \left(0_1, \left(\frac{N}{4} - 2\right)_1, \left(\frac{N}{4} + 2\right)_1\right),$$

$$\left(0_1, \left(\frac{N}{4} - 1\right)_1, \left(\frac{N}{4} + 1\right)_1\right).$$

For $t \geqslant 1$, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right), \left(0_0, 1_1, \left(\frac{N}{2} - 1\right)_1\right), \left(\left(\frac{N}{2} - 2\right)_1, \left(\frac{N}{2} - 1\right)_1, 0_0\right),$$

$$\left(1_1, 0_0, \left(\frac{N}{2} - 2\right)_1\right), \left(0_1, \left(\frac{N}{2} - 3\right)_1, \left(\frac{N}{2} - 2\right)_1\right).$$

For $t=1$, also included are the following and their reverses:

$$\left(0_0,10_1,\left(\frac{N}{2}+2\right)_1\right),\left(0_0,9_1,\left(\frac{N}{2}+3\right)_1\right),\left(0_0,8_1,\left(\frac{N}{2}+4\right)_1\right),$$

$$\left(0_0,13_1,\left(\frac{N}{2}+6\right)_1\right),\left(0_0,12_1,\left(\frac{N}{2}+7\right)_1\right),\left(0_0,5_1,\left(\frac{N}{2}-5\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{6}+s-\frac{4}{3}\right)_1,\left(\frac{N}{3}-s-\frac{20}{3}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{7}{3},$$

$$\left(0_1,\left(\frac{N}{3}+s-\frac{14}{3}\right)_1,\left(\frac{N}{2}-s-11\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{10}{3},$$

$$\left(0_1,\left(\frac{N}{4}-2\right)_1,\left(\frac{5N}{12}-\frac{13}{3}\right)_1\right),\left(0_1,\left(\frac{7N}{24}-\frac{16}{3}\right)_1,\left(\frac{11N}{24}-\frac{26}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{4}-5\right)_1,\left(\frac{5N}{12}-\frac{28}{3}\right)_1\right),\left(0_1,\left(\frac{N}{3}-\frac{17}{3}\right)_1,\left(\frac{5N}{12}-\frac{22}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{2}-9\right)_1,\left(\frac{N}{2}-1\right)_1\right),\left(0_1,\left(\frac{5N}{12}-\frac{37}{3}\right)_1,\left(\frac{5N}{12}-\frac{16}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{4}-7\right)_1,\left(\frac{N}{4}-1\right)_1\right),\left(0_1,\left(\frac{5N}{12}-\frac{34}{3}\right)_1,\left(\frac{5N}{12}-\frac{19}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{4}-8\right)_1,\left(\frac{N}{4}-4\right)_1\right),\left(0_1,\left(\frac{N}{4}-6\right)_1,\left(\frac{N}{4}-3\right)_1\right),$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{31}{3}\right)_1,\left(\frac{5N}{12}-\frac{25}{3}\right)_1\right).$$

For $r \geqslant 2$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{5N}{24}+s-\frac{8}{3}\right)_1,\left(\frac{7N}{24}-s-\frac{19}{3}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{19}{3},$$

$$\left(0_1,\left(\frac{3N}{8}+s-7\right)_1,\left(\frac{11N}{24}-s-\frac{29}{3}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{19}{3}.$$

For $t \geqslant 2$, also included are the following and their reverses:

$$\left(0_0,(s+2)_1,\left(\frac{N}{2}-s-3\right)_1\right)\quad \text{for } s=0,1,\ldots,3t-2,$$

$$\left(0_0, (6t+s+4)_1, \left(\frac{N}{2}+6t-s\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t-2,$$

$$\left(0_0, (6t+3)_1, \left(\frac{N}{2}+6t+2\right)_1\right), (0_0, (3t+1)_1, (6t+1)_1).$$

For t even, $t \geqslant 2$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{N}{6}-2t+s+\frac{5}{3}\right)_1, \left(\frac{N}{3}-4t-s-\frac{2}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24}-\frac{t}{2}-\frac{4}{3},$$

$$\left(0_1, \left(\frac{N}{3}-4t+s+\frac{4}{3}\right)_1, \left(\frac{N}{2}-6t-s-2\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24}-\frac{t}{2}-\frac{7}{3},$$

$$\left(0_1, \left(\frac{3N}{8}-\frac{9t}{2}+s+1\right)_1, \left(\frac{11N}{24}-\frac{11t}{2}-s-\frac{2}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24}-2t-\frac{4}{3},$$

$$\left(0_1, \left(\frac{5N}{24}-\frac{5t}{2}+s+\frac{7}{3}\right)_1, \left(\frac{7N}{24}-\frac{7t}{2}-s-\frac{1}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{N}{24}-2t-\frac{7}{3},$$

$$\left(0_1, \left(\frac{5N}{24}-\frac{5t}{2}+\frac{4}{3}\right)_1, \left(\frac{3N}{8}-\frac{9t}{2}\right)_1\right), \left(0_1, \left(\frac{N}{4}-3t+1\right)_1, \left(\frac{N}{3}-4t+\frac{1}{3}\right)_1\right).$$

For $t \equiv 0 \pmod 4), t \geqslant 4$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{N}{4}-\frac{9t}{2}+s+2\right)_1, \left(\frac{N}{4}-\frac{3t}{2}-s+1\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{2}-2,$$

$$\left(0_1, \left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{2}{3}\right)_1, \left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{4}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4}-1,$$

$$\left(0_1, \left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{11}{3}\right)_1, \left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{1}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4}-2,$$

$$\left(0_1, \left(\frac{N}{4}-\frac{9t}{2}+1\right)_1, \left(\frac{5N}{12}-\frac{13t}{2}+\frac{5}{3}\right)_1\right), \left(0_1, \left(\frac{N}{4}-3t+2\right)_1, \left(\frac{5N}{12}-5t+\frac{5}{3}\right)_1\right).$$

For $t \equiv 2 \pmod 4), t \geqslant 2$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{N}{4}-\frac{9t}{2}+2s+1\right)_1, \left(\frac{N}{4}-\frac{3t}{2}-2s\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4}-\frac{3}{2},$$

$$\left(0_1, \left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{2}{3}\right)_1, \left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{4}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{N}{4}-3t\right)_1,\left(\frac{5N}{12}-5t+\frac{2}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2\right)_1,\left(\frac{5N}{12}-\frac{13t}{2}+\frac{5}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{5N}{12}-5t-\frac{1}{3}\right)_1,\left(\frac{5N}{12}-5t+\frac{8}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-3t+2\right)_1,\left(\frac{N}{4}-3t+4\right)_1\right).$$

For $t \equiv 2 \pmod 4, t \geqslant 6$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+4\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s+1\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{5}{2},$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{11}{3}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{1}{3}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{5}{2}.$$

For t odd, $t \geqslant 3$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{3}-4t+s+\frac{1}{3}\right)_1,\left(\frac{N}{2}-6t-s-2\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{t}{2}-\frac{11}{6},$$

$$\left(0_1,\left(\frac{N}{6}-2t+s+\frac{5}{3}\right)_1,\left(\frac{N}{3}-4t-s-\frac{5}{3}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{t}{2}-\frac{11}{6},$$

$$\left(0_1,\left(\frac{3N}{8}-\frac{9t}{2}+s-\frac{1}{2}\right)_1,\left(\frac{11N}{24}-\frac{11t}{2}-s-\frac{13}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-2t-\frac{4}{3},$$

$$\left(0_1,\left(\frac{5N}{24}-\frac{5t}{2}+s+\frac{5}{6}\right)_1,\left(\frac{7N}{24}-\frac{7t}{2}-s-\frac{11}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-2t-\frac{7}{3},$$

$$\left(0_1,\left(\frac{7N}{24}-\frac{7t}{2}-\frac{5}{6}\right)_1,\left(\frac{11N}{24}-\frac{11t}{2}-\frac{7}{6}\right)_1\right),\left(0_1,\left(\frac{N}{3}-4t-\frac{2}{3}\right)_1,\left(\frac{5N}{12}-5t-\frac{4}{3}\right)_1\right).$$

For $t \equiv 1 \pmod 4, t \geqslant 5$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+\frac{1}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s-\frac{1}{2}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{7}{4},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s-\frac{1}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s-\frac{7}{2}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{7}{4},$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{1}{6}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{11}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{7}{4},$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+2s-\frac{5}{6}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-2s-\frac{29}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{11}{4},$$

$$\left(0_1,\left(\frac{N}{4}-3t-1\right)_1,\left(\frac{5N}{12}-5t-\frac{1}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-\frac{3t}{2}-\frac{3}{2}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-\frac{17}{6}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{4}-3t-2\right)_1,\left(\frac{N}{4}-3t+1\right)_1\right),\left(0_1,\left(\frac{5N}{12}-5t-\frac{13}{3}\right)_1,\left(\frac{5N}{12}-5t-\frac{7}{3}\right)_1\right).$$

For $t \equiv 3 \pmod 4, t \geqslant 3$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+s+\frac{1}{6}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-s-\frac{11}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{2}-\frac{5}{2},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s-\frac{1}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s-\frac{3}{2}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{5}{4},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+\frac{5}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s-\frac{1}{2}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{5}{4},$$

$$\left(0_1,\left(\frac{N}{4}-3t-1\right)_1,\left(\frac{5N}{12}-5t-\frac{1}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+\frac{1}{2}\right)_1,\left(\frac{5N}{12}-\frac{13t}{2}-\frac{5}{6}\right)_1\right).$$

The remaining triples in the cycle of length M are from a cyclic DTS(M).

Let $M = 12t+4$ and $k=5$, so that $N = 60t+20$.

For t even, the base triples include the following:

$$(0_1, 0_0, (30t+10)_1), ((6t+2)_1, 0_0, 0_1), (0_1, (3t+1)_1, (6t+2)_1).$$

For t odd, the base triples include the following:

$$(0_1, 0_0, (30t+10)_1), ((6t+2)_1, 0_0, 0_1), ((3t+1)_1, 0_1, (6t+2)_1).$$

For $t=0$, also included are the following and their reverses:

$$(0_0, 1_1, 7_1), (0_1, 3_1, 8_1), (0_1, 4_1, 11_1).$$

For $t=1$, also included are the following and their reverses:

$$(0_0, 1_1, 47_1), (0_0, 2_1, 46_1), (0_0, 3_1, 45_1), (0_0, 10_1, 55_1), (0_0, 11_1, 54_1),$$

$$(0_0, 12_1, 53_1), (0_0, 4_1, 9_1), (0_1, 19_1, 32_1), (0_1, 17_1, 29_1), (0_1, 16_1, 27_1),$$

$$(0_1, 15_1, 25_1), (0_1, 14_1, 23_1), (0_1, 26_1, 33_1), (0_1, 24_1, 30_1), (0_1, 28_1, 31_1),$$

$$(0_1, 18_1, 20_1), (0_1, 21_1, 22_1).$$

For $t \geqslant 2$, also included are the following and their reverses:

$$(0_0, (1+s)_1, (36t-s+11)_1) \quad \text{for } s=0,1,\ldots,3t-1,$$

$$(0_0, (6t+s+4)_1, (42t-s+13)_1) \quad \text{for } s=0,1,\ldots,3t-1,$$

$$(0_0, (3t+1)_1, (6t+3)_1),$$

$$(0_1,(8t+s+6)_1,(16t-s+7)_1) \quad \text{for } s=0,1,\ldots,t-1,$$

$$(0_1,(16t+s+9)_1,(24t-s+9)_1) \quad \text{for } s=0,1,\ldots,t-2,$$

$$(0_1,(17t+s+9)_1,(23t-s+10)_1) \quad \text{for } s=0,1,\ldots,t-1,$$

$$(0_1,(9t+s+7)_1,(15t-s+7)_1) \quad \text{for } s=0,1,\ldots,t-2,$$

$$(0_1,(10t+6)_1,(18t+9)_1),(0_1,(9t+6)_1,(17t+8)_1),$$

$$(0_1,(16t+8)_1,(20t+10)_1).$$

For $t=2$, also included are the following and their reverses:

$$(0_1,31_1,52_1),(0_1,34_1,54_1),(0_1,27_1,36_1),(0_1,29_1,35_1),(0_1,48_1,53_1),$$

$$(0_1,28_1,32_1),(0_1,30_1,33_1),(0_1,49_1,51_1),(0_1,46_1,47_1).$$

For t even, $t \geqslant 4$, also included are the following and their reverses:

$$(0_1,(10t+s+7)_1,(14t-s+8)_1) \quad \text{for } s=0,1,\ldots,\frac{t}{2}-1,$$

$$(0_1,(18t+s+10)_1,(22t-s+10)_1) \quad \text{for } s=0,1,\ldots,\frac{t}{2}-2,$$

$$\left(0_1,\left(\frac{37t}{2}+s+11\right)_1,\left(\frac{43t}{2}-s+10\right)_1\right) \text{ for } s=0,1,\ldots,\frac{3t}{2}-3,$$

$$(0_1,(12t+7)_1,(20t+12)_1),\left(0_1,\left(\frac{27t}{2}+7\right)_1,\left(\frac{43t}{2}+11\right)_1\right),$$

$$(0_1,(20t+9)_1,(20t+11)_1),\left(0_1,\left(\frac{37t}{2}+9\right)_1,\left(\frac{37t}{2}+10\right)_1\right).$$

For $t \equiv 0 \pmod 4), t \geqslant 4$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{21t}{2}+2s+8\right)_1,\left(\frac{27t}{2}-2s+8\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-1,$$

$$\left(0_1,\left(\frac{21t}{2}+2s+7\right)_1,\left(\frac{27t}{2}-2s+5\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-2,$$

$$(0_1,(12t+5)_1,(12t+8)_1).$$

For $t \equiv 2 \pmod 4), t \geqslant 6$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{21t}{2}+2s+8\right)_1,\left(\frac{27t}{2}-2s+8\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{21t}{2}+2s+7\right)_1,\left(\frac{27t}{2}-2s+5\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{3}{2},$$

$$(0_1,(12t+6)_1,(12t+9)_1).$$

For t odd, $t \geqslant 3$, also included are the following and their reverses:

$$(0_1, (10t+s+7)_1, (14t-s+8)_1) \quad \text{for } s = 0, 1, \ldots, \frac{t}{2} - \frac{3}{2},$$

$$(0_1, (18t+s+10)_1, (22t-s+10)_1) \quad \text{for } s = 0, 1, \ldots, \frac{t}{2} - \frac{3}{2},$$

$$\left(0_1, \left(\frac{21t}{2}+s+\frac{15}{2}\right)_1, \left(\frac{27t}{2}-s+\frac{13}{2}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{2} - \frac{7}{2},$$

$$\left(0_1, \left(\frac{21t}{2}+\frac{13}{2}\right)_1, \left(\frac{37t}{2}+\frac{21}{2}\right)_1\right), \left(0_1, \left(\frac{27t}{2}+\frac{15}{2}\right)_1, \left(\frac{27t}{2}+\frac{17}{2}\right)_1\right).$$

For $t \equiv 1 \pmod 4, t \geqslant 5$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{37t}{2}+2s+\frac{19}{2}\right)_1, \left(\frac{43t}{2}-2s+\frac{19}{2}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{3}{4},$$

$$\left(0_1, \left(\frac{37t}{2}+2s+\frac{25}{2}\right)_1, \left(\frac{43t}{2}-2s+\frac{21}{2}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{7}{4},$$

$$(0_1, (12t+7)_1, (20t+12)_1), (0_1, (12t+5)_1, (12t+9)_1), (0_1, (12t+6)_1, (12t+8)_1).$$

For $t \equiv 3 \pmod 4, t \geqslant 3$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{37t}{2}+2s+\frac{19}{2}\right)_1, \left(\frac{43t}{2}-2s+\frac{19}{2}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{5}{4},$$

$$\left(0_1, \left(\frac{37t}{2}+2s+\frac{25}{2}\right)_1, \left(\frac{43t}{2}-2s+\frac{21}{2}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{9}{4},$$

$$(0_1, (12t+6)_1, (20t+11)_1), (0_1, (20t+9)_1, (20t+13)_1), (0_1, (12t+5)_1, (12t+8)_1),$$

$$(0_1, (12t+7)_1, (12t+9)_1).$$

The remaining triples in the cycle of length M are from a cyclic DTS(M).

Let $M = 12t+4$ and $k = 6r+5$, $k \geqslant 11$.

For t even, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right), ((6t+2)_1, 0_0, 0_1), (0_1, (3t+1)_1, (6t+2)_1).$$

For t odd, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right), ((6t+2)_1, 0_0, 0_1), ((3t+1)_1, 0_1, (6t+2)_1).$$

For $t = 0, k \equiv 5 \pmod{12}$, also included are the following and their reverses:

$$(0_0, 1_1, (k+2)_1),$$

$$\left(0_1, \left(\frac{2k}{3}+s+\frac{8}{3}\right)_1, \left(\frac{4k}{3}-s-\frac{2}{3}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{k}{3} - \frac{11}{3},$$

$$\left(0_1, \left(\frac{4k}{3}+s+\frac{7}{3}\right)_1, (2k-s-2)_1\right) \quad \text{for } s=0,1,\ldots,\frac{k}{3}-\frac{11}{3},$$

$$\left(0_1, k_1, \left(\frac{5k}{3}+\frac{2}{3}\right)_1\right), \left(0_1, \left(\frac{2k}{3}+\frac{5}{3}\right)_1, \left(\frac{4k}{3}+\frac{4}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{4k}{3}+\frac{1}{3}\right)_1, (2k-1)_1\right), \left(0_1, (k+2)_1, \left(\frac{5k}{3}-\frac{1}{3}\right)_1\right).$$

For $t=0, k \equiv 11 \pmod{12}$, also included are the following and their reverses:

$$\left(0_0, 1_1, \left(\frac{5k}{3}+\frac{2}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{4k}{3}+s+\frac{4}{3}\right)_1, (2k-s-2)_1\right) \quad \text{for } s=0,1,\ldots,\frac{k}{3}-\frac{11}{3},$$

$$\left(0_1, \left(\frac{2k}{3}+s+\frac{8}{3}\right)_1, \left(\frac{4k}{3}-s-\frac{5}{3}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{k}{3}-\frac{11}{3},$$

$$\left(0_1, k_1, \left(\frac{5k}{3}+\frac{2}{3}\right)_1\right), \left(0_1, \left(\frac{4k}{3}-\frac{2}{3}\right)_1, (2k-1)_1\right),$$

$$\left(0_1, \left(\frac{2k}{3}+\frac{5}{3}\right)_1, \left(\frac{4k}{3}+\frac{1}{3}\right)_1\right), \left(0_1, (k+1)_1, \left(\frac{5k}{3}-\frac{4}{3}\right)_1\right).$$

For $t \geqslant 1$, also included are the following and their reverses:

$$\left(0_0, (1+s)_1, \left(\frac{N}{2}+6t-s+1\right)_1\right) \quad \text{for } s=0,1,\ldots,3t-1,$$

$$\left(0_0, (6t+s+4)_1, \left(\frac{N}{2}+12t-s+3\right)_1\right) \quad \text{for } s=0,1,\ldots,3t-1,$$

$$(0_0, (3t+1)_1, (6t+3)_1),$$

$$\left(0_1, \left(\frac{N}{6}-2t+s+\frac{8}{3}\right)_1, \left(\frac{N}{3}-4t-s+\frac{1}{3}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{t}{2}-\frac{11}{6},$$

$$\left(0_1, \left(\frac{N}{3}-4t+s+\frac{7}{3}\right)_1, \left(\frac{N}{2}-6t-s-1\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{t}{2}-\frac{17}{6},$$

$$\left(0_1, \left(\frac{5N}{24}-\frac{5t}{2}+s+\frac{17}{6}\right)_1, \left(\frac{7N}{24}-\frac{7t}{2}-s+\frac{7}{6}\right)_1\right) \quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{7t}{2}-\frac{17}{6},$$

$$\left(0_1,\left(\frac{3N}{8}-\frac{9t}{2}+s+\frac{3}{2}\right)_1,\left(\frac{11N}{24}-\frac{11t}{2}-s+\frac{5}{6}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{N}{24}-\frac{7t}{2}-\frac{11}{6},$$

$$\left(0_1,\left(\frac{N}{4}-6t+1\right)_1,\left(\frac{5N}{12}-8t+\frac{2}{3}\right)_1\right),\left(0_1,\left(\frac{5N}{24}-\frac{5t}{2}+\frac{11}{6}\right)_1,\left(\frac{3N}{8}-\frac{9t}{2}+\frac{1}{2}\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{3}-4t+\frac{4}{3}\right)_1,\left(\frac{5N}{12}-5t+\frac{5}{3}\right)_1\right).$$

For $t=1$, also included are the following and their reverses:

$$\left(0_1,(4k-1)_1,\left(\frac{20k}{3}-\frac{4}{3}\right)_1\right),\left(0_1,(4k-3)_1,\left(\frac{20k}{3}-\frac{13}{3}\right)_1\right),$$

$$(0_1,(4k-4)_1,(4k+3)_1),\left(0_1,\left(\frac{20k}{3}-\frac{19}{3}\right)_1,\left(\frac{20k}{3}-\frac{1}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{20k}{3}-\frac{16}{3}\right)_1,\left(\frac{20k}{3}-\frac{7}{3}\right)_1\right),(0_1,(4k-2)_1,(4k)_1),(0_1,(4k+1)_1,(4k+2)_1).$$

For t even, $t\geqslant 2$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-6t+s+2\right)_1,\left(\frac{N}{4}-s+3\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{3t}{2}-1,$$

$$\left(0_1,\left(\frac{5N}{12}-8t+s+\frac{5}{3}\right)_1,\left(\frac{5N}{12}-2t-s+\frac{5}{3}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{3t}{2}-2,$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+s+\frac{8}{3}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-s+\frac{5}{3}\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{3t}{2}-3,$$

$$\left(0_1,\left(\frac{N}{4}-3t+2\right)_1,\left(\frac{5N}{12}-5t+\frac{11}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-\frac{3t}{2}+2\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}+\frac{8}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{5N}{12}-5t+\frac{2}{3}\right)_1,\left(\frac{5N}{12}-5t+\frac{8}{3}\right)_1\right),$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+\frac{2}{3}\right)_1,\left(\frac{5N}{12}-\frac{13t}{2}+\frac{5}{3}\right)_1\right).$$

For $t\equiv 0 \pmod 4), t\geqslant 4$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+3\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s+3\right)_1\right)\quad \text{for } s=0,1,\ldots,\frac{3t}{4}-1,$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+2\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{4}-2,$$

$$\left(0_1,\left(\frac{N}{4}-3t\right)_1,\left(\frac{N}{4}-3t+3\right)_1\right).$$

For $t\equiv 2\ (\mathrm{mod}\ 4), t\geqslant 2$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+3\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s+3\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+2s+2\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-2s\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{N}{4}-3t+1\right)_1,\left(\frac{N}{4}-3t+4\right)_1\right).$$

For t odd, $t\geqslant 3$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{N}{4}-6t+s+2\right)_1,\left(\frac{N}{4}-s+3\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{2}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{5N}{12}-8t+s+\frac{5}{3}\right)_1,\left(\frac{5N}{12}-2t-s+\frac{5}{3}\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{2}-\frac{3}{2},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+s+\frac{5}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}-s+\frac{3}{2}\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{2}-\frac{7}{2},$$

$$\left(0_1,\left(\frac{N}{4}-\frac{9t}{2}+\frac{3}{2}\right)_1,\left(\frac{5N}{12}-\frac{13t}{2}+\frac{13}{6}\right)_1\right),\left(0_1,\left(\frac{N}{4}-\frac{3t}{2}+\frac{5}{2}\right)_1,\left(\frac{N}{4}-\frac{3t}{2}+\frac{7}{2}\right)_1\right).$$

For $t\equiv 1\ (\mathrm{mod}\ 4), t\geqslant 5$, also included are the following and their reverses:

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{7}{6}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-2s+\frac{7}{6}\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{3}{4},$$

$$\left(0_1,\left(\frac{5N}{12}-\frac{13t}{2}+2s+\frac{25}{6}\right)_1,\left(\frac{5N}{12}-\frac{7t}{2}-2s+\frac{13}{6}\right)_1\right)\quad\text{for } s=0,1,\ldots,\frac{3t}{4}-\frac{7}{4},$$

$$\left(0_1,\left(\frac{N}{4}-3t+2\right)_1,\left(\frac{5N}{12}-5t+\frac{11}{3}\right)_1\right),\left(0_1,\left(\frac{N}{4}-3t\right)_1,\left(\frac{N}{4}-3t+4\right)_1\right),$$

$$\left(0_1,\left(\frac{N}{4}-3t+1\right)_1,\left(\frac{N}{4}-3t+3\right)_1\right).$$

For $t \equiv 3 \pmod 4, t \geqslant 3$, also included are the following and their reverses:

$$\left(0_1, \left(\frac{5N}{12} - \frac{13t}{2} + 2s + \frac{7}{6}\right)_1, \left(\frac{5N}{12} - \frac{7t}{2} - 2s + \frac{7}{6}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{5}{4},$$

$$\left(0_1, \left(\frac{5N}{12} - \frac{13t}{2} + 2s + \frac{25}{6}\right)_1, \left(\frac{5N}{12} - \frac{7t}{2} - 2s + \frac{13}{6}\right)_1\right) \quad \text{for } s = 0, 1, \ldots, \frac{3t}{4} - \frac{9}{4},$$

$$\left(0_1, \left(\frac{N}{4} - 3t + 1\right)_1, \left(\frac{5N}{12} - 5t + \frac{8}{3}\right)_1\right), \left(0_1, \left(\frac{5N}{12} - 5t + \frac{2}{3}\right)_1, \left(\frac{5N}{12} - 5t + \frac{14}{3}\right)_1\right),$$

$$\left(0_1, \left(\frac{N}{4} - 3t\right)_1, \left(\frac{N}{4} - 3t + 3\right)_1\right), \left(0_1, \left(\frac{N}{4} - 3t + 2\right)_1, \left(\frac{N}{4} - 3t + 4\right)_1\right).$$

The remaining triples in the cycle of length M are from a cyclic DTS(M). □

6. $M \equiv 7 \pmod{12}$

Lemma 5. *If $v = M + N, N = kM, M \equiv 7 \pmod{12}$ and $k \equiv 2 \pmod 6$, there exists a DTS(v) which admits a bicyclic antiautomorphism where M and N are the lengths of the cycles.*

Proof. Let $M = 12t + 7$.

For $k = 12r + 2$, the base triples include the following and their reverses:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right),$$

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t,$$

$$\left(0_0, (6t + s + 4)_1, \left(\frac{N}{2} + 6t - s + 3\right)_1\right) \quad \text{for } s = 0, 1, \ldots, 3t.$$

For $r + t$ even, also included are the following and its reverse:

$$\left(0_0, (9t+5)_1, \left(\frac{N}{2} + 3t + 2\right)_1\right).$$

The remaining triples in the cycle of length N are formed using an $(A, 24rt + 14r + 2t + 1)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $r + t$ odd, also included are the following and its reverse:

$$\left(0_0, (3t+2)_1, \left(\frac{N}{2} - 3t - 2\right)_1\right).$$

The remaining triples in the cycle of length N are formed using a $(B, 24rt + 14r + 2t + 1)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $k = 12r + 8$, the base triples include the following:

$$\left(0_1, 0_0, \left(\frac{N}{2}\right)_1\right).$$

Also included are the following and their reverses:

$$\left(0_0, (s+1)_1, \left(\frac{N}{2} - s - 1\right)_1\right) \qquad s = 0, 1, \ldots, 3t,$$

$$\left(0_0, (6t+s+4)_1, \left(\frac{N}{2} + 6t - s + 3\right)_1\right) \qquad \text{for}\, s = 0, 1, \ldots, 3t.$$

For $r + t$ even, also included are the following and its reverse:

$$\left(0_0, (9t+5)_1, \left(\frac{N}{2} + 3t + 2\right)_1\right).$$

The remaining triples in the cycle of length N are formed using an $(A, 24rt + 14r + 14t + 8)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M).

For $r + t$ odd, also included are the following and its reverse:

$$\left(0_0, (3t+2)_1, \left(\frac{N}{2} - 3t - 2\right)_1\right).$$

The remaining triples in the cycle of length N are formed using a $(B, 24rt + 14r + 14t + 8)$-system. The triples in the cycle of length M are from the corresponding DTS(M) of a cyclic STS(M). □

7. Conclusion

By the lemmas in the previous sections and [4], we have the following theorem.

Theorem 6. *There exists a DTS(v) which admits a bicyclic antiautomorphism where $v = M + N, N > M, M$ and N being the lengths of the cycles if and only if $M = kN$ and*

(1) $M \equiv 1$ *or* $7 \pmod{12}$ *and* $k \equiv 2 \pmod 6$,
(2) $M \equiv 3$ *or* $5 \pmod 6$ *and* $k = 2$, *or*
(3) $M \equiv 4 \pmod{12}$ *and* $k \equiv 2 \pmod 3$.

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