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# Note

## A Gauss–Lucas Type Theorem on the Location of the Roots of a Polynomial

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In this note, we prove a geometrical relationship between the zeros of a polynomial $p$ of order $m$, say, and the zeros of another polynomial which is derived from $p$ by multiplying each of $p$'s coefficients, call them $\{\alpha_k\}_{k=0}^m$, by a power of $k$ or by $k^2+2k\lambda$ for $\lambda>0$. 

The Gauss–Lucas Theorem (see, for instance, [1]) states that the zeros of the derivative of a polynomial have to lie in the convex hull of the zeros of the polynomial itself. In this note we establish a similar relationship between the zeros of a polynomial $p$ of degree $m$ which is expressed as a linear combination of certain basis polynomials $\varphi_k$, with $\varphi_k$ of degree $k$, $k=0, 1, ..., m$, that span the space of all polynomials of degree $m$, and the zeros of a polynomial $q$ which is obtained from $p$ by multiplying the coefficient of $\varphi_k$ by $k^2$ for all $k$, or by $k^2+2k\lambda$ for positive $\lambda$ for all $k$.

When the basis functions are the Chebyshev polynomials, a result, which is also useful for relating the zeros of the second derivative of an even trigonometric polynomial to the zeros of the polynomial itself, is the following.



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THEOREM 1. *Let* $p=\sum_{k=0}^{m}\alpha_k T_k$ *be a polynomial, written in the Chebyshev basis, and let* $q=\sum_{k=0}^{m}\alpha_k k^2 T_k$. *Then, if* 0, 1, *or* $-1$, *is a zero of* $q$, *it has to lie in the convex hull* $\mathscr{P}'$ *of the zeros of* $p'$, *and therefore in the convex hull* $\mathscr{P}$ *of the zeros of* $p$. *If* $x_0 \notin \{0, 1, -1\}$ *is a zero of* $q$, *it has to lie in the convex hull of* $\mathscr{P}' \cup \{x_0^{-1}\}$, *and therefore in the convex hull of* $\mathscr{P} \cup \{x_0^{-1}\}$.

*Remark* 1. In many cases, the requirement that $x_0$ be in the convex hull of $\mathscr{P}' \cup \{x_0^{-1}\}$ already means that $x_0$ has to be in $\mathscr{P}'$, e.g., if

(i) $x_0^{-1} \in \mathscr{P}'$, or

(ii) the line segment connecting $x_0$ and $x_0^{-1}$ intersects $\mathscr{P}'$, or

(iii) the line through $x_0$ and $x_0^{-1}$ does not intersect $\mathscr{P}'$, or

(iv) the ray from $x_0$ through $x_0^{-1}$ intersects $\mathscr{P}'$.

*Proof.* We note first that by the Gauss–Lucas Theorem $\mathscr{P}' \subset \mathscr{P}$. (This fact has already been used twice in the statement of the theorem.) Now, by the differential equation

$$(1-x^2)\, T_k''(x) - xT_k'(x) + k^2 T_k(x) = 0$$

which is satisfied by the Chebyshev polynomials, it is true that

$$\sum_{k=0}^{m} \alpha_k k^2 T_k(x) = (x^2-1) \sum_{k=0}^{m} \alpha_k T_k''(x) + x \sum_{k=0}^{m} \alpha_k T_k'(x),$$

and we therefore have

$$q(x) = (x^2-1)\, p''(x) + xp'(x). \tag{1}$$

Suppose that $x_0 = 0$ is a zero of $q$. Then, by (1), $p''(x_0) = 0$, whence 0 is in the convex hull of the zeros of $p''$, and therefore it is in $\mathscr{P}'$, by the Gauss–Lucas Theorem, as required. Suppose that $x_0 = 1$ or $x_0 = -1$ is a zero of $q$. Then, again by (1), $p'(x_0) = 0$, which implies $x_0 \in \mathscr{P}'$. In all other cases, $q(x_0) = 0$ implies

$$\frac{p''(x_0)}{p'(x_0)} = \frac{x_0}{1-x_0^2}, \tag{2}$$

where we assume that $x_0$ is not already a zero of $p'$, because in that case the result follows immediately. We can rewrite (2) as

$$\sum_{j=1}^{m-1} \frac{1}{x_0 - x_j} = \frac{x_0}{1-x_0^2},$$

where $\{x_j \mid 1 \leqslant j \leqslant m-1\}$ are the zeros of $p'$ (multiple zeros being counted multiply). Hence,

$$\sum_{j=1}^{m-1} \frac{\overline{x_0 - x_j}}{|x_0 - x_j|^2} = \frac{1}{x_0^{-1} - x_0} = \frac{\overline{x_0^{-1} - x_0}}{|x_0 - x_0^{-1}|^2},$$

and therefore

$$\sum_{j=1}^{m} \frac{x_0 - x_j}{|x_0 - x_j|^2} = 0, \tag{3}$$

where we let $x_m := x_0^{-1}$. Now let us define

$$\mu_j := \frac{|x_0 - x_j|^{-2}}{\sum_{l=1}^{m} |x_0 - x_l|^{-2}}$$

for all $1 \leqslant j \leqslant m$. Then (3) implies

$$x_0 = \sum_{j=1}^{m} \mu_j x_j, \tag{4}$$

where

$$\sum_{j=1}^{m} \mu_j = 1 \qquad \text{and} \qquad \mu_j > 0 \qquad \text{for all } j. \tag{5}$$

Expressions (4) and (5) imply the theorem. ∎

*Remark* 2. The assertion of the theorem remains true if we replace Chebyshev polynomials by any ultraspherical polynomials $P_k^{(\lambda)}$, where $q$ now becomes $q = \sum_{k=0}^{m} \alpha_k k(k+2\lambda)\, P_k^{(\lambda)}$ and where $\lambda$ is a positive constant.

COROLLARY. *Let $p$ and $q$ be as in the statement of the theorem or of Remark 2. Then the following statements are valid:*

(i) *If all the roots of $p$ are real, so are the roots of $q$.*

(ii) *If all the roots of $p$ are in the upper (lower) half-plane, then so are the roots of $q$.*

(iii) *If all the roots of $p$ are inside a closed disk $\mathscr{D}$ about the origin of radius $r \geqslant 1$, so are the roots of $q$.*

*Proof.* We prove (i): If the roots of $p$ are real, then $\mathscr{P}'$ is a subset of the real line. Suppose $q(x_0) = 0$. If $x_0$ is real, we are done. Otherwise $x_0^{-1}$ lies in the other half-plane than $x_0$, i.e., the imaginary parts of $x_0$ and $x_0^{-1}$ have opposite signs, thus contradicting the theorem. The second claim is established in a similar way as is the first one. We prove the last claim. Suppose $x_0$ is a root of $q$. If it is inside the closed disk $\mathscr{D}$, there is nothing to prove. Otherwise, $x_0^{-1}$ will be inside $\mathscr{D}$, and so $x_0$ cannot be in the convex hull of $\mathscr{P}' \cup \{x_0^{-1}\}$, thus contradicting the assertion of our theorem. The corollary is proved. ▮

In case $p$ is expressed as a linear combination of monomials, which can be considered as the limiting case of the one studied in Remark 2 for $\lambda \to \infty$, we have the following result.

THEOREM 2. *Let* $p(x) = \sum_{k=0}^{m} \alpha_k x^k$ *and* $q_n(x) = \sum_{k=0}^{m} \alpha_k k^n x^k$ *for a positive integer n. Then all zeros of* $q_n$ *lie in the convex hull of* $\mathscr{P}' \cup \{0\}$.

*Proof.* We argue inductively, using the simple identity

$$q_n(x) = x q'_{n-1}(x) \tag{6}$$

which is true for positive $n$. For $n = 1$, the assertion of the theorem follows directly from (6) because $q_0 = p$ and therefore $q'_0 = p'$. If the assertion is true for $q_{n-1}$, then (6) and the Gauss–Lucas theorem imply that it also holds for $q_n$. The theorem is proved. ▮

## REFERENCE

1. G. PÓLYA AND G. SZEGŐ, "Problems and Theorems in Analysis," Vol. 1, p. 108, Springer-Verlag, Berlin/Heidelberg/New York, 1972.

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