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About a question concerning $\mathbb{Q}$-groups

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Abstract

A $\mathbb{Q}$-group is a finite group all of whose ordinary complex representations have rationally valued characters. Let G be a solvable $\mathbb{Q}$-group so that the Schur index $m_{\mathbb{R}}(\chi)=1$ for all $\chi\in \mathrm{Irr}(G)$. In [3, Note 1, p. 285] Gow asks if not, under these conditions, already $m_{\mathbb{Q}}(\chi)=1$ for all $\chi\in\mathrm{Irr}(G)$. In this paper we shall prove that the answer of this question is positive. The notations and definitions will be those of [6]. © 1999 Elsevier Science B.V. All rights reserved.

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A $\mathbb{Q}$-group is a finite group all of whose ordinary complex representations have rationally valued characters. Let G be a solvable $\mathbb{Q}$-group so that the Schur index $m_{\mathbb{R}}(\chi)=1$ for all $\chi\in\mathrm{Irr}(G)$. In [3, Note 1, p. 285] Gow asks if not, under these conditions, already $m_{\mathbb{Q}}(\chi)=1$ for all $\chi\in\mathrm{Irr}(G)$. In this paper we shall prove that the answer of this question is positive. The notations and definitions will be those of [6].

The following theorem is a version of Brauer–Witt theorem, obtained from maximality arguments of a kind used by Benard in [1, Section 3].

Theorem 1 (Benard [1], Gow [5]). *Let χ be an irreducible character such that $m_{\mathbb{R}}(\chi)=1$ of the $\mathbb{Q}$-group G. There is a subgroup W of G and a real-valued irreducible character φ of W for which (φ,χ_W) is odd and $[\mathbb{Q}(\varphi):\mathbb{Q}]$ is odd. W can be taken either to be a Sylow 2-subgroup of G or to have the form AH, where A is a cyclic subgroup of odd order generated by an element a and H is a Sylow*

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2-subgroup of $N_G(\langle a\rangle)$. In the second case it can be assumed that A is not in the kernel of φ.

Theorem 2 (Gow [3, Theorem 1, Note 1]). *Let G be a solvable $\mathbb{Q}$-group such that $m_{\mathbb{R}}(\chi)=1$ for all $\chi \in \mathrm{Irr}(G)$. Then $|G|=2^a 3^b$.*

Theorem 3. *Suppose G is a 2-group. If $\chi \in \mathrm{Irr}(G)$ is rational valued and has Schur index $m_{\mathbb{R}}(\chi)=1$, then $m_{\mathbb{Q}}(\chi)=1$.*

Proof. Since $m_{\mathbb{Q}_p}(\chi)=1$ for all prime numbers $p \neq 2$ and $m_{\mathbb{R}}(\chi)=1$, by Minkowski–Hasse theorem and using the Hilbert symbol, it follows that $m_{\mathbb{Q}}(\chi)=1$ (see [2]). □

Main Theorem. *Suppose G is a solvable $\mathbb{Q}$-group and $m_{\mathbb{R}}(\chi)=1$ for all $\chi \in \mathrm{Irr}(G)$. Then $m_{\mathbb{Q}}(\chi)=1$ for all $\chi \in \mathrm{Irr}(G)$.*

Proof. By Theorem 2 we have $|G|=2^c 3^b$. If $|G|=2^c$ the statement follows by Theorem 3. Let $\chi \in \mathrm{Irr}(G)$ and φ, W, A, H, a be as in Theorem 1. Let K be a field such that $\mathbb{Q}(\varphi) \subseteq K \subseteq \mathbb{Q}(\varepsilon)$, for ε a primitive $|G|$th root of unity such that $|\mathbb{Q}(\varepsilon):K|$ is the 2-part of $|\mathbb{Q}(\varepsilon):\mathbb{Q}(\varphi)|$. Let $H_0=C_H(a)$.

Remark 1. By the choice of K and since $|\mathbb{Q}(\varphi):\mathbb{Q}|$ is odd, $|K:\mathbb{Q}|$ is odd.

Remark 2. Since G is a $\mathbb{Q}$-group, $N_G(A)/C_G(A) \simeq \mathrm{Aut}(A)$, hence a Sylow 2-subgroup of $\mathrm{Aut}(A)$ is isomorphic to H/H_0. Let ω be a primitive $|A|$th root of unity. By the choice of K, a^s, a^t are conjugate in H only if ω^s, ω^t are conjugate under the action of $\mathrm{Gal}(K(\varepsilon), K)$. Hence $W=AH$ is K-elementary with respect to 2.

Case 1: W is a Sylow 2-subgroup of G. Since φ is a character of a 2-group and $[K:\mathbb{Q}]$ is odd by Remark 1, φ is $\mathbb{Q}$-valued.

Since (φ, χ_W) is odd and $m_{\mathbb{R}}(\chi)=1$ by Brauer–Speiser theorem (which affirms that if $\xi \in \mathrm{Irr}(G)$ is real valued then $m_{\mathbb{Q}}(\xi) \leq 2$, see [6, p. 171]) it follows that $m_{\mathbb{R}}(\varphi)=1$. Then, by Theorem 3, $m_{\mathbb{Q}}(\varphi)=1$, so that $m_{\mathbb{Q}}(\chi)=1$.

Case 2: $W=AH$, where A is nontrivial.

(a) If $m_K(\varphi)=1$ then $m_{\mathbb{Q}}(\chi)=1$.

Proof. Since (χ, φ^G) is odd, it follows that $m_K(\chi)=1$. It follows from [6, Lemma 10.4, p. 162] for $F=\mathbb{Q}$ that $m_{\mathbb{Q}}(\chi)$ divides $m_{\mathbb{Q}}(\varphi)|\mathbb{Q}(\varphi):\mathbb{Q}|(\chi, \varphi^G)$. Now, $|\mathbb{Q}(\varphi):\mathbb{Q}|$ divides $|K:\mathbb{Q}|$, which is odd. In view of the Brauer–Speiser theorem and the fact that $|\mathbb{Q}(\varphi):\mathbb{Q}|(\chi, \varphi^G)$ is odd, the previous relation implies that $m_{\mathbb{Q}}(\chi)$ divides $m_{\mathbb{Q}}(\varphi)=m_{\mathbb{Q}(\varphi)}(\varphi)=m_K(\varphi)=1$ (see also [6, p. 161]). □

Remark 3. In the sequel we shall determine a splitting field for φ. Let X be a $\mathbb{C}$-representation of φ. By [6, Lemma 2.19, p. 23], $g \in \ker(X)$ if and only if $g \in \ker(\varphi)$.

So, if Y is a $\mathbb{C}$-representation of $\varphi/\ker(\varphi)$ then $X(kg)=Y(g)$ is a $\mathbb{C}$-representation of φ. Hence, $m_F(\varphi)=m_F(\varphi/\ker(\varphi))$ for any complex number field F. Also, in the sequel χ appears only by χ_W and the used properties of φ and χ_W do not fail true factorization by $\ker(\varphi)$. Thus we can suppose that φ is faithful.

(b) By Remark 2 and Theorem 4.3 of [1] we have that:
(1) there is $\lambda\times\mu\in\mathrm{Irr}(AH_0)$ such that $(\lambda\times\mu)^{AH}=\varphi$.
(2) μ is rational valued and is H invariant.
(3) λ and μ are faithful.
(c) Suppose that there is $\tau\in\mathrm{Irr}(H)$ an extension of μ. If $m_{\mathbb{Q}}(\tau)=1$ then $m_{\mathbb{Q}(\tau)}(\chi)=1$ and if besides $\mathbb{Q}(\tau)=\mathbb{Q}$ then $m_{\mathbb{Q}}(\chi)=1$.

Proof. Let τ be an extension of μ to H. We have that $m_{\mathbb{Q}(\tau)}(\varphi)$ divides $m_{\mathbb{Q}(\tau)}(\tau)$ $|\mathbb{Q}(\varphi,\tau):\mathbb{Q}(\tau)|(\varphi,\tau^W)$ by [6, Lemma 10.4]. It follows from Mackey's theorem that $\varphi_H=\mu^H$. Hence $(\varphi,\tau^W)=(\varphi_H,\tau)=(\mu^H,\tau)=(\mu,\tau_{H_0})=1$. Now suppose that $m_{\mathbb{Q}(\tau)}(\tau)=m_{\mathbb{Q}}(\tau)=1$. Then since $m_{\mathbb{Q}}(\chi)$ divides $m_{\mathbb{Q}}(\varphi)$ and by the previous relation we have that $m_{\mathbb{Q}(\tau)}(\chi)$ divides $|\mathbb{Q}(\varphi,\tau)|$. The latter number is odd by assumption, so the Brauer–Speiser theorem implies that $m_{\mathbb{Q}(\tau)}(\chi)=1$. □

(d) There is an extension τ of μ to H and if $m_{\mathbb{Q}}(\mu)=1$ then $m_{\mathbb{Q}}(\tau)=1$.

Proof. Since H/H_0 is cyclic ($|H/H_0|=2$. since $|a|=3^d$) by [6, Theorem 1.22, p. 186] it follows that there is an extension τ of μ to H. Since $(\tau_{H_0},\mu)=1$ if $\mathbb{Q}$ is a splitting field for μ it follows that $m_{\mathbb{Q}}(\tau)=1$. □

(e) $m_{\mathbb{Q}}(\mu)=1$ iff τ is real valued.

Proof. Let $1,\ t\in H$ be a transversal of H_0 in H such that $t^2\in H_0$ and $tat^{-1}=a^{-1}$. Since φ is real valued and $m_{\mathbb{R}}(\varphi)=1$, the Frobenius–Schur invariant $\nu_2(\varphi)=\nu_2((\lambda\times\mu)^{AH})=1$.

A direct computation shows that

$$\nu_2((\lambda\times\mu)^{AH})=(1/|H_0|)\sum_{h\in H_0}\mu((ht)^2).$$

Then,

$$\nu_2(\tau)=(1/|H|)\sum_{x\in H}\tau(x^2)=(1/|H|)\sum\mu(x^2)=\left(1/(2|H_0|)\left(\sum_{h\in H}\mu(h^2)+\mu((ht)^2)\right)\right).$$

Hence $\nu_2(\tau)$ is 0 or 1. □

(f) To compute the Schur index of $(\lambda\times\mu)^{AH}$ over the field K, we can assume that $(\lambda\times\mu)^{AH}$ is K-primitive (i.e. is not induced from a K-valued character ν of a proper subgroup AH' of AH). Indeed, if there is a K-valued irreducible character φ_1

of a proper subgroup W_1 of W such that $\varphi_1^W = \varphi$ then by the general properties of the Schur index $m_K(\varphi) = m_K(\varphi_1)$.

Then, applying Theorem 4.6 of [1] it follows that $H_0' = H_0 \cap H'$ is a dihedral or quaternion group of order 8, or $|H_0'| \le 2$.

(g) If $|H_0'| \le 2$ then μ is linear. If $|H_0'| = 1$, or $H' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ there is a rational-valued extension τ of μ such that $m_{\mathbb{Q}}(\tau) = 1$ and the statement follows by (c).

If $|H_0'| = 2$, since $v_2(\varphi) = 1$ and μ is linear, an element of $H' - H_0'$ which inverts a is an involution. Thus $H' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(h) Let H_0 be dihedral or quaternion group. Then $|H| = 16$. They are nine nonabelian groups of order 16.

(1) $H = H_1 \times H_2$, where H_1 is the dihedral group of order 8 and $|H_2| = 2$. Then τ is rationally valued and $m_{\mathbb{Q}}(\tau) = 1$ and the statement follows by (c).

(2) $H = H_1 \times H_2$, where H_1 is the quaternion group of order 8 and $|H_2| = 2$. Then $v_2(\varphi) = -1$. Thus this case is impossible.

(3) H is the generalized quaternion group. Then $v_2(\varphi) = -1$. Hence this case is impossible.

(4) H is the semidihedral group and H_0 is dihedral. Then $v_2(\varphi) = -1$, impossible.

(5) H is semidihedral and H_0 is quaternionic. Then τ extends to a faithful character τ of H with $\mathbb{Q}(\tau) = \mathbb{Q}(i\sqrt{2})$ and $m_{\mathbb{Q}}(\tau) = 1$. Since $\mathbb{Q}_3 = \mathbb{Q}_3(i\sqrt{2})$ it follows by (c) that $m_{\mathbb{Q}_3}(\varphi) = 1$. Since $m_{\mathbb{Q}_\infty}(\varphi) = 1$, by Hasse's theorem (see [2]) it follows that $m_{\mathbb{Q}}(\varphi) = 1$.

(6) $H = \langle \{u, v, w \mid u^4 = v^2 = w^2 = 1,\ uv = vu,\ uw = wu,\ vwv = wu^2\} \rangle$ and H_0 is dihedral. Then $v_2(\varphi) = -1$.

(7) $H = H_0\langle x \rangle$, with $H_0 = \langle u, v \rangle$ dihedral, $u^4 = v^2 = 1$, $x^2 = u$ and $x^{-1}vx = uv$. Then $v_2(\varphi) = -1$.

(8) $H = \langle \{u, v, w \mid u^4 = v^2 = w^2 = 1,\ uv = vu,\ uw = wu,\ vwv = wu^2\} \rangle$ and H_0 is quaternionic or H is generalized dihedral and H_0 is dihedral. In both cases $m_{\mathbb{Q}}(\varphi) = 2$ but we shall prove using methods of [4], that these cases are impossible.

Let $|A| = 2n + 1$ and $\beta_1, \ldots, \beta_n$ be the characters of the n nontrivial irreducible real representations of A. Then these characters can be extended to n characters $\sigma_1, \ldots, \sigma_n$ of real representations of W. To prove that, let ω a primitive $(2n+1)$ root of unity. Then $\beta_i(a^j) = \omega^{ij} + \omega^{-ij}$. Take α_i to be the complex linear character of A defined by $\alpha_i(a^j) = \omega^{ij}$. Each α_i may be extended to an irreducible character γ_i of AH_0 by putting $\alpha_i(a^j h) = \alpha_i(a^j)$, for every h in H_0. The n induced characters $\sigma_i = \gamma_i^W$ are real-valued irreducible characters of W which extend the β_i. Now it is easy to show that σ_i are realizable in $\mathbb{R}$ by computing the Frobenius–Schur invariant $v_2(\sigma_i)$.

Let $\theta_i = \sigma_i^G$. Clearly θ_i are rational valued. Let Φ be an irreducible constituent of θ_i. Since $m_{\mathbb{R}}(\Phi) = 1$, Φ must occur with even multiplicity, hence $\theta_i(a) \equiv 0 \pmod 2$.

Let D be the $n \times n$ matrix with entries $d_{ij} = \beta_j(a^i)$. D is a $n \times n$ submatrix of the real character table of A. Using the orthogonality relations we may show that $(\det D)^2 = |A|^{n-1}$.

Let $\mathbb{Q}_{3^b}$ the field obtained by adjoining a primitive 3^b root of unity to $\mathbb{Q}$. Let S be the ring of algebraic integers in $\mathbb{Q}_{3^b}$ and let P be a maximal ideal of S containing 2.

Then there is a matrix E with entries in $\mathbb{Q}_{3^b}$ such that $DE = \alpha I$, where $\alpha \in \mathbb{Q}_{3^b}$, and α is not congruent to $0 \pmod{P}$. Set $\rho = \sum_{i=1}^{n} e_i\theta_i$, where $e_1, \ldots, e_n$ are the entries of the first column of E. From the definition of induced characters, we have

$$\rho(a) = (1/|W|) \sum_{x \in G} \sum_{i} e_i \theta_i(xax^{-1}),$$

where $\theta_i(xax^{-1}) = 0$ if xax^{-1} is not in W. If $xax^{-1} \in W$, then $xax^{-1} \in A$, hence $\theta_i(xax^{-1}) = \beta_i(a^j)$. By the choice of $e_1, \ldots, e_n$, the value of the inner sum in the above formula is 0 unless $a^j = a$ or a^{-1}. It follows that $\rho(a) = \alpha|W : H|$, hence $\rho(a) \not\equiv 0 \pmod{P}$. Since $\theta_i(a) \equiv 0 \pmod{P}$ this is impossible. □

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