On the Dirichlet problem for the prescribed mean curvature equation over general domains

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1. Introduction

In this paper we study and solve the Dirichlet problem for \( n \)-dimensional graphs of prescribed mean curvature in \( \mathbb{R}^{n+1} \): Given a domain \( \Omega \subset \mathbb{R}^n \) and Dirichlet boundary values \( g \in C^0(\partial \Omega, \mathbb{R}) \) we want to find a solution \( f \in C^2(\Omega, \mathbb{R}) \cap C^0(\bar{\Omega}, \mathbb{R}) \) of

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = nH(x, f) \quad \text{in} \ \Omega, \quad f = g \quad \text{on} \ \partial \Omega.
\]

The given function \( H: \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) is called the prescribed mean curvature. At each point \( x \in \Omega \) the geometric mean curvature of the graph \( f \), defined as the average of the principal curvatures, is equal to the value \( H(x, f(x)) \), thus a solution \( f \) is also called a graph of prescribed mean curvature \( H \).

For the minimal surface case, i.e. \( H \equiv 0 \), it is known that the mean convexity of the domain \( \Omega \) yields a necessary and sufficient condition for the Dirichlet problem to be solvable for all Dirichlet boundary values (see [5]). Here, mean convexity means that \( \hat{H}(x) \geq 0 \) for the mean curvature of \( \partial \Omega \) w.r.t. the inner normal. For the prescribed mean curvature case, a stronger assumption is needed on the domain \( \Omega \) in order to solve the boundary value problem for all Dirichlet boundary values \( g \). A necessary condition on the domain \( \Omega \) and the prescribed mean curvature \( H \) is

\[
|H(x, z)| \leq \frac{n-1}{n} \hat{H}(x) \quad \text{for} \ (x, z) \in \partial \Omega \times \mathbb{R}
\]

(see [3, Corollary 14.13] and [12, formula (93)]). Additionally requiring a smallness condition on \( H \) implying the existence of a \( C^0 \)-estimate (such as [3, (10.32)]) Gilbarg and Trudinger could then solve the Dirichlet problem in case \( H = H(x) \) (see [3, Theorem 16.9]).
It is now a natural question to ask if we can relax the mean convexity assumption (2) if we only consider certain boundary values, for example zero boundary values. This is indeed possible, as our first existence result demonstrates.

**Theorem 1.** Assumptions:

a) Let the bounded $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ satisfy a uniform exterior sphere condition of radius $r > 0$ and be included in the annulus $\{x \in \mathbb{R}^n : r < |x| < r + d\}$ for some constant $d > 0$.

b) Let the prescribed mean curvature $H = H(x, z) \in C^{1+\alpha}(\bar{\Omega} \times \mathbb{R})$ satisfy $H \geq 0$ and the smallness assumption

$$h := \sup_{x \in \Omega} |H(x, 0)| < \frac{2(2r)^{n-1}}{(2r + d)^n - (2r)^n}. \quad (3)$$

Then the Dirichlet problem (1) has a unique solution $f \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R})$ for zero boundary values.

For dimension $n = 2$ and constant mean curvature, similar existence theorems, again for zero boundary values, can be found in [8,9] or [10]. Note that Theorem 1 can be applied in particular to the annulus $\Omega := \{x \in \mathbb{R}^n : r < |x| < r + d\}$ which does not satisfy the mean convex assumption (2). Given any bounded $C^2$-domain $\Omega$ we can find constants $r > 0$ and $d > 0$ such that assumption a) of Theorem 1 is satisfied for a suitable translation of $\Omega$.

The uniqueness part of Theorem 1 follows directly from the assumption $H \geq 0$ together with the maximum principle. However, $H \geq 0$ is not only needed for the uniqueness but also for the existence of a solution. More precisely, it is needed to obtain a global gradient estimate for solutions of Dirichlet problem (1) (see Theorem 4).

The smallness condition (3) is required for two reasons: first to obtain an estimate of the solution while (3) does. This follows from a comparison with spherical caps of constant mean curvature [1].

Furthermore, the smallness condition on $h$ in Theorem 1 cannot solely depend on the radius $r$ of the exterior sphere condition.

Note that some kind of smallness assumption on $h$ in Theorem 1 is needed since there exists the following necessary condition: If there exists a graph of constant mean curvature $h > 0$ over a domain $\Omega$ containing a disc of radius $\rho > 0$, then we have necessarily $h < \frac{\rho}{2}$. This follows from a comparison with spherical caps of constant mean curvature $\frac{\rho}{2}$, together with the maximum principle (for dimension $n = 2$ compare [2, Corollary on page 156] and the references given there).

Consequently, the smallness condition on $h$ in Theorem 1 cannot solely depend on the radius $r$ of the exterior sphere condition.

**Corollary 1.** Let a bounded convex $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ be given such that $\bar{\Omega}$ is included within the strip $\{x \in \mathbb{R}^n : 0 < x_1 < d\}$ of width $d > 0$. Let the prescribed mean curvature $H \in C^{1+\alpha}(\bar{\Omega} \times \mathbb{R})$ satisfy $H \geq 0$ as well as

$$h := \sup_{\bar{\Omega}} |H(x, 0)| < \frac{2}{nd}. \quad (4)$$

Then the Dirichlet problem (1) has a unique solution $f \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R})$ for zero boundary values.

Note that in Corollary 1 the diameter of the domain $\Omega$ can be arbitrarily large, while in Theorem 1 the diameter is bounded by $2(r + d)$. Additionally, we can choose the volume $|\Omega|$ of the domain $\Omega$ arbitrarily large so that the smallness assumption (4) will not be satisfied.

In case of arbitrary boundary values $g$, Williams [13] could show that the Dirichlet problem (1) for $H \equiv 0$ is still solvable over domains not being mean convex domains, if one requires certain smallness assumptions on $g$. More precisely he showed: For any Lipschitz constant $0 < L < \frac{1}{\sqrt{1 + \epsilon}}$ there exists some $\epsilon = \epsilon(L, \Omega) > 0$ such that the Dirichlet problem (1) is solvable for the minimal surface equation if the boundary values $g$ satisfy

$$|g(x) - g(y)| \leq L|x - y| \quad \text{for } x, y \in \partial \Omega \quad \text{and} \quad |g(x)| \leq \epsilon \quad \text{for } x \in \partial \Omega. \quad (5)$$

Note that the boundary values are only required to be Lipschitz continuous and they are not of class $C^{2+\alpha}$. Hence, also the solution will be at most Lipschitz continuous up to the boundary. For the proof Williams first considers weak solutions of
the minimal surface equation. Constructing suitable barriers he then shows that these weak solutions are continuous up to
the boundary and that the Dirichlet boundary values are attained.

Schulz and Williams [11] generalised the result of Williams [13] from the minimal surface case to the prescribed mean
curvature case \( H = H(x, z) \). However, two more assumptions are needed there: As in Theorem 1, the prescribed mean
curvature function \( H \) must satisfy the monotonicity assumption \( H_z \geq 0 \). This assumption is needed for the existence of
weak solutions (see [7]). Moreover, they require the existence of an initial solution \( f_0 \in C^2(\Omega, \mathbb{R}) \cap C^1(\partial \Omega, \mathbb{R}) \) for Dirichlet
boundary values \( g_0 \), which must be Lipschitz continuous with a Lipschitz constant smaller than \( \frac{1}{\sqrt{n-1}} \).

Using our solution of Theorem 1 and Corollary 1 as an initial solution with zero boundary values, we can apply the
result of Schulz and Williams to solve the Dirichlet problem for Lipschitz continuous boundary values as well:

**Theorem 2.** Let the assumptions of Theorem 1 or Corollary 1 be satisfied. Then for any Lipschitz constant \( 0 < L < \frac{1}{\sqrt{n-1}} \) there exists
some \( \epsilon = \epsilon(\Omega, H, L) > 0 \) such that the Dirichlet problem (1) has a solution \( f \in C^{2+\epsilon}(\Omega, \mathbb{R}) \cap C^0(\partial \Omega, \mathbb{R}) \) for all Lipschitz continuous
boundary values \( g : \partial \Omega \to \mathbb{R} \) satisfying assumption (5).

As demonstrated in [11], the smallness assumption on the Lipschitz constant \( L \) is sharp. In case of the minimal surface
equation, Theorem 2 will be false for any Lipschitz constant \( L > \frac{1}{\sqrt{n-1}} \) and any domain \( \Omega \) which is not mean convex (see
[13, Theorem 4]).

This paper is organized as follows: In Section 2 we first we show that solutions satisfy a height as well as a boundary
gradient estimate. As barriers we use a piece of a rotationally symmetric surface of constant mean curvature \( h \), a so-called
Delaunay nodoid. This surface is constructed in Proposition 1 by solving an ordinary differential equation. There we need
a smallness assumption on \( h \) corresponding to assumption (3) of Theorem 1. In Section 3 we first give a global gradient
estimate in terms of the boundary gradient (see Theorem 4). The monotonicity assumption \( H_z \geq 0 \) plays an important role
there. We then give the proof of Theorem 1 and Corollary 1 using the Leray–Schauder method from [3].

2. Estimates of the height and the boundary gradient

To obtain a priori \( C^0 \) estimates as well as boundary gradient estimates for solutions of problem (1), it is essential to
have certain super and subsolutions at hand serving us upper and lower barriers. In this paper we will use a rotationally
symmetric surface of constant mean curvature \( h \), a so-called Delaunay nodoid. This surface is constructed in Proposition 1 by solving an ordinary differential equation. There we need
a smallness assumption on \( h \) corresponding to assumption (3) of Theorem 1. In Section 3 we first give a global gradient
estimate in terms of the boundary gradient (see Theorem 4). The monotonicity assumption \( H_z \geq 0 \) plays an important role
there. We then give the proof of Theorem 1 and Corollary 1 using the Leray–Schauder method from [3].

**Proposition 1.** Let the numbers \( r > 0 \), \( h \geq 0 \) and \( R > r \) be given satisfying

\[
h < \frac{2(2r)^{n-1}}{(R + r)^n - (2r)^n}.
\]

Then there exists a function \( p \in C^2([r, R], [0, +\infty)) \) with \( p(r) = 0 \) and \( p(t) > 0 \) for \( t \in (r, R) \) such that the rotationally symmetric
graph \( f(x) := p(|x|) \) defined on the annulus \( r \leq |x| \leq R \) has constant mean curvature \(-h\). Furthermore, there exists some \( t_0 \in (r, R) \) such that \( p(t) \) is increasing for \( t \in [r, t_0] \) and decreasing for \( t \in [t_0, R] \).

**Proof.**

1) Inserting \( p(|x|) = f(x) \) into the mean curvature equation

\[
\text{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = -nh
\]

we obtain for \( p \) the second order differential equation

\[
\frac{p''}{(1 + p'^2)^{\frac{3}{2}}} + \frac{(n-1)p'}{t(1 + p'^2)^{\frac{3}{2}}} = -nh.
\]

Multiplying this equation by \( t^{n-1} \) and integrating this yields the first order differential equation

\[
\frac{t^{n-1}p'}{\sqrt{1 + p'^2}} = c - ht^n
\]
where \( c \in \mathbb{R} \) is some integration constant serving as a parameter. We focus here on the case \( c > 0 \), corresponding to the choice of a nodoid. The case \( c = 0 \) yields a sphere and \( c < 0 \) an unduloid. Solving Eq. (7) for \( p' \) we obtain
\[
p'(t) = \frac{c - h t^n}{\sqrt{\frac{2}{2n-2} - (c - h t^n)^2}}.
\]
(8)

Clearly, (8) is only well defined for those \( t \in (0, +\infty) \) for which the term under the root in the denominator is positive. We will later determine for which \( t \) this is the case. Integrating (8) we can now define
\[
p(t) := \int_1^t \frac{c - h s^n}{\sqrt{\frac{2}{2n-2} - (c - h s^n)^2}} \, ds
\]
with \( p(r) = 0 \).

2) Let us first study the case \( h = 0 \). The denominator of (8) has exactly one zero \( a > 0 \) given as solution of \( a^{n-1} = c \) and \( p'(t) \) is defined for all \( t \in (a, +\infty) \). For the integral (9) to be defined, we need to have that \( r \in (a, +\infty) \), which is equivalent to \( c < r^{n-1} \). For example, we can set \( c := \frac{1}{2}r^{n-1} \). The function \( p(t) \) is now defined for all \( t \in [r, +\infty) \) and also \( p'(t) > 0 \) for all \( t \in [r, +\infty) \). The claim of the proposition now follows with \( t_0 = R \).

3) In case \( h > 0 \), the denominator of (8) has precisely two positive zeros \( 0 < a < b \) given as solutions of the equations
\[
hr^n + a^{n-1} = c, \quad hb^n - b^{n-1} = c.
\]

Now \( p'(t) \) is defined for all \( t \in (a, b) \) and formally we have \( p'(a) = +\infty, p'(b) = -\infty \). Note that for
\[
t_0 := (ch^{-1})^\frac{1}{n} \in (a, b)
\]
we have
\[
p'(t_0) = 0, \quad p'(t) > 0 \quad \text{for} \quad t \in (a, t_0) \quad \text{and} \quad p'(t) < 0 \quad \text{for} \quad t \in (t_0, b),
\]
as desired. Now for the integral (9) to be defined, we need to have \( a < r < t_0 \), which is equivalent to restricting the parameter \( c \) such that
\[
h r^n < c < h r^n + r^{n-1}.
\]
(10)

We then obtain \( p \in C^2([r, b), \mathbb{R}) \).

4) We will now show the inequality
\[
p'(t_0 - s) > |p'(t_0 + s)| \quad \text{for all} \quad s \in (0, t_0 - a).
\]
(11)

Together with \( p(r) = 0 \) this will yield \( p(t) > 0 \) for all \( t \in (r, r + 2(t_0 - r)) \). In fact, after some computation (11) turns out to be equivalent to
\[
q(t_0 - s) + q(t_0 + s) > 0 \quad \text{for} \quad s \in (0, t_0 - a)
\]
for the function \( q(t) := (c - h t^n)r^{1-n} = c r^{1-n} - h t \). This however is a direct consequence of the inequality
\[
c(t_0 + s)^{1-n} + c(t_0 - s)^{1-n} > 2ht_0
\]
which holds for all \( s \in (0, t_0) \), proving (11).

5) We now set
\[
R' = R'(c) := r + 2(t_0 - r) = 2t_0 - r = 2(ch^{-1})^\frac{1}{n} - r < b.
\]

From 4) we conclude the positivity \( p(t) > 0 \) for all \( t \in (r, R') \). Keeping in mind the restriction (10) on \( c \) we obtain the limit
\[
R'(c) \to 2(r^{n} + h^{-1}r^{n-1})^\frac{1}{n} = 2rc(1 + h^{-1}r^{-1})^\frac{1}{n} - r
\]
if we let \( c \to hr^n + r^{n-1} \). This proves the claim of the proposition whenever
\[
R < 2(1 + h^{-1}r^{-1})^\frac{1}{n} - r
\]
is satisfied. An easy computation, however, asserts that this inequality is indeed equivalent to assumption (6). □

Fig. 1 shows the graph of the function \( p(t) \) for \( n = 2, h = \frac{1}{2}, a = 1 \) and \( b = 4 \).

Remarks.

a) For \( h = 0 \) and \( n = 2 \) the function \( p(t) \) has the explicit form \( p(t) = c \text{arcosh}(t/c) \), i.e. the well known catenary. If either \( h > 0 \) or \( n \geq 3 \) the function \( p(t) \) can only be represented by the elliptic integral given in the proof of Proposition 1.

Lemma 1. Consider the annulus \( \Omega := \{ x \in \mathbb{R}^n : \varepsilon < |x| < 1 \} \). Then given any constant \( h > 0 \) there exists some \( \varepsilon = \varepsilon(h) \in (0, 1) \), such that a graph \( f \in C^2(\Omega, \mathbb{R}) \cap C^0(\partial \Omega, \mathbb{R}) \) of constant mean \( h \) with zero boundary values does not exist.

Proof. We will show that such a graph of constant mean curvature \( -h \) does not exist for sufficiently small \( \varepsilon > 0 \). By a reflection argument, then a graph of constant mean curvature \( h \) does not exist either. Assume to the contrary that a graph \( f = f_\varepsilon \) does exist for each \( \varepsilon > 0 \). Because \( f_\varepsilon \) has constant mean curvature \( -h < 0 \) and zero boundary values, the maximum principle yields \( f_\varepsilon(x) \geq 0 \) for \( x \in \Omega \). Now note that the domain \( \Omega \) and the boundary values of \( f_\varepsilon \) are rotationally symmetric. Hence, the solution \( f_\varepsilon \) is also rotationally symmetric, following from the uniqueness of the Dirichlet problem. But then we can write \( f_\varepsilon(x) = p_\varepsilon(|x|) \) where \( p_\varepsilon(t) \) satisfies \( p_\varepsilon(t) \geq 0 \) for \( t \in [\varepsilon, 1] \) and \( p_\varepsilon(\varepsilon) = p_\varepsilon(1) = 0 \). From (8) we conclude

\[
p_\varepsilon(t) = \int_\varepsilon^t \frac{c - hs^n}{\sqrt{s^{2n-2} - (c - hs^n)^2}} \, ds
\]

where \( c = c(\varepsilon) \in \mathbb{R} \) is a suitable constant. We set \( k := c - h\varepsilon^n \) and claim \( k \geq 0 \). Otherwise \( p'_\varepsilon(t) < 0 \) would hold for all \( t \in (\varepsilon, 1) \), contradicting \( p_\varepsilon(\varepsilon) = p_\varepsilon(1) = 0 \). Note that the expression under the root must be nonnegative for all \( s \in [\varepsilon, 1] \), in particular for \( s = \varepsilon \) we get

\[
\varepsilon^{2n-2} - (c - h\varepsilon^n)^2 = \varepsilon^{2n-2} - k^2 \geq 0
\]

or equivalently \( k^2 \varepsilon^{2n-2} \geq 1 \). For any \( t \in [\varepsilon, 1] \) we now estimate

\[
p_\varepsilon(t) = \int_\varepsilon^t \frac{c - hs^n}{\sqrt{s^{2n-2} - (c - hs^n)^2}} \, ds \leq \int_\varepsilon^t \frac{c - h\varepsilon^n}{\sqrt{\varepsilon^{2n-2} - (c - h\varepsilon^n)^2}} \, ds
\]

\[
= \int_{\varepsilon}^{t/\varepsilon} \frac{k}{\sqrt{(\varepsilon \tau)^{2n-2} - k^2}} \varepsilon \, d\tau = \int_{1}^{t/\varepsilon} \frac{1}{\sqrt{k^2 \varepsilon^{2n-2} - k^2}} \frac{1}{\sqrt{\varepsilon^{2n-2} - k^2}} \, d\tau
\]

\[
\leq \varepsilon \int_{1}^{t/\varepsilon} \frac{1}{\sqrt{\varepsilon^{2n-2} - 1}} \, d\tau.
\]
In case of dimension $n \geq 3$ we conclude that $\lim_{\varepsilon \to 0} p_\varepsilon(t) = 0$, which follows from
\[
\int_{1}^{+\infty} \frac{1}{\sqrt{\tau^{2n-2} - 1}} \, d\tau < +\infty
\]
for dimension $n \geq 3$. In case of $n = 2$, the above integral is infinite. However, the explicit computation
\[
p_\varepsilon(t) \leq \varepsilon \int_{1}^{1/\varepsilon} \frac{1}{\sqrt{\tau^{2} - 1}} \, d\tau = \varepsilon \left[ \text{arcosh}(t) \right]_{1/\varepsilon}^{1} = \varepsilon \text{arcosh}(1/\varepsilon)
\]
again shows $\lim_{\varepsilon \to 0} p_\varepsilon(t) = 0$. This implies that the family $f_\varepsilon(x) = p_\varepsilon(|x|)$ converges uniformly to $f_0(x) \equiv 0$ on every compact subset of $\{x \in \mathbb{R}^n: 0 < |x| \leq 1\}$. Then, after extracting some subsequence, all first and second derivatives of $f_\varepsilon$ will converge to zero by interior gradient estimates for the constant mean curvature equation. Hence, also the mean curvature of $f_\varepsilon$ must converge to zero. This yields a contradiction as the mean curvature of $f_\varepsilon$ is $-h$ for each $\varepsilon > 0$. \hfill $\Box$

We can now show the a priori estimates of the height and boundary gradient of solutions of (1).

**Theorem 3. Assumptions:**

a) Let the bounded $C^2$-domain $\Omega \subset \mathbb{R}^n$ satisfy a uniform exterior sphere condition of radius $r > 0$ and be included in the annulus $\{x \in \mathbb{R}^n: r < |x| < r + d\}$ for some constant $d > 0$.

b) Let the prescribed mean curvature $H = H(x, z) \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy $H_\varepsilon \geq 0$ in $\Omega \times \mathbb{R}$ as well as the smallness assumption $\|H(x, 0)\| \leq h$ for some constant $h < \frac{2(2r)^{n-1}}{(2r+d)^n - (2r)^n}$.

c) Let $f \in C^2(\Omega \times \mathbb{R})$ be a solution of problem (1) for zero boundary values.

Then there exists a constant $C = C(h, r, d)$ such that $f$ satisfies the estimates
\[
\|f\|_{C^0(\Omega)} \leq C \quad \text{and} \quad \sup_{\partial \Omega} |\nabla f(x)| \leq C.
\]

**Proof.**

1) We first show the $C^0$-estimate. Because of $\Omega \subset \{x \in \mathbb{R}^n: r < |x| < r + d\}$ the rotationally symmetric graph $\eta(x) := p(|x|)$ is well defined and has constant mean curvature $-h$. Here, $p(t)$ is the function defined by Proposition 1 for $R := r + d$. From $\|H(x, 0)\| \leq h$ together with $H_\varepsilon \geq 0$ we conclude $H(x, z) \geq -h$ for $x \in \Omega, z \geq 0$ and $H(x, z) \leq h$ for $x \in \Omega, z \leq 0$. (12)

We now choose $c > 0$ minimal such that $f(x) \leq \eta(x) + c$ holds in $\Omega$. We claim that $c = 0$. Otherwise there would be a point $x_0 \in \Omega$ with $f(x_0) = \eta(x_0) + c > 0$. From (12) together with the strong maximum principle we then would have $f(x) \equiv \eta(x) + c$ in $\Omega$, contradicting $f(x) = 0$ on $\partial \Omega$. Hence we have shown $f(x) \geq \eta(x)$ in $\Omega$. Similarly, we obtain $f(x) \geq -\eta(x)$. Combining these estimates we have
\[
\|f\|_{C^0(\Omega)} = \sup_{\Omega} |f(x)| \leq \sup_{\Omega} |\eta(x)| \leq \sup_{\partial \Omega} |\eta(x)| = |\eta(x_0)| =: C_1.
\]
Here, $t_0$ defined by Proposition 1 is the argument for which the function $p$ achieves its maximum. Note that $p$ only depends on $r, d$ and hence $C_1 = C_1(r, d, h)$.

2) Given some point $x_0 \in \partial \Omega$ we show the boundary gradient estimate at $x_0$. Since $\Omega$ satisfies a uniform exterior sphere condition of radius $r$, we may assume that
\[
\Omega \cap B_r(0) = \emptyset \quad \text{and} \quad x_0 \in \partial B_r(0) \cap \partial \Omega
\]
holds after a suitable translation. We define the annulus $U := \{x \in \mathbb{R}^n: r < |x| < t_0\}$ and consider the graph
\[
\eta \in C^2(I \times \mathbb{R}), \quad \eta(x) := p(|x|) \quad \text{for} \quad x \in U.
\]
From $f(x) = 0$ on $\partial \Omega$ together with $f(x) \leq p(t_0) = \eta(x)$ for $|x| = t_0$ we conclude $f(x) \leq \eta(x)$ on $\partial (\Omega \cap U)$. As in part 1), the maximum principle gives $f(x) \leq \eta(x)$ in $\Omega \cap U$ as well as $f(x) \geq -\eta(x)$ in $\Omega \cap U$. From $x_0 \in \partial (\Omega \cap U)$ and $f(x_0) = \eta(x_0)$ we obtain
\[
|\nabla f(x_0)| = |\frac{\partial}{\partial \nu} f(x_0)| \leq |\frac{\partial}{\partial \nu} \eta(x_0)| = |p'(r)| =: C_2,
\]
where $\nu$ is the outward normal to $\partial \Omega$ at $x_0$. \hfill $\Box$
Remark. A closer inspection of the proof shows that Theorem 3 also holds without the assumption $H_2 \geq 0$ if one requires $|H(x, z)| \leq h$ in $\Omega \times \mathbb{R}$ instead of $|H(x, 0)| \leq h$ in $\Omega$. However, we will essentially need the assumption $H_2 \geq 0$ in the next section to prove a global gradient estimate.

3. Global gradient estimate and the proof of Theorem 1

In the previous section we have shown a $C^0$-estimate together with a boundary gradient estimate, thus we can assume

$$|f(x)| \leq M \quad \text{in} \quad \hat{\Omega}$$

(13)

given a solution $f \in C^{2+\alpha}(\hat{\Omega}, \mathbb{R})$ of problem (1). It now remains to establish a global gradient estimate in terms of the $C^0$-norm and the boundary gradient. This is derived in [3, Theorem 15.2] for a fairly large class of quasilinear elliptic equations. This includes the prescribed mean curvature equation in case of $H = H(x)$, as verified in example (ii) after [3, Theorem 15.2]. We will show that [3, Theorem 15.2] continues to hold in case $H = H(x, z)$, if we assume the monotonicity condition $H_2 \geq 0$. Let us first write the prescribed mean curvature equation in the form

$$\Delta f - \frac{\partial_i f \partial_j f}{1 + |\nabla f|^2} \partial_j f - n H(x, f) \sqrt{1 + |\nabla f|^2} = 0.$$ 

Now quantities $\alpha, \beta, \gamma$ are defined by [3, (15.27)], which in our case are

$$\alpha = -1 + \frac{1}{1 + |p|^2}, \quad \beta = \frac{n H(x, z) \sqrt{1 + |p|^2}}{|p|^2},$$

$$\gamma = -n \frac{(1 + |p|^2)^{3/2}}{|p|^2} \left[ H_2(x, z) + \sum_{i=1}^{n} \frac{p_i}{|p|^2} H_{\kappa_i}(x, z) \right] \quad \text{for} \quad x \in \Omega, \quad |z| \leq M, \quad p \in \mathbb{R}^n$$

(compare with example (ii) in Chapter 15.2 of [3]). We now compute the limits

$$a := \limsup_{|p| \to \infty} \alpha = -1, \quad b := \limsup_{|p| \to \infty} \beta \leq n \sup_{\Omega \times [-M, M]} |H(x, z)|,$$

$$c := \limsup_{|p| \to \infty} \gamma \leq n \sup_{\Omega \times [-M, M]} |\nabla H(x, z)|$$

(14)

using $H_2 \geq 0$ for the last limit. Because of $a = -1$ together with $b, c < +\infty$ we may apply [3, Theorem 15.2] to obtain

**Theorem 4.** Let the prescribed mean curvature $H \in C^1(\hat{\Omega} \times \mathbb{R})$ satisfy

$$H_2(x, z) \geq 0, \quad |H(x, z)| + |\nabla H(x, z)| \leq h_0 \quad \text{for} \quad x \in \Omega, \quad |z| \leq M.$$ 

Let $f \in C^2(\hat{\Omega}, \mathbb{R})$ be a solution Dirichlet problem (1) satisfying $\|f\|_{C^0(\Omega)} \leq M$. Then the estimate

$$\sup_{x \in \Omega} |\nabla f(x)| \leq C$$

holds with a constant $C$ depending only on $n, h_0, M, \Omega$ and $\sup_{\partial \Omega} |\nabla f|$.

**Remark.** If we do not assume $H_2 \geq 0$, then we will obtain $c = +\infty$ in (14) and [3, Theorem 15.2] will not be applicable. In fact, the following example shows that a gradient estimate is false if one does not require $H_2 \geq 0$.

**Example 1.** Given some parameter $\varepsilon > 0$ let $\beta(z) := z^2 + \varepsilon z$ for $z \in I := [-1, 1]$. Noting $\beta'(z) = 2z^2 + \varepsilon > 0$ in $I$, there exists a smooth inverse $\beta^{-1}: I \to \mathbb{R}$. From $\beta(-1) \leq -1$ and $\beta(1) \geq 1$ we conclude $\beta^{-1}: I \to I$. We now consider the one-dimensional graph $f_\varepsilon(x) := \beta^{-1}(x)$ for $x \in I$ with its parameterisation $X(x) = (x, f_\varepsilon(x))$. Substituting $z = f_\varepsilon(x)$ we obtain the reparametrisation $\tilde{X}(z) = (\beta(z), z)$ and we can compute the curvature $H = H(z)$ by

$$H(z) := H_\varepsilon(z) = \frac{\beta''}{(1 + (\beta')^2)^{3/2}} = -\frac{6z}{(1 + (3z^2 + \varepsilon)^2)^{3/2}}.$$ 

Hence, $f_\varepsilon$ is a graph of prescribed mean curvature $H_\varepsilon(z)$. We can find a constant $C$ such that

$$|H_\varepsilon(z)| + |\nabla H_\varepsilon(z)| \leq C \quad \text{for all} \quad z \in [-1, 1], \quad 0 < \varepsilon \leq 1.$$ 

Additionally, we have the $C^0$-estimate and boundary gradient estimate

$$|f_\varepsilon(x)| \leq 1 \quad \text{for} \quad x \in I \quad \text{and} \quad |\nabla f_\varepsilon(x)| \leq 1 \quad \text{for} \quad x \in \partial I = [-1, 1].$$
However, there is no uniform gradient bound for $f_\varepsilon$ in $I$ because

$$|\nabla f_\varepsilon(0)| = |f_\varepsilon'(0)| = \frac{1}{|\beta'(0)|} = \frac{1}{\varepsilon} \to \infty \text{ if } \varepsilon \to 0.$$  

In this example, all of the assumptions of Theorem 4 are satisfied except for $H_z \geq 0$. Even though this example was purely one-dimensional, a generalisation to higher dimensions $n \geq 2$ is easily possible.

We can now give the

**Proof of Theorem 1.** For $t \in [0, 1]$ consider the family of Dirichlet problems

$$f \in C^{2+\alpha}(\tilde{\Omega}, \mathbb{R}), \quad \text{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = tnH(x, f) \quad \text{in } \Omega \quad \text{and } f = 0 \quad \text{on } \partial \Omega.$$  

Let $f$ be such a solution for some $t \in [0, 1]$. By Theorems 3 and 4 we have the estimate

$$\|f\|_{C^1(\Omega)} \leq C$$

with some constant $C$ independent of $t$. The Leray–Schauder method [3, Theorem 13.8] yields a solution of the Dirichlet problem (15) for each $t \in [0, 1]$. For $t = 1$ we obtain the desired solution of (1). \qed

**Proof of Corollary 1.** Corollary 1 is obtained as the limit case of Theorem 1 by increasing the radius $r$ of the exterior sphere condition to infinity. First, since $\tilde{\Omega}$ is bounded and included within the strip $\{x \in \mathbb{R}^n: 0 < x_1 < d\}$, after a suitable translation it will also be included within the annulus $\{x \in \mathbb{R}^n: r < |x| < r + d\}$ for sufficiently large $r > 0$. To show which smallness condition on $h$ is required in order to apply Theorem 1 we have to compute the limit

$$\lim_{r \to \infty} \frac{2(2r)^{n-1}}{(2r + d)^n - (2r)^n}.$$  

To do this, we calculate

$$\lim_{r \to \infty} \frac{(2r + d)^n - (2r)^n}{2(2r)^{n-1}} = \lim_{r \to \infty} \frac{(2r)^n + n(2r)^{n-1}d + O(r^{n-2}) - (2r)^n}{2(2r)^{n-1}} = \frac{nd}{2}.$$  

We see that the limit in (16) is equal to $\frac{2}{nd}$ and hence the smallness condition $h < \frac{2}{nd}$ is required. Alternatively we could prove Corollary 1 also directly, by proving an analogue result to Theorem 3 for convex domains. Instead of using the nodoid we would then use a cylinder as barrier whose axis is lying in the $x_1, \ldots, x_n$ hyperplane. Note that the cylinder $\{x \in \mathbb{R}^{n+1}: x_1^2 + \cdots + x_n^2 = (\frac{2}{d})^2\}$ has constant mean curvature $h = \frac{4}{nd}$, corresponding to the smallness condition from above. \qed

**Remarks.**

a) Using the methods from [1], it is also possible to generalise Theorem 1 and Corollary 1 to the case of prescribed anisotropic mean curvature

$$\text{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = nH(x, f, N) \quad \text{in } \Omega.$$  

Here, the prescribed mean curvature does not only depend on the point $(x, f(x))$ in space but also on the normal $N(x)$ of the graph. Within this situation, $H_z \geq 0$ can be relaxed to weaker assumption allowing nonuniqueness of solutions.

b) The results can also be generalised in another direction: Define the boundary part

$$\Gamma_+ := \left\{ x \in \partial \Omega : \left| H(x, z) \right| \leq \frac{1}{n} \tilde{H}(x) \text{ for all } z \in \mathbb{R} \right\}$$

where $\tilde{H}(x)$ is the mean curvature of $\partial \Omega$ at $x$ w.r.t. the inner normal. Now choose a subset $\Gamma \subseteq \Gamma_+$ such that $\text{dist}(\Gamma, \partial \Omega \setminus \Gamma_+) > 0$. On $\Gamma$ we can use the standard boundary gradient estimate (see [3, Corollary 14.8]) and prescribe $C^{2+\alpha}$ boundary values $g$ there. Our boundary gradient estimate of Theorem 3, requiring zero boundary values, is then only needed on $\partial \Omega \setminus \Gamma$. This way, Theorem 1 and Corollary 1 also hold for Dirichlet boundary values $g \in C^{2+\alpha}(\partial \Omega, \mathbb{R})$ with $g(x) = 0$ on $\partial \Omega \setminus \Gamma$ and $|g(x)| \leq \varepsilon$, where $\varepsilon = \varepsilon(\Omega, \Gamma, H) > 0$ is some constant determined by the height of the nodoid constructed in Proposition 1.
References