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# A comparison of a posteriori error estimates for biharmonic problems solved by the FEM

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ABSTRACT

The classical a posteriori error estimates are mostly oriented to the use in the finite element *h*-methods while the contemporary higher-order *hp*-methods usually require new approaches in a posteriori error estimation. These methods hold a very important position among adaptive numerical procedures for solving ordinary as well as partial differential equations arising from various technical applications.

In the paper, we are concerned with a review and comparison of error estimation procedures for the biharmonic and some more general fourth order partial differential problems with special regards to the needs of the *hp*-method. We point out some advantages and drawbacks of analytical and computational a posteriori error estimates.



## 1. Introduction

The development of numerical computation has been accompanied with the construction of *a priori error estimates* as both the approximate result and its error (or some estimate of its error) are important quantities. These error estimates are very useful in theory but usually include constants that are completely unknown and can be estimated in rare cases. In particular, the development of adaptive finite element methods stimulated the development of reliable and computable *a posteriori error indicators* and *estimators* that depend only on the approximate solution just computed. This is the means for the local mesh refinement in the *h*-version, for the increase of the polynomial degree in the *p*-version, and for both the mesh refinement and polynomial degree increase in the *hp*-version.

We introduce several a posteriori error estimators in the paper to assess the error of the approximate solution. The quality of such an estimator is often measured by its *effectivity index*, i.e. the ratio of some norm of the error estimate and the true error. Of course, the effectivity index can be computed only for those problems for which the exact solution is known [1]. An error estimator is called *effective* if both its effectivity index and the inverse of the index remain bounded for all meshsizes of triangulation. It is called *asymptotically exact* if its effectivity index converges to 1 as the meshsize tends to 0.

The *hp*-adaptive computation is mostly carried out using the following general scheme:

1. Generate the initial mesh $\mathcal{T}_h$, $h$ being the maximum size of all elements of the mesh.
2. Solve the problem on $\mathcal{T}_h$.
3. Compute the (global) error estimate. If the estimate is below the tolerance given, stop.
4. Compute an (analytical or computational) error indicator $\eta_T$ for every element $T \in \mathcal{T}_h$.

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5. Construct a new mesh $\mathcal{T}_{\widetilde{h}}$, where $\widetilde{h}$ is the maximum size of all elements of this new mesh, by refining some elements and by increasing polynomial degrees of some elements in the parts of the domain with the largest error indicated.
6. Replace $\mathcal{T}_h$ by $\mathcal{T}_{\widetilde{h}}$ and go to Step 2.

The stopping criterion in Step 3 can be modified properly if the adaptivity required is *goal-oriented*.

Note that a posteriori error estimation plays two different roles: in Step 3 it helps decide *when to stop the algorithm* while in Step 4 it indicates *where the mesh should be refined or polynomial degree increased*. In practice, the same error indicator is often used for both the purposes. This, however, may not always be optimal, in particular for problems with *non-local error sources (pollution error)*. Such a situation can occur also in coupled PDE systems.

In Step 3, the (possibly global) error estimator should provide a *guaranteed error upper bound*, i.e., to guarantee that the actual error is below the tolerance given. On the other hand, the (local) error indicator in Step 4 need not be guaranteed. It should assume large values in areas that *cause the largest error* (which need not always be the areas *where the largest error occurs*).

*Computational estimates* (based on *reference solutions*) in Step 4 are usually obtained by solving the problem in a systematically refined mesh and, at the same time, with the polynomial degree of all elements increased by 1 (see, e.g., [2,3]).

Undoubtedly, obtaining efficient and computable a posteriori error estimates is not easy. (*Computable* means in this context that the degree of piecewise polynomials approximating the solution is high enough.) Analytical estimates need not be the sharpest. The local nature of the estimates is very advantageous. It provides for the local mesh refinement or the local increase of the polynomial degree and usually implies lower time requirements of the computation. The papers [4,5] by Babuška and Rheinboldt represent the pioneering work in this field. The books [6] or [7] are surveys of the state of the art a few years ago.

There are several classes of a posteriori error indicators and estimators based on different approaches and their names slightly vary in the literature. Let us name residual estimators: explicit, implicit (based on the solution of local problems), and hierarchic (multilevel) estimators, further also gradient recovery based estimators (that employ the averaging of gradient), and some other. They can, e.g., use the mixed finite element formulation of the problem or utilize functional-analytical properties of the approximate solution constructed.

In the paper, we are concerned with several formulations of the biharmonic problem and a general 4th order elliptic problem on a 2D domain: a Dirichlet problem for biharmonic equation (Section 3), a mixed formulation of the same problem (Section 4), a second problem (with the homogeneous boundary condition for the solution $u$ as well the function $\Delta u$) for the same biharmonic equation (Section 5), and a Dirichlet problem for a more general 4th order elliptic equation (Section 6). We present analytical a posteriori error estimates of different nature found in literature for the above mentioned problems and, in Section 7, we assess the advantages and drawbacks of the analytical as well as computational estimates in general.

## 2. Notation and preliminaries

Some common notation is introduced in this section. We write $C(S)$ for the space of all functions continuous on the set $S$ and $C_m(S)$ for that of all functions continuous together with their $m$ derivatives.

Let $\Omega \subset R^n$, $n \geq 2$, be a connected bounded domain with boundary $\Gamma$. We use the notation $\|\cdot\|_0$ for the $L_2(\Omega)$ norm, $\|\cdot\|_\infty$ for the $L_\infty(\Omega)$ norm, and $\|\cdot\|_1$ and $\|\cdot\|_2$ for the $H^1(\Omega)$ and $H^2(\Omega)$ norms for real scalar, vector, and matrix-valued functions. Analogically, we introduce the seminorm $|\cdot|_k$ in the space $H^k(\Omega)$.

In this connection, we need some simple definitions. Let $F = [f_i]$ and $G = [g_i]$ be two $n$-vectors from $R^n$. We introduce their *inner product* $F \cdot G \in R$ by

$$F \cdot G = \sum_{i=1}^{n} f_i g_i.$$

Moreover, let $\Phi = [\varphi_{ik}]$ and $\Psi = [\psi_{ik}]$ be two $n \times n$ matrices, $\Phi, \Psi \in R^{n\times n}$. We introduce their *elementwise matrix product (double-dot product)* $\Phi \odot \Psi \in R$ by

$$\Phi \odot \Psi = \sum_{i=1}^{n} \sum_{k=1}^{n} \varphi_{ik} \psi_{ik}$$

and the *Frobenius* or *Schur norm* [8] of the matrix $\Phi$ as $\|\Phi\|_F = \sqrt{\Phi \odot \Phi}$. We write

$$\mathrm{tr}(\Phi) = \sum_{i=1}^{n} \varphi_{ii}$$

for the *trace* of the matrix $\Phi$ and

$$(\Phi, \Psi)_0 = \int_\Omega \Phi \odot \Psi$$

for the $L_2$ inner product of the matrices $\Phi$ and $\Psi$.

The norm or seminorm may be restricted to any open set $\omega \subset \Omega$ with Lipschitz boundary $\gamma$. We write $\|\cdot\|_{0;\omega}$ for the $L_2(\omega)$ norm, $\|\cdot\|_{1;\omega}$ for the $H^1(\omega)$ norm, etc., and $\|\cdot\|_{0;\gamma}$ for the $L_2(\gamma)$ norm. On $\Omega$, $\omega$, and $\gamma$, we further consider the spaces $W^{k,s}(\Omega)$, $W^{k,s}(\omega)$, and $L_s(\gamma)$ with an integer $k$ and $1 \le s \le \infty$. We also employ the spaces $H_0^1(\Omega)$, $H_0^2(\Omega)$, etc. and the adjoint spaces $H^{-k}(\Omega)$, $k > 0$, of linear functionals. We often omit the symbol $\Omega$ if $\Omega$ is the domain concerned.

We use the notation

$$\operatorname{div} A = \nabla \cdot A = \sum_{s=1}^{n} \frac{\partial a_s}{\partial x_s} \in R$$

for the divergence of a differentiable vector-valued function $A = [a_1, \dots, a_n]$. We put $\nabla A = \nabla \otimes A \in R^{n\times n}$, where $\otimes$ is the tensor product, for the vector-valued function $A$ and $\nabla b = \operatorname{grad} b \in R^n$ for the gradient of a differentiable scalar-valued function $b$. Further, for a differentiable matrix-valued function $\Theta = [\vartheta_{ij}]_{i,j=1}^n$ we introduce its divergence as a vector-valued function

$$\operatorname{Div} \Theta = \nabla \cdot \Theta = \sum_{j=1}^{n} \frac{\partial \vartheta_{ij}}{\partial x_j} \in R^n.$$

Let $R_s^{n\times n}$ be the space of real symmetric $n \times n$ matrices. We consider also the space $H(\operatorname{div}, \Omega) = \{Y \in L_2(\Omega, R^n) \mid \operatorname{div} Y \in L_2(\Omega)\}$ of vector-valued functions $Y$ and the space $H(\operatorname{Div}, \Omega) = \{\Theta \in L_2(\Omega, R_s^{n\times n}) \mid \operatorname{Div} \Theta \in L_2(\Omega, R^n)\}$ of symmetric matrix-valued functions $\Theta$.

For a matrix-valued function $\Phi : \Omega \to R^{n\times n}$, $\Phi = [\varphi_{ik}]$, we put

$$\operatorname{div}^2 \Phi = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \varphi_{ik}}{\partial x_i \partial x_k} \in R$$

provided these derivatives exist.

Finally, let

$$H(\operatorname{div}^2, \Omega) = \{\Phi \in L_2(\Omega, R^{n\times n}) \mid \operatorname{div}^2 \Phi \in L_2(\Omega)\},$$
$$H(\operatorname{div}\operatorname{Div}, \Omega) = \{\Phi \in L_2(\Omega, R_s^{n\times n}) \mid \operatorname{div}\operatorname{Div} \Phi \in L_2(\Omega)\}$$

be the spaces of matrix-valued and symmetric matrix-valued functions, respectively.

Symbols $c, c_1, \dots$ are generic. They may represent different quantities (depending possibly on other quantities) at different occurrences.

*Finite element mesh notation*

The notation is based mostly on [9]. Let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a family of triangulations $\mathcal{T}_h$ of $\Omega$. For any triangle $T \in \mathcal{T}_h$ we denote by $h_T$ its diameter, while $h$ indicates the maximum size of all the triangles in the mesh. We further denote by $\rho_T$ the diameter of the largest ball inscribed into $T$ and by $|T|$ the area of $T$. Let $\mathcal{E}(T)$ be the set of all edges and $\mathcal{N}(T)$ the set of all nodes of $T$. We set

$$\mathcal{E}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T), \qquad \mathcal{N}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{N}(T).$$

We split $\mathcal{E}_h$ in the form $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\Gamma}$ (cf. [10]) with

$$\mathcal{E}_{h,\Omega} = \{E \in \mathcal{E}_h \mid E \subset \Omega\}, \qquad \mathcal{E}_{h,\Gamma} = \{E \in \mathcal{E}_h \mid E \subset \Gamma\}.$$

For $T \in \mathcal{T}_h$ we define

$$\omega_T = \bigcup_{\mathcal{E}(T)\cap\mathcal{E}(T')\neq\emptyset} T'. \tag{2.1}$$

The length of $E \in \mathcal{E}_h$ is denoted by $h_E$. Further, with every edge $E \in \mathcal{E}_h$ we associate a unit normal vector $n_E$ [10]. If $E \in \mathcal{E}_{h,\Gamma}$ then $n_E$ is equal to the unit outer normal vector to $\Gamma$. If $E \in \mathcal{E}_{h,\Omega}$ the choice of the outer direction of $n_E$ is arbitrary but fixed.

Let $T_+$ and $T_-$ be any two triangles with a common edge $E \in \mathcal{E}_{h,\Omega}$, the subscripts $+$ and $-$ being chosen in such a way that the unit outer normal to $T_-$ at $E$ corresponds to $n_E$. Given a piecewise continuous scalar-valued function $w$ on $\Omega$, call $w^+$ or $w^-$ its trace $w|_{T_+}$ or $w|_{T_-}$ on $E$. The jump of $w$ across $E$ in the direction of $n_E$ is given by

$$[w]_E = w^+ - w^-. \tag{2.2}$$

For a vector-valued function, the jump is defined componentwise.

Further, we write $P_k(T)$ for the space of polynomials of degree $k$ on $T$. Finally, let $f_h$ be an approximation of a function $f \in L_2(\Omega)$ on a triangle $T \in \mathcal{T}_h$. We then put

$$e_T = \|f - f_h\|_{0;T}. \tag{2.3}$$

## 3. Dirichlet problem for biharmonic equation

*Problem*

Let the domain $\Omega \subset R^2$ have a polygonal boundary $\Gamma$. We consider the two-dimensional biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{3.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{3.2}$$

with $f \in L_2(\Omega)$ that models, e.g., the vertical displacement of the mid-surface of a clamped plate subject to bending.

*Weak solution*

Let $X$ and $Y$ be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $\mathcal{L}(X, Y)$ denote the Banach space of continuous linear maps of $X$ on $Y$ and $\mathrm{Isom}(X, Y) \subset \mathcal{L}(X, Y)$ an open subset of linear homeomorphisms of $X$ onto $Y$. Let $Y^* = \mathcal{L}(Y, R)$ be the dual space of $Y$ and $\langle\cdot,\cdot\rangle$ the corresponding duality pairing.

Let us put, in particular,

$$X = Y = H_0^2(\Omega), \qquad \|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_2,$$

$$\langle F(u), v\rangle = \int_\Omega \Delta u \Delta v - \int_\Omega f v. \tag{3.3}$$

We then say that $u \in X$ is the weak solution of the problem (3.1), (3.2) if

$$\langle F(u), v\rangle = 0 \tag{3.4}$$

for all $v \in Y$.

Since the bilinear form

$$\{u, v\} \to \int_\Omega \Delta u \Delta v$$

is continuous and coercive on $X$ (cf. [9]) we have $\mathrm{D}F(u) \in \mathrm{Isom}(X, Y^*)$ for all $u \in X$, where $\mathrm{D}F$ is the derivative.

*Approximate solution*

Let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a regular family of triangulations of $\Omega$ (see, e.g., [9]). For the discretization of the problem (3.1), (3.2) we assume that $X_h \subset X$ and $Y_h \subset Y$ are $C_1$ finite element spaces corresponding to $\mathcal{T}_h$ and consisting of piecewise polynomials. These conditions imply in particular that the functions in $X_h$ and $Y_h$ are of class $C_1$. Denote by $k$, $k \geq 1$, the maximum polynomial degree of the functions in $X_h$. Further, put

$$f_h = \sum_{T\in\mathcal{T}_h} \pi_{l,T} f,$$

where $P_l$, $l \geq 0$ fixed, is the space of polynomials of degree at most $l$ and $\pi_{l,S}$, $S \in \mathcal{T}_h \cup \mathcal{E}_h$, is the $L_2$ projection of the Lebesgue space $L_1(S)$ onto $P_{l|S}$.

Replacing $f$ in the definition (3.3) by $f_h$ to get the functional $F_h$, we say that $u_h \in X_h$ is the approximate solution of the problem (3.1), (3.2) if

$$\langle F_h(u_h), v_h\rangle = 0$$

for all $v_h \in Y_h$.

*Error indicator*

Using the notation (2.3) for $e_T$ and defining the *local residual a posteriori error indicator*

$$\eta_{V,T} = \left( h_T^4 \|\Delta^2 u_h - f_h\|_{0;T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} (h_E \|[\Delta u_h]_E\|_{0;E}^2 + h_E^3 \|[n_E \cdot \nabla\Delta u_h]_E\|_{0;E}^2) \right)^{1/2}$$

for all $T \in \mathcal{T}_h$, we have the following theorem [10].

**Theorem 3.1.** *Let $u \in X$ be the unique weak solution of the problem (3.1), (3.2), i.e. of (3.4), and let $u_h \in X_h$ be an approximate solution of the corresponding discrete problem. Then we have the a posteriori estimates*

$$\|u - u_h\|_2 \leq c_1 \left( \sum_{T\in\mathcal{T}_h} \eta_{V,T}^2 \right)^{1/2} + c_2 \left( \sum_{T\in\mathcal{T}_h} h_T^4 e_T^2 \right)^{1/2} + c_3 \|F(u_h) - F_h(u_h)\|_{Y_h^*} + c_4 \|F_h(u_h)\|_{Y_h^*}$$

*and*

$$\eta_{V,T} \le c_5 \|u-u_h\|_{2;\omega_T} + c_6 \left( \sum_{T' \subset \omega_T} h_{T'}^4 e_{T'}^2 \right)^{1/2}$$

*for all $T \in \mathcal{T}_h$. The quantities $\|F(u_h) - F_h(u_h)\|_{Y_h^*}$ and $\|F_h(u_h)\|_{Y_h^*}$ represent the consistency error of the discretization and the residual of the discrete problem, and the quantities $c_1, \ldots, c_6$ may depend only on $h_T/\rho_T$, and the integers $k$ and $l$.*

The proof is given in [10]. This estimate is likely to be the first a posteriori error estimate for 4th order problems published.

**Remark 3.1.** In order to avoid the use of $C_1$ elements it is possible to consider mixed finite element approximations of the problem (3.1), (3.2) obtained from the weak formulation of an appropriate 2nd order elliptic system, see Sections 4 and 5. This discretization involves only $C_0$ elements and a posteriori error estimates can be constructed using the means for standard 2nd order problems.

## 4. Dirichlet problem for biharmonic equation in mixed finite element formulation

*Problem*

Let the domain $\Omega \subset R^2$ be a convex polygon with boundary $\Gamma$. We consider the two-dimensional biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{4.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{4.2}$$

with $f \in L_2(\Omega)$ that serves both for linear plate analysis and incompressible flow simulation. The same problem with $f \in H^{-1}(\Omega)$ is treated in [11].

*Weak solution*

Put $V = H_0^1(\Omega)$ and $X = H^1(\Omega)$ and define the continuous bilinear forms

$$a(w, z) = \int_\Omega wz \quad \text{on } X \times X \quad \text{and} \quad b(z, u) = \int_\Omega \nabla z \cdot \nabla u \quad \text{on } X \times V$$

with scalar-valued functions $u$, $w$, and $z$.

The Ciarlet–Raviart weak formulation [12] of (4.1) and (4.2) then reads: Find $\{w, u\} \in X \times V$ such that

$$a(w, z) + b(z, u) = 0 \quad \text{for all } z \in X, \tag{4.3}$$

$$b(w, v) + \int_\Omega f v = 0 \quad \text{for all } v \in V. \tag{4.4}$$

This mixed variational formulation is known to be ill-posed. Nevertheless, the existence and uniqueness of the solution $\{w = \Delta u, u\}$ of the problem (4.3) and (4.4) are proven in [13].

*Approximate solution*

We construct the conforming second order discretization according to [14]. We assume that $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ is a uniformly regular family of triangulations of $\Omega$ (i.e., it is regular and satisfies the inverse assumption, see [9], to guarantee that the inequality (4.5) holds), even though it is not easy to satisfy this assumption in the presence of mesh refinements. In the sequel, $\Pi_h$ denotes the $L_2$ orthogonal projection on the set of piecewise constant functions on the triangulations.

The finite element spaces $X_h$ and $V_h$ are then

$$X_h = \{x_h \in X \mid x_h|_T \in P_2(T) \text{ for all } T \in \mathcal{T}_h\},$$
$$V_h = \{v_h \in V \mid v_h|_T \in P_2(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Our assumption of uniform regularity of the triangulation family $\mathcal{F}$ implies that there is a positive constant $c$ such that the *inverse inequality*

$$|x_h|_{m;T} \le c h^{l-m} |x_h|_{l;T} \tag{4.5}$$

holds for all integers $l$ and $m$, $l \le m$, and all $x_h \in X_h$ and $T \in \mathcal{T}_h$.

The discrete formulation of the problem (4.3) and (4.4) now reads: Find $\{w_h, u_h\} \in X_h \times V_h$ such that

$$a(w_h, z_h) + b(z_h, u_h) = 0 \quad \text{for all } z_h \in X_h, \tag{4.6}$$

$$b(w_h, v_h) + \int_\Omega f v_h = 0 \quad \text{for all } v_h \in V_h. \tag{4.7}$$

*Error indicators*

Let $e_T$ be given by (2.3) where we put $f_h = \Pi_h f$. For the local residuals we use the notation

$$\mathcal{P}_T(u_h) = -\Delta u_h + w_h, \qquad \mathcal{R}_T(w_h) = -\Delta w_h + f_h,$$
$$\mathcal{P}_E(u_h) = \left[\frac{\partial u_h}{\partial n_E}\right]_E, \qquad \mathcal{R}_E(w_h) = \left[\frac{\partial w_h}{\partial n_E}\right]_E.$$

If $E \in \mathcal{E}_{h,\Gamma}$ the definition (2.2) of jump holds but we assume that the value outside $\Omega$ is zero. We introduce the *local residual a posteriori error indicators*

$$\eta^2_{C,T} = |T|\;\|\mathcal{P}_T(u_h)\|^2_{0;T} + \frac{1}{2}\sum_{E\in\mathcal{E}(T)} h_E\|\mathcal{P}_E(u_h)\|^2_{0;E},$$

$$\widetilde{\eta}^2_{C,T} = |T|\;\|\mathcal{R}_T(u_h)\|^2_{0;T} + \frac{1}{2}\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_E\|\mathcal{R}_E(u_h)\|^2_{0;E} + |T|e_T^2.$$

Then the following theorem holds [11].

**Theorem 4.1.** *Let $\{w, u\} \in X \times V$ be the unique weak solution of the problem (4.1) and (4.2), i.e. of (4.3) and (4.4), and let $\{w_h, u_h\} \in X_h \times V_h$ be an approximate solution of the corresponding discrete problem (4.6) and (4.7). Then we have the a posteriori estimates*

$$\|u-u_h\|_1 + h\|w-w_h\|_0 \le C_1\left(\left(\sum_{T\in\mathcal{T}_h}\eta^2_{C,T}\right)^{1/2} + h^2\left(\sum_{T\in\mathcal{T}_h}\widetilde{\eta}^2_{C,T}\right)^{1/2}\right)$$

*with some positive constant $C_1$ independent of $h$ and*

$$\eta_{C,T} + h_T^2\widetilde{\eta}_{C,T} \le C_2\left(|u-u_h|_{1;\omega_T} + h_T\|w-w_h\|_{0;\omega_T} + h_T^3\sum_{T'\subset\omega_T} e_{T'}\right)$$

*for $T \in \mathcal{T}_h$ and with some positive constant $C_2$ independent of $h$.*

The proof is given in [11].

## 5. Second problem for biharmonic equation in mixed finite element formulation

*Problem*

Let the domain $\Omega \subset R^2$ be a convex polygon with boundary $\Gamma$. We consider the two-dimensional second biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{5.1}$$
$$u = \Delta u = 0 \quad \text{on } \Gamma \tag{5.2}$$

with $f \in L_2(\Omega)$ that models the deformation of a simply supported thin elastic plate. Putting $w = \Delta u$, we can rewrite the problem (5.1), (5.2) as the system of two Poisson equations, both with the homogeneous Dirichlet boundary condition,

$$\Delta w = f \quad \text{in } \Omega,$$
$$\Delta u = w \quad \text{in } \Omega,$$
$$u = w = 0 \quad \text{on } \Gamma.$$

*Weak solution*

Put $V = H^1_0(\Omega)$ and define the continuous bilinear forms

$$a(w,z) = \int_\Omega wz \quad \text{and} \quad b(z,u) = \int_\Omega \nabla z\cdot\nabla u \quad \text{on } V\times V$$

with scalar-valued functions $u$, $w$, and $z$.

The Ciarlet–Raviart weak formulation [12] of (5.1) and (5.2) then reads: Find $\{w, u\} \in V \times V$ such that

$$a(w,z) + b(z,u) = 0 \quad \text{for all } z \in V, \tag{5.3}$$

$$b(w,v) + \int_\Omega fv = 0 \quad \text{for all } v \in V. \tag{5.4}$$

*Approximate solution*

Let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a quasi-uniform family of triangular or rectangular partitions of $\Omega$ [6]. Put

$$V_h = \{z \in C(\overline{\Omega}) \mid z|_T \in P_k(T),\ k \ge 1,\ \text{for all } T \in \mathcal{T}_h\} \cap H^1_0(\Omega).$$

The discrete formulation of the problem (5.3) and (5.4) now reads: Find $\{w_h, u_h\} \in V_h \times V_h$ such that

$$a(w_h, z_h) + b(z_h, u_h) = 0 \quad \text{for all } z_h \in V_h, \tag{5.5}$$

$$b(w_h, v_h) + \int_\Omega f v_h = 0 \quad \text{for all } v_h \in V_h. \tag{5.6}$$

*Error estimator*

Let the basis function $v_{h,N}$ from $V_h$ be associated with the node $N \in \mathcal{N}_{h,\Omega} = \mathcal{N}_h \cap \Omega$. Put $\omega_N = \operatorname{supp} v_{h,N}$, $h_N = \max_{T \subset \omega_N} h_T$, and introduce the quantities

$$\varepsilon_1^2 = \sum_{N \in \mathcal{N}_{h,\Omega}} h_N^2 \int_{\omega_N} [(f - \bar{f}_N)^2 + (w_h - \overline{(w_h)}_N)^2 + (\Delta w_h - \overline{(\Delta w_h)}_N)^2 + (\Delta u_h - \overline{(\Delta u_h)}_N)^2],$$

$$\varepsilon_2^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T [(f - \bar{f}_T)^2 + (w_h - \overline{(w_h)}_T)^2],$$

where

$$\bar{q}_N = |\omega_N|^{-1} \int_{\omega_N} q, \qquad \bar{q}_T = |T|^{-1} \int_T q.$$

We introduce the *gradient recovery operator* $G v_h : V_h \to V_h \times V_h$ in the following way [15]. Assume that

$$v_h(x) = \sum_{N \in \mathcal{N}_{h,\Omega}} \beta_N v_{h,N}(x), \quad x \in \Omega,$$

with some coefficients $\beta_N$ and put

$$\widetilde{G} v_{h,N} = \sum_{T \cap \omega_N \neq \emptyset} \alpha_N^T (\nabla v_{h,N})|_T, \quad \text{where} \sum_{T \cap \omega_N \neq \emptyset} \alpha_N^T = 1$$

and $0 \le \alpha_N^T \le 1$ are chosen weights. Note that the vector $\nabla v_{h,N}$ is constant on each triangle. Finally we set

$$G v_h(x) = \sum_{N \in \mathcal{N}_{h,\Omega}} \widetilde{G} v_{h,N} v_{h,N}(x), \quad x \in \Omega.$$

Define the *gradient recovery a posteriori error estimator*

$$\varepsilon_{\mathrm{L}}^2 = \sum_{T \in \mathcal{T}_h} (\|G w_h - \nabla w_h\|_{0;T}^2 + \|G u_h - \nabla u_h\|_{0;T}^2).$$

Then the following theorem holds [15].

**Theorem 5.1.** *Let $\{w, u\} \in V \times V$ be the unique weak solution of the problem (5.1) and (5.2), i.e. of (5.3) and (5.4), and let $\{w_h, u_h\} \in V_h \times V_h$ be an approximate solution of the corresponding discrete problem (5.5) and (5.6). Then we have the a posteriori estimates*

$$c\varepsilon_{\mathrm{L}}^2 - C_2 \varepsilon_2^2 \le |w - w_h|_1^2 + |u - u_h|_1^2 \le C\varepsilon_{\mathrm{L}}^2 + C_1 \varepsilon_1^2$$

*with some positive constants $c$, $C$, $C_1$, and $C_2$ independent of $h$.*

The proof is given in [15].

## 6. Dirichlet problem for fourth order elliptic equation

*Problem*

Let $\mathrm{D}^2 u$ denote the Hessian matrix of a function $u : \Omega \to R$, $u \in H^2(\Omega)$. Let the matrix-valued function $\Lambda = [\lambda_{ik}]$, $\Lambda : \Omega \times R^{n \times n} \to R^{n \times n}$ be measurable and bounded with respect to the variable $x \in \Omega$ and of class $C_2$ with respect to the matrix variable $\Theta \in R^{n \times n}$.

Let the domain $\Omega \subset R^n$ have a piecewise $C_1$ boundary. We consider the fourth order problem

$$\operatorname{div}^2 \Lambda(x, \mathrm{D}^2 u) = f \quad \text{in } \Omega, \tag{6.1}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \tag{6.2}$$

with $f \in L_2(\Omega)$.

*Weak solution*

Let us assume that the Jacobian arrays

$$\Lambda'(x, \Theta) = \frac{\partial \Lambda(x, \Theta)}{\partial \Theta} = \left[\frac{\partial \lambda_{rs}(x, \Theta)}{\partial \vartheta_{ik}}\right]_{i,k,r,s=1}^{n} \in R^{(n\times n)^2}$$

are symmetric, i.e. $\partial\lambda_{rs}/\partial\vartheta_{ik} = \partial\lambda_{ik}/\partial\vartheta_{rs}$ for all $i, k, r, s$, and that there are constants $0 < m \le M$ such that

$$m\|\Phi\|_{\mathrm{F}}^2 \le (\Lambda'(x, \Theta)\Phi) \odot \Phi \le M\|\Phi\|_{\mathrm{F}}^2 \quad \text{for all } x \in \Omega,\ \Theta, \Phi \in R^{n\times n}. \tag{6.3}$$

Further let the mapping $\Lambda' : \Omega \times R^{n\times n} \to R^{(n\times n)^2}$ be Lipschitz continuous in the matrix variable $\Theta \in R^{n\times n}$ with a Lipschitz constant $L$.

It can be shown [16] that the counterpart of the Friedrichs inequality is

$$\|w\|_0 \le C_\Omega \|\mathrm{D}^2 w\|_0 \tag{6.4}$$

for all $w \in H_0^2(\Omega)$ and a suitable constant $C_\Omega > 0$. Let us introduce some more notation. We write $\Lambda^{-1}$ for the inverse of $\Lambda$ with respect to $\Theta \in R^{n\times n}$, i.e.

$$\Lambda(x, \Theta) = \Phi \Rightarrow \Lambda^{-1}(x, \Phi) = \Theta.$$

$\Lambda^{-1}$ exists by virtue of the assumptions on $\Lambda$ and $\Lambda'$ [16].

The above assumptions imply that the problem (6.1), (6.2) has a unique weak solution $u \in H_0^2(\Omega)$ that satisfies

$$\int_\Omega \Lambda(x, \mathrm{D}^2 u) \odot \mathrm{D}^2 v - \int_\Omega f v = 0 \quad \text{for all } v \in H_0^2(\Omega).$$

*Error estimators*

Let $\bar{u}$ be a function from $H_0^2(\Omega)$ considered as an approximation of the weak solution $u$. In [16], no specification of the way $\bar{u}$ has been computed is required, it is just an arbitrary function of the admissible class.

We measure the error of the approximate solution $\bar{u}$ by the functional

$$\begin{aligned} E(\bar{u}) &= \int_\Omega (\Lambda(x, \mathrm{D}^2\bar{u}) - \Lambda(x, \mathrm{D}^2 u)) \odot (\mathrm{D}^2\bar{u} - \mathrm{D}^2 u) \\ &= \int_\Omega \Lambda(x, \mathrm{D}^2\bar{u}) \odot (\mathrm{D}^2\bar{u} - \mathrm{D}^2 u) - \int_\Omega f(\bar{u} - u). \end{aligned}$$

According to [16], the inequality

$$\|\bar{u} - u\|_2^2 \le mE(\bar{u})$$

holds with the positive constant $m$ from (6.3).

Define the *global a posteriori error estimator*

$$\begin{aligned} \varepsilon_{\mathrm{K}}(\Psi, w, \bar{u}) = \Big(& m^{-1/2} C_\Omega \|\mathrm{div}^2\Psi - f\|_0 + \frac{1}{2} L m^{-3/2} \delta(\Psi, w, \bar{u}) + ((\Lambda(x, \mathrm{D}^2\bar{u}) - \Psi, \mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi))_0 \\ &+ \frac{1}{2} L m^{-1} \delta(\Psi, w, \bar{u}) \|\mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi)\|_0)^{1/2}\Big)^2, \end{aligned} \tag{6.5}$$

where

$$\delta(\Psi, w, \bar{u}) = (M\|\Lambda^{-1}(x, \Psi) - \mathrm{D}^2 w\|_0 + C_\Omega \|\mathrm{div}^2\Psi - f\|_0) \|\mathrm{D}^2\bar{u} - \Lambda^{-1}(x, \Psi)\|_\infty.$$

$\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ is an arbitrary matrix-valued function and $w \in H_0^2(\Omega)$ an arbitrary scalar-valued function, $m$ and $M$ are the constants from (6.3), $C_\Omega$ from (6.4), and $L$ the Lipschitz continuity constant of $\Lambda'$. Then the following theorem holds [16].

**Theorem 6.1.** *Let $u \in H_0^2(\Omega)$ be the unique weak solution of the problem (6.1), (6.2) and $\bar{u} \in W^{2,\infty}(\Omega)$ an arbitrary function. Then*

$$E(\bar{u}) \le \varepsilon_{\mathrm{K}}(\Psi, w, \bar{u})$$

*for any $\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n})$ and $w \in H_0^2(\Omega)$.*

The proof of the theorem is based on a more general statement proven in [16].

**Remark 6.1.** Following [17], the term $C_\Omega \|\mathrm{div}^2 \Psi - f\|_0$ in the expression (6.5) can be replaced by

$$C_\Omega \|\mathrm{div}\, \Psi - V\|_0 + C_\Omega \|\mathrm{div}\, V - f\|_0,$$

where $V \in H(\mathrm{div}, \Omega)$ is some further arbitrary vector-valued function and the assumption on $\Psi$ can be weakened to the condition that the individual rows of $\Psi$ belong to $H(\mathrm{div}, \Omega)$.

To avoid the computation of $\Lambda^{-1}$ we can introduce another global a posteriori error estimator $\widetilde{\varepsilon}_{\mathrm{K}}(\Phi, w, \bar{u})$, where $\Phi = \Lambda^{-1}(x, \Psi)$, i.e. $\Psi = \Lambda(x, \Phi)$, and reformulate Theorem 6.1. Put

$$\widetilde{\varepsilon}_{\mathrm{K}}(\Phi, w, \bar{u}) = \Bigg( m^{-1/2} C_\Omega \|\mathrm{div}^2 \Lambda(x, \Phi) - f\|_0 + \frac{1}{2} L m^{-3/2} \widetilde{\delta}(\Phi, w, \bar{u}) + \Bigg( (\Lambda(x, \mathrm{D}^2\bar{u}) \\ - \Lambda(x, \Phi), \mathrm{D}^2\bar{u} - \Phi)_0 + \frac{1}{2} L m^{-1} \widetilde{\delta}(\Phi, w, \bar{u}) \|\mathrm{D}^2\bar{u} - \Phi\|_0 \Bigg)^{1/2} \Bigg)^2,$$

where

$$\widetilde{\delta}(\Phi, w, \bar{u}) = (M\|\Phi - \mathrm{D}^2 w\|_0 + C_\Omega \|\mathrm{div}^2 \Lambda(x, \Phi) - f\|_0) \|\mathrm{D}^2\bar{u} - \Phi\|_\infty.$$

$\Phi \in L_\infty(\Omega, R^{n\times n})$ is an arbitrary matrix-valued function such that $\Lambda(x, \Phi) \in H(\mathrm{div}^2, \Omega)$, $w \in H_0^2(\Omega)$ is an arbitrary scalar-valued function, $m$ and $M$ are the constants from (6.3), $C_\Omega$ from (6.4), and $L$ the Lipschitz continuity constant of $\Lambda'$. Then the following two theorems proven in [16] hold.

**Theorem 6.2.** *Let $u \in H_0^2(\Omega)$ be the unique weak solution of the problem (6.1), (6.2) and $\bar{u} \in W^{2,\infty}(\Omega)$ an arbitrary function. Then*

$$E(\bar{u}) \le \widetilde{\varepsilon}_{\mathrm{K}}(\Phi, w, \bar{u})$$

*for any $\Phi \in L_\infty(\Omega, R^{n\times n})$ such that $\Lambda(x, \Phi) \in H(\mathrm{div}^2, \Omega)$ and $w \in H_0^2(\Omega)$.*

**Theorem 6.3.** *If the weak solution $u \in W^{2,\infty}(\Omega)$ then the estimator $\eta_{\mathrm{K}}$ of (6.5) is sharp, i.e.*

$$\min_{\Psi \in H(\mathrm{div}^2, \Omega) \cap L_\infty(\Omega, R^{n\times n}),\ w \in H_0^2} \varepsilon_{\mathrm{K}}(\Psi, w, \bar{u}) = E(\bar{u}).$$

## 7. Conclusion

Automatic *hp*-adaptivity belongs to the most advanced topics in the higher-order finite element technology and it is subject to active research. We refer to, e.g., [2,18,3] and references therein.

We have met, in the individual sections of the paper, several analytical a posteriori error indicators and estimators that appear in inequalities, usually with some unknown constants on the right-hand part. They are easily computable from the approximate solution only, the computation is fast (i.e. cheap), but their quantitative properties cannot be easily assessed. Some analytical error estimators are constructed only for the lowest-order polynomial approximation.

There are, however, global analytical error estimates for some classes of problems (see, e.g., [16,17,19]) that require as few unknown constants as possible. Moreover, some papers provide for the estimation of these constants. Exceptionally, there are analytical estimates containing really no unknown constants (see, e.g., [20] for a 2D linear 2nd order elliptic problem).

The best situation (cf. nonlinear parabolic equations) occurs if the analytical estimator is asymptotically exact. However, the asymptotic exactness of estimator could be of little practical advantage. Fortunately enough, a lot of asymptotically exact estimators behave on many classes of problems very properly: they give sharp estimates not only in the limit, but for particular finite *h*, too.

The automatic *hp*-adaptivity gives many *h* as well *p* possibilities for the next step of the solution process. A single number provided by the analytical a posteriori error estimator for each mesh element need not be enough information for the decision. This is the reason for using the computational error estimate (reference solution), that is employed as the standard approach also in solving ordinary differential equations.

The computation of the reference solution is rather time-consuming but need not be carried out at each step of the adaptive solution process. The reference solution is obtained by the same software that is used to compute the approximate solution. Reference solutions are used as robust error estimators to control the adaptive (in particular, *hp*-adaptive) strategies.

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