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Monoid deformations and group representations

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Abstract

The purpose of this paper is to introduce the concept of monoid deformations in connection with group representations. The underlying philosophy for finite reductive monoids M is that while M is contained in a modular representation of the unit group G, a deformation $M(q)$ is contained in a complex representation of G. This is worked out in detail in the case of the Steinberg representation. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

This paper is part of a general program of the author to make linear representation theory of finite monoids relevant to group representation theory. We introduce in this paper the concept of monoid deformations. Though not directly related to our approach, we note that John Rhodes [10] (early 1970s) had an idea of 'resetting' zero products in a finite semigroup. For a finite monoid M with zero, we consider generic 'monoids' $M(t)$ in the indeterminate t. For a scalar c, we call $M(c)$ a deformation of $M = M(0)$. Classical monoid representation theory is extended to a representation theory of monoid deformations.

For a finite reductive group G over $\mathbb{F}_q$, the canonical monoid $\mathscr{M} = \mathscr{M}(G)$ was constructed by Renner and the author [8]. $\mathscr{M}$ is the abstract finite analogue of the canonical compactification of reductive groups in the theory of embeddings of homogeneous spaces [4]. It turns out that $\mathscr{M}$ can also be constructed within the modular Steinberg representation of G. We construct here a bigger monoid $\tilde{\mathscr{M}}$ within the modular Steinberg representation and show that a q-deformation $\tilde{\mathscr{M}}(q)$ of $\tilde{\mathscr{M}}$ is contained

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within the ordinary Steinberg representation. This is facilitated by a description of the Steinberg representation via an associative multiplication on Chevalley's big cell.

1. Abstract monoid deformations

Let M be a finite regular ($a \in aMa$ for all $a \in M$) monoid with zero 0 and unit group G. Let $\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ denote the usual Green's relations on $M : a\mathscr{J}b$ if $MaM = MbM$, $a\mathscr{R}b$ if $aM = bM$, $a\mathscr{L}b$ if $Ma = Mb$, $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$. Let $\mathscr{U} = \mathscr{U}(M)$ denote the set of non-zero $\mathscr{J}$-classes of M. If $X \subseteq M$, let $E(X)$ denote the set of idempotents of X. For $J \in \mathscr{U}$, choose $e_J \in E(J)$ and let $H_J = H(e_J)$ denote the $\mathscr{H}$-class of e_J, i.e. the unit group of $e_J M e_J$.

Let $J \in \mathscr{U}$. Then $J^0 = J \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{otherwise} \end{cases}$$

is a semigroup and $M(J) = G \cup J^0$ is a monoid. Let R_J, L_J denote the $\mathscr{R}$ and $\mathscr{L}$-classes of e, respectively. Choose $\mathscr{L}$-class representatives $X_J = \{1 = a_1, \ldots, a_m\}$ in R_J and $\mathscr{R}$-class representatives $Y_J = \{1 = b_1, \ldots, b_n\}$ in L_J. Then $\Gamma_J = (a_i b_j)$ is a $m \times n$ matrices with entries in $H_J \cup \{0\}$. Γ_J is called the *sandwich matrix* of J. We refer to [2] for details.

We next briefly review semigroup representation theory [2; Ch. 5]. Let F be a field and let FM denote the contracted monoid algebra of M, i.e. the zero of M is the zero of FM. Hence $M\backslash\{0\}$ is a basis of FM. Similarly, let FJ denote the contracted semigroup algebra of J^0. It has basis J. If I is an ideal of M, then FI is an ideal of FM. If $\operatorname{rad} \mathscr{A}$ denotes the radical of an algebra $\mathscr{A}$, then

$$FM/\operatorname{rad} FM \cong \bigoplus_{J \in \mathscr{U}} FJ/\operatorname{rad} FJ.$$

Now, FJ is isomorphic to the *Munn algebra* over FH_J with sandwich matrix Γ_J. If Γ_J is $m \times n$, then this is the algebra of $n \times m$ matrices over FH_J with multiplication given by

$$A \circ B = A\Gamma_J B.$$

Let $\mathit{Irr}\, H_J$ denote the set of irreducible representations of H_J. Let $\theta \in \mathit{Irr}\, H_J$ of degree d and let $\mathscr{A}_\theta$ denote the Munn algebra over the matrix algebra $M_d(F)$ with sandwich matrix $\theta(\Gamma_J)$. Then

$$\mathscr{A}_\theta/\operatorname{rad} \mathscr{A}_\theta \cong M_r(F),$$

where r is the rank of $\theta(\Gamma_J)$. Clearly,

$$FJ/\operatorname{rad} FJ \cong \bigoplus_{\theta \in \mathit{Irr}\, H_J} \mathscr{A}_\theta/\operatorname{rad} \mathscr{A}_\theta.$$

By a representation of M we mean a homomorphism $\varphi : M \to M_n(F)$ such that $\varphi(1)=1$ and $\varphi(0)=0$. The representations of M are in 1–1 correspondence with those of FM. In particular, every representation of M is completely reducible if and only if FM is semisimple. By the above, the set of irreducible representations ($Irr\,M$) of M is in 1–1 correspondence with the set of irreducible representations ($Irr\,H_J$) of H_J as J ranges through $\mathscr{U}$. Let $\tilde{\theta} \in Irr\,M$ correspond to $\theta \in Irr\,H_J$. Then

$$\deg \tilde{\theta} = rk\theta(\Gamma_J),$$

where deg denotes degree and rk denotes rank. In particular, FM is semisimple if and only if FH_J is semisimple and Γ_J is invertible over FH_J for all $J \in \mathscr{U}$. If $\mathscr{U}$ has a least element J_0 with $H_{J_0} = \{e_{J_0}\}$, then there is a unique irreducible representation φ of M such that $\varphi(e_{J_0}) \neq 0$. We call φ the *principal representation* of M (over F).

We now consider *generic* 'monoids' $M(t)$ in the indeterminate t. By this we mean an associative 'operation'

$$\xi : M \times M \to \mathbb{C}(t)M,$$

where $\mathbb{C}(t)$ is the field of rational functions in t, such that:

(1) If $a, b \in M$, $ab \neq 0$ in M, then $\xi(a,b) = ab$.

(2) If $a, b \in M$, $ab = 0$, then $\xi(a,b) = f(t)u$ for some $f(t) \in \mathbb{C}(t)$ with $f(0) = 0$ and $u \in MaM \cap MbM$.

We will write ab for $\xi(a,b)$. M with this new 'operation' is denoted by $M(t)$. So $M(0) = M$. If $c \in \mathbb{C}$ such that c is not a pole of any of the coefficients of $M(t)$, then we call $M(c)$ a *deformation* of $M = M(0)$. The corresponding complex algebra over $\mathbb{C}$ (with basis $M \backslash \{0\}$) is denoted by $\mathbb{C}M(c)$. By a representation of $M(c)$, we mean a map $\varphi : M \to M_n(\mathbb{C})$ such that:

(1) $\varphi(1) = 1$, $\varphi(0) = 0$.

(2) If a, b, $u \in M$, $\alpha \in \mathbb{C}$, such that $ab = \alpha u$ in $M(c)$, then $\varphi(a)\varphi(b) = \alpha\varphi(u)$.

Clearly there is a 1–1 correspondence between the representations of $M(c)$ and those of $\mathbb{C}M(c)$. In particular, every representation of $M(c)$ is completely reducible if and only if $\mathbb{C}M(c)$ is semisimple.

Let $J \in \mathscr{U}$. The multiplication in $J^0(t) = J \cup \{0\}$ is as follows. If $a, b \in J$, $ab = f(t)u$ in $M(t)$, then

$$a \circ b = \begin{cases} f(t)u & \text{if } u \in J, \\ 0 & \text{otherwise.} \end{cases}$$

If $X_J = \{1 = a_1, \ldots, a_m\}$, $Y_J = \{1 = b_1, \ldots, b_n\}$, then the *generic sandwich matrix*,

$$\Gamma_J(t) = (a_i\, b_j)$$

is a matrix over $\mathbb{C}(t)H_J$. The deformation $\Gamma_J(c)$ is a matrix over $\mathbb{C}H_J$.

If I is an ideal of M, then it follows from the definition of $M(t)$ that $\mathbb{C}I(c)$ is an ideal of $\mathbb{C}M(c)$. Hence,

$$\mathbb{C}M(c)/\text{rad}\,\mathbb{C}M(c) \cong \bigoplus_{J \in \mathscr{U}} \mathbb{C}J(c)/\text{rad}\,\mathbb{C}J(c),$$

where $\mathbb{C}J(c)$ is the contracted semigroup algebra of $J^0(c)$. Now, $\mathbb{C}J(c)$ is the Munn algebra over $\mathbb{C}H_J$ with sandwich matrix $\Gamma_J(c)$. Hence, we have:

Theorem 1.1. (i) *The irreducible representations of* $\mathbb{C}M(c)$ *are in* 1–1 *correspondence with those of* $\mathbb{C}H_J, J \in \mathscr{U}$. *If* $\theta \in Irr\, H_J$ *corresponds to* $\tilde{\theta} \in Irr\, M(c)$, *then* $deg\, \tilde{\theta} = rk\, \theta(\Gamma_J(c))$.

(ii) $\mathbb{C}J(c)$ *is semisimple if and only if* $\Gamma_J(c)$ *is invertible over* $\mathbb{C}H_J$.

(iii) $\mathbb{C}M(c)$ *is semisimple if and only if* $\Gamma_J(c)$ *is invertible over* $\mathbb{C}H_J$ *for all* $J \in \mathscr{U}$.

Suppose $\mathbb{C}M$ is semisimple and let $J \in \mathscr{U}$. Then for $\theta \in Irr\, H_J$, $\theta(\Gamma_J)$ is invertible and hence has non-zero determinant. Let $f(t) \in \mathbb{C}(t)$ denote the determinant of $\theta(\Gamma_J(t))$. Then $f(0) \neq 0$. So $f(t) \neq 0$. Thus, $f(c) \neq 0$ for all but finitely many $c \in \mathbb{C}$. Hence,

Corollary 1.2. *Suppose* $\mathbb{C}M$ *is semisimple. Then* $\mathbb{C}M(c)$ *is semisimple for all but finitely many* $c \in \mathbb{C}$.

Example 1.3. Let

$$M = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

denote the symmetric inverse monoid of degree 2. Then $\mathscr{U}(M) = \{G, J\}$, where

$$J = \left\{ e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

We construct the generic 'monoid' $M(t)$ by defining

$$ef = \frac{t^2 + t}{t^4 + 1} a, \qquad fe = t^2 b.$$

Then by associativity,

$$eb = \frac{t^2 + t}{t^4 + 1} e, \qquad fa = t^2 f, \qquad ae = t^2 e,$$

$$a^2 = t^2 a, \qquad bf = \frac{t^2 + t}{t^4 + 1} f, \qquad b^2 = \frac{t^2 + t}{t^4 + 1} b.$$

All other products are as in M. The generic sandwich matrix

$$\Gamma_J(t) = \begin{bmatrix} 1 & (t^2 + t)/(t^4 + 1) \\ t^2 & 1 \end{bmatrix}.$$

For $c \in \mathbb{C}$ with $c^4 \neq -1$, $M(c)$ is a deformation of M. $M = M(0)$, $M(1)$, $M(-1)$ are actual monoids. $M(0)$ is an inverse monoid. $M(1)$ is an orthodox monoid that is not an inverse monoid. $M(-1)$ is a regular monoid that is not an orthodox monoid. $\mathbb{C}M(c)$ is semisimple if and only if c is not a cube root of 1.

If $\mathscr{U}$ has a least element J_0 with $H_{J_0}=\{e_{J_0}\}$, then there is a unique irreducible representation of $M(c)$ such that $\varphi(e_{J_0})\neq 0$. We will then call φ the *principal representation* of $M(c)$.

2. Canonical monoids

Let G be a Chevalley group over $\mathbb{F}_q$, $q=p^\delta$, of adjoint type (such as $PGL_n(\mathbb{F}_q)$), cf. [1]. So G has trivial center. Let B, B^- be opposite Borel subgroups of $G, T=B\cap B^-$. Let $W=N/T$ denote the Weyl group of G with set of simple reflections S. If $x\in W$, let $x=\dot{x}T$, $\dot{x}\in N$. If $I\subseteq S$, let $W_I=\langle I\rangle, P_I=BW_IB, P_I^-=B^-W_IB^-, L_I=P_I\cap P_I^-$. Let $U=\mathrm{O_p}(B)$, $U^-=\mathrm{O_p}(B^-)$, $U_I=\mathrm{O_p}(P_I)$, $U_I^-=\mathrm{O_p}(P_I^-)$ denote the unipotent radicals of B, B^-, P_I, P_I^-, respectively. Then $|U|=|U^-|=q^m$ where m denotes the number of positive roots. If $g\in P_I^-P_I=U_I^-L_IU_I$, then let g_I be defined as

$$g_I\in L_I,\quad g\in U_I^-g_IU_I.$$

In [8] a universal canonical monoid $\mathscr{M}^+=\mathscr{M}^+(G)$ (denoted in [8] by $\mathscr{M}$) is constructed. $\mathscr{M}^+$ has zero 0, unit group G, non-zero $\mathscr{J}$-classes $J_I(I\subseteq S)$, $e_I\in E(J_I)$, such that

$$\begin{aligned} &J_I=Ge_IG, U_Ie_I=\{e_I\}=e_IU_I^-,\\ &H(e_I)=e_IL_I=L_Ie_I\cong L_I. \end{aligned}\tag{1}$$

Moreover for I, $K\subseteq S$,

$$\begin{aligned} &e_Ie_K=e_Ke_I=e_{I\cap K},\\ &e_Ige_K=0\quad \text{for } g\in G\backslash P_I^-P_K. \end{aligned}\tag{2}$$

Let the *cross-section lattice*

$$\Lambda=\{e_I\mid I\subseteq S\}\cong\mathscr{U}(\mathscr{M})\cong 2^S.$$

The monoid $\mathscr{M}^+$ is of basic importance in the theory of monoids of Lie type.

Our focus will be on the *fundamental* canonical monoid $\mathscr{M}=\mathscr{M}(G)$, where for $I\subseteq S$,

$$\begin{aligned} &e_Ig=ge_I=e_I\quad \text{for } g\in Z(L_I),\\ &H_I=H(e_I)=L_I/Z(L_I). \end{aligned}\tag{3}$$

In particular, $H_\emptyset=Be_\emptyset=\{e_\emptyset\}=e_\emptyset B^-$. Since $Z(G)=\{1\}, G$ is the unit group of $\mathscr{M}$. $\mathscr{M}$ has a Bruhat decomposition

$$\mathscr{M}=\bigsqcup_{r\in R}BrB,$$

where R is the Renner monoid of $\mathscr{M}$, cf. [7, 9]:

$$R=\langle N,\Lambda\rangle/T=\bigsqcup_{e\in\Lambda}WeW\cup\{0\}.$$

In R, $H(e_I)=W_I$, $I\subseteq S$. Moreover, R is an inverse monoid (i.e. a regular monoid with commuting idempotents) and

$$E(R)=\{x^{-1}ex \mid e\in\Lambda, x\in W\}\cup\{0\}.$$

By [8; Corollary 2.6], we have:

Theorem 2.1. *The principal representation of $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ is faithful and restricts to the modular Steinberg representation of G.*

Thus, $\mathcal{M}$ can be found within the modular Steinberg representation of G. We now consider an equivalence relation on 2^S that arises naturally in the theory of cuspidal representations of G, cf. [1; Ch. 9]. If $I,I'\subseteq S$, define

$$I\sim I' \quad \text{if } x^{-1}Ix=I' \quad \text{for some } x\in W.$$

Then $x^{-1}L_Ix=L_I'$ and

$$e_I\mathcal{M}e_I\cong\mathcal{M}(H_I)\cong\mathcal{M}(H_{I'})\cong e_{I'}\mathcal{M}e_{I'}.$$

It follows from [5] that e_I and $e_{I'}$ are in the same $\mathcal{J}$-class of the universal fundamental monoid of $\mathcal{M}$. For our purposes, we will construct an intermediate monoid $\tilde{\mathcal{M}}$ with $e_I\,\mathcal{J}\,e_{I'}$, whenever $I\sim I'$. Note that e_I, $e_{I'}$ are not in the same $\mathcal{J}$-class of $\mathcal{M}$ if $I\neq I'$. $\tilde{\mathcal{M}}$ will also be contained in the modular Steinberg representation of G. We will see in the next section that a deformation $\tilde{\mathcal{M}}(q)$ is contained in the original (characteristic 0) Steinberg representation of G.

In $\mathcal{M}$, let $\mathcal{X}=Ge_\emptyset\cup\{0\}$. Let $\mathcal{T}$ denote the monoid (with respect to composition) of all maps $\alpha:\mathcal{X}\to\mathcal{X}$ such that $\alpha(0)=0$. Now $\mathcal{M}$ acts on $\mathcal{X}$ on the left. If $I\subseteq S$, then

$$e_Ixe_\emptyset=\begin{cases} x_Ie_\emptyset & \text{if } x\in P_I^-P_I,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, Λ and hence $\mathcal{M}$ acts faithfully on $\mathcal{X}$. We identify $\mathcal{M}$ with its image in $\mathcal{T}$. For I, $I'\subseteq S$, let

$$\tilde{W}_{I,I}=\{x\in W \mid x^{-1}W_Ix=W_{I'}\}.$$

Thus,

$$\tilde{W}_{I,I'}\neq\emptyset\Leftrightarrow I\sim I'.$$

Let

$$\tilde{W}_I=\tilde{W}_{I,I}=N_W(W_I)\supseteq W_I.$$

If $I\sim I'$, let

$$\tilde{L}_{I,I'}=L_I\tilde{W}_{I,I'}=\tilde{W}_{I,I'}L_{I'}.$$

Let

$$\tilde{L}_I = \tilde{L}_{I,I} = L_I N_W(W_I).$$

Usually, $\tilde{L}_I$ is just $N_G(L_I)$ (see [1; Section 3.6]). Let $I \sim I'$, $g \in \tilde{L}_{I,I'}$. Then $g \in L_I z = zL_{I'}$ with $z \in W$ being of minimum length in $W_I z = zW_{I'}$. Let $\varphi_g^{I,I'} \in \mathscr{M}$ be defined as

$$\varphi_g^{I,I'}(xe_\emptyset) = \begin{cases} gx_{I'}\dot{z}^{-1}e_\emptyset & \text{if } x \in P_{I'}^- P_{I'}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

By [1; Proposition 2.3.3], $z(B \cap L_{I'})z^{-1} \subseteq B$. Hence $\varphi_g^{I,I'}$ is well defined. By (1), (2),

$$e_I \varphi_g^{I,I'} = \varphi_g^{I,I'} = \varphi_g^{I,I'} e_{I'},$$

$$l\varphi_g^{I,I'} = \varphi_{lg}^{I,I'},\ \varphi_g^{I,I'} l' = \varphi_{gl'}^{I,I'} \quad \text{for } l \in L_I,\ l' \in L_{I'}. \tag{5}$$

Also if $K \subseteq I$, then $K' = z^{-1}Kz \subseteq I'$ and

$$\varphi_{\dot{z}}^{I,I'} e_{K'} = \varphi_{\dot{z}}^{K,K'} = e_K \varphi_{\dot{z}}^{I,I'}. \tag{6}$$

Let

$$\tilde{H}_{I,I'} = \{\varphi_g^{I,I'} \mid g \in \tilde{L}_{I,I'}\}.$$

If $g \in \tilde{L}_I$, let $\varphi_g^I = \varphi_g^{I,I}$ and let $\tilde{H}_I = \tilde{H}_{I,I}$. Then

$$\varphi_l^I = le_I = e_I l \quad \text{for all } l \in L_I \tag{7}$$

and

$$H_I \subseteq \tilde{H}_I = \{\varphi_g^I \mid g \in \tilde{L}_I\}.$$

If $I \sim I' \sim I''$, then

$$\varphi_g^{I,I'} \varphi_h^{I',I''} = \varphi_{gh}^{I,I''} \quad \text{for } g \in \tilde{L}_{I,I'},\ h \in \tilde{L}_{I',I''}. \tag{8}$$

In particular, $\tilde{H}_I$ is a group with identity element e_I. Let

$$\tilde{\mathscr{M}} = \tilde{\mathscr{M}}(G) = \bigsqcup_{\substack{I,I' \subseteq S \\ I \sim I'}} G\tilde{H}_{I,I'}G \cup \{0\}.$$

Theorem 2.2. (i) *$\tilde{\mathscr{M}}$ is a regular monoid containing $\mathscr{M}$ and $E(\mathscr{M}) = E(\tilde{\mathscr{M}})$.*
(ii) *The $\mathscr{J}$-class of e_I in $\tilde{\mathscr{M}}$ is*

$$\tilde{J}_I = \bigsqcup_{I' \sim I} G\tilde{H}_{I,I'}G.$$

In particular, $\mathscr{U}(\tilde{\mathscr{M}}) \cong 2^S/\sim$.

(iii) *The* $\mathscr{H}$*-class of* e_I *in* $\tilde{\mathscr{M}}$ *is*

$$\tilde{H}_I \cong \tilde{L}_I/\tilde{L}_I \cap C_G(L_I'),$$

where L_I' *is the subgroup of* L_I *generated by its unipotent* ($= p-$) *elements.*

(iv) *The principal representation of* $\tilde{\mathscr{M}}$ *over* $\overline{\mathbb{F}}_p$ *is faithful and restricts to the modular Steinberg representation of G.*

Proof. (i), (ii) follows from the repeated use of (1)–(8). Let $g \in \tilde{L}_I$. Then $g \in L_I z$ for some $z \in N_W(W_I)$ of minimum length in $W_I z$. So $g = l\dot{z}$ for some $l \in L_I$. Suppose $\varphi_g^I = e_I$. Then

$$e_\emptyset = \varphi_g^I(e_\emptyset) = g\dot{z}^{-1}e_\emptyset = l\,e_\emptyset.$$

Let $k \in L_I$. Then

$$k\,e_\emptyset = e_I k\,e_\emptyset = \varphi_g^I(k\,e_\emptyset) = g\,k\,\dot{z}^{-1}e_\emptyset = g\,k\,\dot{z}^{-1}l^{-1}e_\emptyset = g\,k\,g^{-1}e_\emptyset.$$

Hence, $k^{-1}g\,k\,g^{-1} \in B \cap L_I$. If $l_1 \in L_I$, then by (5), $\varphi_{g'}^I = e_I$ where $g' = l_1\,g\,l_1^{-1}$. It follows that $k^{-1}g\,k\,g^{-1}$ is an element of every Borel subgroup of L_I. Hence, $k^{-1}g\,k\,g^{-1} \in Z(L_I)$ for all $k \in L_I$. In particular, if k is unipotent, then $k^{-1}g\,k\,g^{-1} = 1$. Thus $g \in C_G(L_I')$.

Conversely suppose, $g \in C_G(L_I') \cap \tilde{L}_I$. Then $l \in T$. So for $x \in L_I'$,

$$\varphi_g^I(x\,e_\emptyset) = g\,x\,\dot{z}^{-1}e_\emptyset = x\,g\,\dot{z}^{-1}e_\emptyset = x\,l\,e_\emptyset = x\,e_\emptyset.$$

Since $L_I = L_I' \cdot Z(L_I)$, we see that $\varphi_g^I = e_I$. This proves (iii).

(iv) By (ii), $\tilde{J}_\emptyset = J_\emptyset$. Hence by Theorem 2.1 and [2; Ch. 5], the principal representation of $\tilde{\mathscr{M}}$ over $\overline{\mathbb{F}}_p$ restricts to the modular Steinberg representation of G. Since the representation is faithful on $\mathscr{M}$, it is faithful on $\mathscr{X} \subseteq \mathscr{M}$. Since $\tilde{\mathscr{M}}$ acts faithfully on the left on $\mathscr{X}$, the representation is faithful on $\tilde{\mathscr{M}}$. □

Let Γ_I denote the sandwich matrix of J_I in $\mathscr{M}$. Let $\tilde{\Gamma}_I$ denote the sandwich matrix of $\tilde{J}_I$ in $\tilde{\mathscr{M}}$. Then

$$\tilde{\Gamma}_I = \bigoplus_{I' \sim I} \Gamma_{I'},$$

where the entries of $\Gamma_{I'}$ are changed from $H_{I'} \cup \{0\}$ to $\tilde{H}_I \cup \{0\}$ via

$$H_{I'} \subseteq \tilde{H}_{I'} \cong \tilde{H}_I.$$

Now, $\mathbb{C}\mathscr{M}$ is semisimple by [6]. Hence each $\Gamma_{I'}$ is invertible over $\mathbb{C}H_{I'}$ and hence over $\mathbb{C}\tilde{H}_I$. Thus, we have:

Theorem 2.3. $\mathbb{C}\tilde{\mathscr{M}}$ *is a semisimple algebra.*

If $I \sim I', z \in \tilde{W}_{I,I'}$, let $\varphi_z^{I,I'} = \varphi_{\dot{z}}^{I,I'}T$. Let

$$\tilde{V}_{I,I'} = \{\varphi_z^{I,I'} \mid z \in \tilde{W}_{I,I'}\}.$$

Let

$$\tilde{R} = \bigsqcup_{I, I' \subseteq S} W\tilde{V}_{I,I'}W \cup \{0\}.$$

Theorem 2.4. (i) *$\tilde{R}$ is an inverse monoid containing R with $E(\tilde{R}) = E(R)$ and $\mathscr{U}(\tilde{R}) \cong 2^S/\sim$.*

(ii) *$\tilde{\mathscr{M}} = \bigsqcup_{r \in \tilde{R}} BrB$.*

Proof. (i) follows in the same way as Theorem 2.2. So we prove (ii). Let $a \in G\tilde{H}_{I,I'}G$. Then by (5), $a \in G\varphi_z^{I,I'}G$ with $z \in W$ of minimum length in $W_I z = zW_{I'}$. By (5),

$$l\varphi_z^{I,I'} = \varphi_z^{I,I'}\dot{z}^{-1}l\dot{z} \quad \text{for } l \in L_I. \tag{9}$$

By (1), (5), (9) and the Bruhat decomposition, there exist $x, y \in W$, $l \in L_I$ such that $a \in Bxl\varphi_z^{I,I'}yB$. Now $B_1 = x^{-1}Bx \cap L_I$ and $B_2 = z(yBy^{-1} \cap L_{I'})z^{-1}$ are Borel subgroups of L_I. By the Bruhat decomposition for L_I, $l \in B_1wB_2$ for some $w \in W_I$. By (9), $a \in BrB$ where $r = xw\varphi_z^{I,I'}y \in \tilde{R}$.

Next, we show that the union is disjoint. Let $r_i = x_i\varphi_{z_i}^{I,I'}y_i \in \tilde{R}$ with x_i being of minimum length in x_iW_I and y_i being of minimum length in $W_{I'}y_i$, $i = 1,2$. Suppose $Br_1B = Br_2B$. Then for some $b, b' \in B$,

$$\dot{x}_1\varphi_{z_1}^{I,I'}\dot{y}_1 = b\dot{x}_2\varphi_{z_2}^{I,I'}\dot{y}_2b'.$$

By (5), $\dot{x}_1^{-1}b\dot{x}_2 \in P_I$. So $Bx_2 \cap x_1P_I \neq \emptyset$. Hence, $x_1 = x_2$ and $\dot{x}_1^{-1}b\dot{x}_1 \in P_I$. Similarly $y_1 = y_2$ and $\dot{y}_1b'\dot{y}_1^{-1} \in P_{I'}^-$. Hence for some $u_1 \in x_1^{-1}Ux_1 \cap L_I$, $b_1 \in y_1By_1^{-1} \cap L_{I'}$,

$$\varphi_{z_1}^{I,I'} = u_1\varphi_{z_2}^{I,I'}b_1.$$

By Theorem 2.2, there exist $y \in N_W(W_I)$, $t \in T$ such that $t\dot{y} \in C_G(L_I')$ and

$$\dot{z}_1 = t\dot{y}\,u_1\dot{z}_2b_1 = u_1t\,\dot{y}\,\dot{z}_2b_1.$$

By the Bruhat decomposition $z_1 = yz_2$. It follows that $\varphi_{z_1}^{I,I'} = \varphi_{z_2}^{I,I'}$. Hence $r_1 = r_2$. □

3. Steinberg representation

We wish to find a deformation of $\tilde{\mathscr{M}}$ within the ordinary Steinberg representation ψ of G with the cross-section lattice Λ being represented as

$$e_I \to \psi\left(\sum U_I^- U_I\right), \quad I \subseteq S.$$

To facilitate this we begin by considering a variation of the original approach of Steinberg [11] for the Steinberg representation. In particular, the Steinberg representation ψ is obtained via an associative multiplication on the variant U^-U of Chevalley's big cell B^-B.

For $X \subseteq G$, let $\sum X = \sum_{x\in X} x \in \mathbb{C}G$
and

$$\varepsilon = \frac{1}{|B|}\sum B.$$

Let

$$C = \sum_{x\in W}(-1)^{l(x)}\varepsilon x \varepsilon = \frac{1}{|B|}\sum_{x\in W}(-q)^{-l(x)}\sum BxB. \tag{10}$$

By [11; Lemma 2],

$$C^2 = \sum_{x\in W} q^{-l(x)} C$$

and hence by [11; Theorem 2],

$$e = \frac{1}{\sum_{x\in W} q^{-l(x)}}\, C \tag{11}$$

is a primitive idempotent of $\mathbb{C}G$. Moreover, the ideal

$$\mathscr{C} = \mathbb{C}Ge\mathbb{C}G$$

is a simple algebra of dimension q^{2m}. Also by [3; Theorem 5.7],

$$ex\varepsilon = \varepsilon xe = (-q)^{-l(x)}e \quad \text{for all } x \in W. \tag{12}$$

In particular,

$$(e\,w_0)^2 = (-q)^{-m} e\,w_0,$$

where w_0 is the longest element of W. For our purposes, we need to consider the primitive idempotent

$$f = (-q)^m e\,w_0 \tag{13}$$

of $\mathscr{C}$. Clearly,

$$bf = f = fb' \quad \text{for all } b \in B,\ b' \in B^-. \tag{14}$$

By (12),

$$fxf = (-q)^{l(x)} f \quad \text{for all } x \in W. \tag{15}$$

By the Bruhat decomposition and (14), (15), we see that for all $g \in G$,

$$\begin{aligned} gf &= fgf \Leftrightarrow g \in B, \\ fg &= fgf \Leftrightarrow g \in B^-. \end{aligned} \tag{16}$$

Next, we claim that for all $x \in W$,

$$xe = (-1)^{l(x)}\sum [U^- \cap xUx^{-1}]e. \tag{17}$$

First, assume that $x=s\in S$. Let X_s, X_s^- denote the respective positive and negative root subgroups associated with s. Then

$$BsB=X_s sB. \tag{18}$$

Since $X_s^-\cap B=\{1\}$ and $X_s^-\subseteq B\cup BsB$, we have (as in [11; Lemma 1]),

$$X_s^- B=B\cup(X_s-\{1\})sB.$$

So

$$\sum X_s^- B=\sum B+\sum X_s sB-\sum sB. \tag{19}$$

Let

$$C_0=\frac{1}{|B|}\left(\sum B-\frac{1}{q}\sum BsB\right).$$

Then by (18),

$$sC_0=\frac{1}{|B|}\left(\sum sB-\frac{1}{q}\sum X_s^- B\right). \tag{20}$$

By (19),

$$\sum \dot{s}X_s\sum X_s^- B=q\sum sB+q\sum X_s^- B-\sum X_s^- B.$$

Hence by (20),

$$\begin{aligned}\sum X_s^- C_0&=\sum \dot{s}X_s sC_0\\&=\frac{1}{|B|}\left(\sum X_s^- B-\sum sB-\sum X_s^- B+\frac{1}{q}\sum X_s^- B\right)\\&=\frac{1}{|B|}\left(\frac{1}{q}\sum X_s^- B-\sum sB\right)\\&=-sC_0.\end{aligned}$$

By (10), (11),

$$C_0e=e+\frac{1}{q}e=\left(1+\frac{1}{q}\right)e.$$

Thus,

$$\sum X_s^- e=-se \quad \text{for all } s\in S. \tag{21}$$

We now prove (17) by induction on $l(x)$. If $l(x)=0$, this is obvious. So let $l(x)\geq 1$. Then $x=sy$ for some $y\in W$, $s\in S$ with $l(x)=l(y)+1$. Then by [1; Ch. 2],

$$y^{-1}X_s y\subseteq U, \qquad x^{-1}X_s^- x\subseteq U.$$

It follows that

$$U^- \cap xUx^{-1} = s[U^- \cap yUy^{-1}]sX_s^-. \tag{22}$$

So by (21), (22) and the induction hypothesis,

$$\begin{aligned} xe &= sye \\ &= (-1)^{l(y)} s \sum [U^- \cap yUy^{-1}]e \\ &= (-1)^{l(y)} s \sum [U^- \cap yUy^{-1}]sse \\ &= (-1)^{l(y)+1} \sum s[U^- \cap yUy^{-1}]s \sum X_s^- e \\ &= (-1)^{l(x)} \sum [U^- \cap xUx^{-1}]e. \end{aligned}$$

This establishes (17). Dually for all $x \in W$,

$$ex = (-1)^{l(x)} e \sum [U^- \cap x^{-1}Ux]. \tag{23}$$

By the Bruhat decomposition $\mathscr{C}$ has a basis

$$vfu, \quad v \in U^-, \ u \in U. \tag{24}$$

By (12),

$$\begin{aligned} e\sum_{g\in G} geg^{-1}e &= \sum_{x\in W}\sum_{g\in BxB} egeg^{-1}e = \sum_{x\in W} |BxB| exex^{-1}e \\ &= \sum_{x\in W} |BxB| q^{-2l(x)} e = |B| \sum_{x\in W} q^{-l(x)} e. \end{aligned}$$

By Schur's lemma $\sum_{g\in G} geg^{-1}$ is a scalar element of $\mathscr{C}$. Hence, the unity ξ of $\mathscr{C}$ is given by

$$\begin{aligned} \xi &= \frac{1}{|B| \sum_{x\in W} q^{-l(x)}} \sum_{g\in G} geg^{-1} \\ &= \frac{q^m}{|B| \sum_{x\in W} q^{l(x)}} \sum_{g\in G} geg^{-1}, \quad \text{since } l(xw_0) = m - l(x). \end{aligned}$$

Let $h \in G$. We wish to determine $\psi(h) = h\xi$ in terms of the basis (24) of $\mathscr{C}$. By the Bruhat decomposition

$$G = \bigsqcup_{y\in W} h^{-1}B^- y^{-1}B.$$

Hence,

$$G = \bigsqcup_{x,y\in W} [BxB \cap h^{-1}B^- y^{-1}B].$$

Fix $x,\ y\in W$. Then

$$BxB\cap h^{-1}B^-y^{-1}B=[(U\cap xU^-x^{-1})xB]\cap[h^{-1}(U^-\cap y^{-1}U^-y)y^{-1}B].$$

If $g\in BxB\cap h^{-1}B^-y^{-1}B$, then there exist unique $v\in U^-\cap y^{-1}U^-y$, $u\in U\cap xU^-x^{-1}$ such that

$$g\in h^{-1}vy^{-1}B\cap u^{-1}xB.$$

Let $\sigma(g)=(u,v)$. Then

$$uh^{-1}v\in xBy,\quad hgeg^{-1}=vy^{-1}ex^{-1}u. \tag{25}$$

Let

$$\mathscr{A}=\{(u,v)\,|\,u\in U\cap xU^-x^{-1},\ v\in U^-\cap y^{-1}U^-y,\ uh^{-1}v\in xBy\}.$$

If $(u,v)\in\mathscr{A}$ then $uh^{-1}v=\dot{x}b\dot{y}$ for some $b\in B$. Let

$$g_1=h^{-1}v\dot{y}^{-1}=u^{-1}\dot{x}b\in h^{-1}vy^{-1}B\cap u^{-1}xB.$$

Then $\sigma(g_1)=(u,v)$. So σ is onto. Let $g\in h^{-1}vy^{-1}B\cap u^{-1}xB$ so that $\sigma(g)=(u,v)$. Then

$$g=h^{-1}v\dot{y}^{-1}b_1=u^{-1}\dot{x}b_2\quad\text{for some } b_1,b_2\in B.$$

So $h^{-1}v\dot{y}^{-1}=u^{-1}\dot{x}b_2b_1^{-1}$. Hence $b=b_2b_1^{-1}$. Thus,

$$|\sigma^{-1}(u,v)|=|B|. \tag{26}$$

Let

$$\mathscr{B}=\{(u,v)\,|\,u\in U,\ v\in U^-,\ uh^{-1}v\in xBy\}.$$

Let $u\in U$, $v\in U^-$. Then

$$u=u_1u_0,\quad u_0\in U\cap xU^-x^{-1},\ u_1\in U\cap xUx^{-1},$$

$$v=v_0v_1,\quad v_0\in U^-\cap y^{-1}U^-y,\ v_1\in U^-\cap y^{-1}Uy.$$

Then $x^{-1}u_1x,\ yv_1y^{-1}\in U$. Hence,

$$uh^{-1}v\in xBy\quad\text{if and only if}\quad u_0h^{-1}v_0\in xBy.$$

Thus,

$$(u,v)\in\mathscr{B}\quad\text{if and only if}\quad(u_0,v_0)\in\mathscr{A}. \tag{27}$$

Also by (23),

$$\begin{aligned}ex^{-1}&=ex^{-1}w_0w_0\\&=(-1)^{l(xw_0)}e\sum[U^-\cap w_0xUx^{-1}w_0]w_0\\&=(-1)^m(-1)^{l(x)}ew_0\sum[U\cap xUx^{-1}].\end{aligned}$$

Hence by (17), (25)–(27),

$$\sum_{g\in G} hgeg^{-1} = |B| \sum_{x,y\in W} (-1)^m(-1)^{l(x)+l(y)} \sum_{\substack{u\in U\\ v\in U^-\\ uh^{-1}v\in xBy}} vew_0u.$$

Thus by (13),

$$\psi(h) = h\xi = \frac{1}{\sum_{z\in W} q^{l(z)}} \sum_{\substack{u\in U\\ v\in U^-}} \left[\sum_{\substack{x,y\in W\\ uh^{-1}v\in xBy}} (-1)^{l(x)+l(y)} \right] vfu. \tag{28}$$

Thus by (14), (15), (24), (28),

Theorem 3.1. *On $D = U^-U$, define*

$$vu\cdot v'u' = (-q)^{l(x)}vu' \quad \textit{if } uv' \in B^-xB,\ x\in W.$$

Then $\mathbb{C}D$ is a simple algebra with unity

$$\frac{1}{\sum_{z\in W} q^{l(z)}} \sum_{\substack{u\in U\\ v\in U^-}} \left[\sum_{\substack{x,y\in W\\ uv\in xBy}} (-1)^{l(x)+l(y)} \right] vu.$$

The map $\psi : G \to \mathbb{C}D$ given by

$$\psi(g) = \frac{1}{\sum_{z\in W} q^{l(z)}} \sum_{\substack{u\in U\\ v\in U^-}} \left[\sum_{\substack{x,y\in W\\ ug^{-1}v\in xBy}} (-1)^{l(x)+l(y)} \right] vu$$

is the Steinberg representation of G.

Let us continue to view the Steinberg representation as $\psi : \mathbb{C}G \to \mathscr{C}$. We begin the tideous process (culminating in Theorem 3.2) of constructing a deformation of the monoid $\tilde{\mathscr{M}}$ of Theorem 2.2 within $\mathscr{C} \cong \mathbb{C}D$. For $I \subseteq S$, let

$$e_I = \frac{1}{\sum_{z\in W_I} q^{l(z)}} \sum_{\substack{u\in U\cap L_I\\ v\in U^-\cap L_I}} \left[\sum_{\substack{x,y\in W_I\\ uv\in xBy}} (-1)^{l(x)+l(y)} \right] vfu. \tag{29}$$

Then $e_\emptyset = f$ and $e_S = \xi$ is the unity of $\mathscr{C}$. By Theorem 3.1 applied to L_I,

$$\begin{aligned} e_I &= e_I^2 \quad \text{has rank } q^{m_I}, \\ e_I l &= l e_I \quad \text{for all } l \in L_I, \end{aligned} \tag{30}$$

where m_I is the number of positive roots of L_I. Now,

$$\sum UU^- f = \sum U w_0 \sum U w_0 f = q^m \sum U w_0 \varepsilon w_0 f$$
$$= (-1)^m \sum U w_0 f \quad \text{by (12)} = (-q)^m \varepsilon w_0 f = f \quad \text{by (12)}.$$

Similarly,

$$\sum (U \cap L_I)(U^- \cap L_I) f = f.$$

Since,

$$UU^- = U_I U_I^- (U \cap L_I)(U^- \cap L_I),$$

we get

$$\sum U_I U_I^- f = f \tag{31}$$

and dually,

$$f \sum U_I U_I^- = f. \tag{32}$$

Since L_I normalizes U_I and U_I^-, we see by (29), (31), (32), that

$$\sum U_I U_I^- e_I = e_I = e_I \sum U_I U_I^-. \tag{33}$$

If $u \in U$, $u \neq 1$, then by [1; Theorem 6.4.7], $\psi(u)$ has trace zero. Hence,

$$f_I = \frac{1}{|U_I|} \psi\left(\sum U_I\right) = q^{m_I - m} \psi\left(\sum U_I\right)$$

is an idempotent in $\mathscr{C}$ of rank q^{m_I}. Since $f_I \psi(\sum U_I U_I^-) = \psi(\sum U_I U_I^-)$, we see by (30), (33) that $\psi(\sum U_I U_I^-)$ has rank q^{m_I}. Hence by (30), (33),

$$e_I = \psi\left(\sum U_I U_I^-\right) \quad \text{for } I \subseteq S. \tag{34}$$

In particular,

$$u e_I = e_I = e_I v \quad \text{for } u \in U_I v \in U_I^-. \tag{35}$$

Let c, $c' \in \mathscr{C}$. Then we see by (24) that

$$cvf = c'vf \text{ for all } v \in U^- \Rightarrow c = c'. \tag{36}$$

Now, let $I \subseteq S$, $x \in W$ such that x is of minimum length in $W_I x$. Let $u \in U \cap L_I$. Then by [1; Proposition 2.3.3], $\dot{x}^{-1} u \dot{x} \in B$. Hence by (14), (15), (29),

$$e_I x f = (-q)^{l(x)} f \quad \text{for } x \text{ of min. length in } W_I x. \tag{37}$$

Next, let I, $I' \subseteq S$, $x \in W$ of minimum length in $W_I x W_{I'}$. Let $K = I \cap xI'x^{-1}$, $K' = I' \cap x^{-1}Ix$. Then $K' = x^{-1}Kx$ and $K \sim K'$. Let $v \in U^-$. Then

$$
\begin{aligned}
e_I \dot{x} e_{I'} v f &= e_I \dot{x} e_{I'} v_{I'} f \quad \text{by (35)}\\
&= e_I \dot{x} v_{I'} f \quad \text{by (30), (37)}\\
&= e_I \dot{x} v_{I'} \dot{x}^{-1} \dot{x} f\\
&= e_I \dot{x} v_{K'} \dot{x}^{-1} \dot{x} f \quad \text{by (35)}\\
&= \dot{x} v_{K'} \dot{x}^{-1} e_I \dot{x} f \quad \text{by (30)}\\
&= (-q)^{l(x)} \dot{x} v_{K'} \dot{x}^{-1} f \quad \text{by (37)}\\
&= \dot{x} v_{K'} \dot{x}^{-1} e_K \dot{x} f \quad \text{by (37)}\\
&= e_K \dot{x} v_{K'} \dot{x}^{-1} \dot{x} f \quad \text{by (30)}\\
&= e_K \dot{x} v_{K'} f\\
&= e_K \dot{x} e_{K'} v_{K'} f \quad \text{by (30), (37)}\\
&= e_K \dot{x} e_{K'} v f \quad \text{by (35).}
\end{aligned}
$$

By (36), $e_I \dot{x} e_{I'} = e_K \dot{x} e_{K'}$. Hence,

$$e_I \dot{x} e_{I'} = e_K \dot{x} e_{K'} \quad \text{with } K \subseteq I, K' \subseteq I', K \sim K'. \tag{38}$$

In particular,

$$e_I e_{I'} = e_{I \cap I'} \quad \text{for } I, I' \subseteq S. \tag{39}$$

Next let I, $I' \subseteq S$, $I \sim I'$. Let $x \in W$ be of minimum length in $W_I x = x W_{I'}$. Let $K \subseteq I$. Then $K' = x^{-1}Kx \subseteq I'$ and $K \sim K'$. By (38),

$$e_I \dot{x} e_{K'} = e_K \dot{x} e_{K'} = e_K \dot{x} e_{I'}. \tag{40}$$

Now, let $I, I', I'' \subseteq S$ such that $I \sim I' \sim I''$. Let $x, y \in W$ be of minimum lengths in $W_I x = x W_{I'}$ and $W_{I'} y = y W_{I''}$, respectively. Let $v \in U^-$. Then,

$$
\begin{aligned}
e_I \dot{x} e_{I'} y e_{I''} v f &= e_I \dot{x} e_{I'} \dot{y} v_{I''} f \quad \text{by (30), (35)}\\
&= e_I \dot{x} e_{I'} \dot{y} v_{I''} \dot{y}^{-1} \dot{y} f\\
&= e_I \dot{x} \dot{y} v_{I''} \dot{y}^{-1} e_{I'} \dot{y} f \quad \text{by (30)}\\
&= (-q)^{l(y)} e_I \dot{x} \dot{y} v_{I''} \dot{y}^{-1} f \quad \text{by (37)}\\
&= (-q)^{l(y)} e_I \dot{x} \dot{y} v_{I''} \dot{y}^{-1} \dot{x}^{-1} \dot{x} f\\
&= (-q)^{l(y)} \dot{x} \dot{y} v_{I''} \dot{y}^{-1} \dot{x}^{-1} e_I \dot{x} f \quad \text{by (30)}\\
&= (-q)^{l(x)+l(y)} \dot{x} \dot{y} v_{I''} \dot{y}^{-1} \dot{x}^{-1} f \quad \text{by (37).}
\end{aligned}
$$

Also,

$$
\begin{aligned}
e_I\,\dot{x}\dot{y}\,e_{I''}vf &= e_I\,\dot{x}\dot{y}\,v_{I''}f \quad \text{by (30), (35)}\\
&= e_I\,\dot{x}\dot{y}v_{I''}\dot{y}^{-1}\dot{x}^{-1}\dot{x}\dot{y}f\\
&= \dot{x}\dot{y}v_{I''}\dot{y}^{-1}\dot{x}^{-1}e_I\,\dot{x}\dot{y}f \quad \text{by (30)}\\
&= (-q)^{l(xy)}\dot{x}\dot{y}v_{I''}\dot{y}^{-1}\dot{x}^{-1}f \quad \text{by (37).}
\end{aligned}
$$

Hence by (36),

$$e_I\,\dot{x}e_{I'}\,\dot{y}e_{I''} = (-q)^{l(x)+l(y)-l(xy)}e_I\,\dot{x}\dot{y}e_{I''}. \tag{41}$$

Let $l \in L_I,\ l' \in L_{I'}$. Then $g = l\,\dot{x} \in \tilde{L}_{I,I'},\ h = l'\,\dot{y} \in \tilde{L}_{I',I''}$ and

$$
\begin{aligned}
e_I\,ge_{I'}he_{I''} &= l\,e_I\,\dot{x}l'e_{I'}\,\dot{y}e_{I''} \quad \text{by (30)}\\
&= l\,e_I\,\dot{x}l'\dot{x}^{-1}\dot{x}e_{I'}\,\dot{y}e_{I''}\\
&= l\dot{x}l'\dot{x}^{-1}e_I\,\dot{x}e_{I'}\,\dot{y}\,e_{I''} \quad \text{by (30)}
\end{aligned}
$$

and

$$
\begin{aligned}
e_I\,ghe_{I''} &= e_I\,l\dot{x}\,l'\,\dot{y}e_{I''}\\
&= e_I\,l\dot{x}l'\dot{x}^{-1}\dot{x}\dot{y}e_{I''}\\
&= l\dot{x}l'\dot{x}^{-1}e_I\,xye_{I''} \quad \text{by (30).}
\end{aligned}
$$

Hence by (41),

$$e_I\,ge_{I'}he_{I''} = (-q)^{l(x)+l(y)-l(xy)}e_I\,ghe_{I''}. \tag{42}$$

Let $I \sim I',\ g \in \tilde{L}_{I,I'}$. Then $g \in L_Iz = zL_{I'}$ with $z \in W$ being of minimum length in $W_Iz = z\,W_{I'}$. Let

$$\varphi_g^{I,I'} = (-q)^{-l(z)}e_I\,g\,e_{I'}. \tag{43}$$

Then by (30),

$$
\begin{aligned}
&e_I\,\varphi_g^{I,I'} = \varphi_g^{I,I'} = \varphi_g^{I,I'}e_{I'}\\
&l\varphi_g^{I,I'} = \varphi_{lg}^{I,I'},\ \varphi_g^{I,I'}l' = \varphi_{gl'}^{I,I'} \quad \text{for } l \in L_I,\ l' \in L_{I'}.
\end{aligned} \tag{44}
$$

Also if $K \subseteq I$, then $K' = z^{-1}Kz \subseteq I'$ and $K \sim K'$. So by (40),

$$\varphi_{\dot{z}}^{I,I'}e_{K'} = \varphi_{\dot{z}}^{K,K'} = e_K\,\varphi_{\dot{z}}^{I,I'}. \tag{45}$$

If $g \in \tilde{L}_I$, let $\varphi_g^I = \varphi_g^{I,I}$. Then by (30),

$$\varphi_l^I = le_I = e_I\,l \quad \text{for all } l \in L_I. \tag{46}$$

If $I \sim I' \sim I''$, then by (42),

$$\varphi_g^{I,I'} \varphi_h^{I',I''} = \varphi_{gh}^{I,I''} \quad \text{for } g \in \tilde{L}_{I,I'},\ h \in \tilde{L}_{I',I''}. \tag{47}$$

For $I \sim I'$, let

$$\tilde{H}_{I,I'} = \{\varphi_g^{I,I'} \mid g \in \tilde{L}_{I,I'}\}\ .$$

Then by (47) $\tilde{H}_I = \tilde{H}_{I,I}$ is a group. Let $g \in \tilde{L}_I$. We claim that

$$\varphi_g^I = e_I \ \Leftrightarrow\ g \in C_G(L_I'), \tag{48}$$

where L_I' is the subgroup of L_I generated its unipotent (= p-) elements. Now $g = l\dot{z}$ for some $l \in L_I$, $z \in N_W(W_I)$ of minimum length in $W_I z$. Suppose first that $\varphi_g^I = e_I$. Then

$$\begin{aligned} f = e_I\, f = \varphi_g^I f &= (-q)^{-l(z)} e_I\, g e_I\, f \\ &= (-q)^{-l(z)} e_I\, l z f \\ &= l f \quad \text{by (30), (37).} \end{aligned}$$

Hence for $k \in L_I$,

$$\begin{aligned} kf = e_I\, kf = \varphi_g^I kf &= (-q)^{-l(z)} e_I\, g e_I\, kf = (-q)^{-l(z)} e_I\, l\dot{z}kf \\ &= (-q)^{-l(z)} e_I\, l\dot{z}k\dot{z}^{-1}\dot{z}f = (-q)^{-l(z)} l\dot{z}k\dot{z}^{-1} e_I\, \dot{z}f \quad \text{by (30)} \\ &= l\dot{z}k\dot{z}^{-1} f \quad \text{by (37)} = l\dot{z}k\dot{z}^{-1}l^{-1} f = gkg^{-1} f. \end{aligned}$$

By (16), $k^{-1}gkg^{-1} \in B \cap L_I$ for all $k \in L_I$. If $l_1 \in L_I$, then by (30), $\varphi_{g'}^I = e_I$ where $g' = l_1 g l_1^{-1}$. It follows that $k^{-1}gkg^{-1}$ is an element of every Borel subgroup of L_I. Hence, $k^{-1}gkg^{-1} \in Z(L_I)$ for all $k \in L_I$. In particular, if k is unipotent, $k^{-1}gkg^{-1} = 1$. Thus, $g \in C_G(L_I')$. Conversely, if $g \in C_G(L_I')$, then $l \in T$ and for $v \in U^-$,

$$\begin{aligned} \varphi_g^I v f &= (-q)^{-l(z)} e_I\, g e_I\, v f = (-q)^{-l(z)} e_I\, g v_I f \quad \text{by (30), (35)} \\ &= (-q)^{-l(z)} e_I\, v_I g f = (-q)^{-l(z)} e_I\, v_I l\dot{z}\, f = (-q)^{-l(z)} v_I l\, e_I\, \dot{z} f \quad \text{by (30)} \\ &= v_I l f \quad \text{by (37)} = v_I f \quad \text{by (14)} = e_I\, v f \quad \text{by (30), (35),} \end{aligned}$$

Hence by (36), $\varphi_g^I = e_I$.

Let

$$\tilde{\mathscr{M}} = \bigsqcup_{\substack{I,I' \subseteq S \\ I \sim I'}} G \tilde{H}_{I,I'} G \cup \{0\}.$$

Let $\tilde{\mathcal{M}}' = \tilde{\mathcal{M}} \setminus \{0\}$. Then by the repeated use of (35), (38), (43)–(47),

$$\tilde{\mathcal{M}}'\tilde{\mathcal{M}}' \subseteq \bigsqcup_{i=0}^{m} (-q)^i \tilde{\mathcal{M}}'. \tag{49}$$

Replacing q by an indeterminate t, we have a generic monoid $\tilde{\mathcal{M}}(t)$. By (35)–(49), $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(0)$ is as in Section 2 and $\tilde{\mathcal{M}}(q)$ is as above. We have proved:

Theorem 3.2. (i) *$\tilde{\mathcal{M}}(t)$ is a generic 'monoid' with $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(0)$ as in Theorem* 2.2.

(ii) *The principal complex representation of $\tilde{\mathcal{M}}(q)$ is faithful and restricts to the Steinberg representation of G.*

For $z \in \tilde{W}_{I,I'}$, let $\varphi_z^{I,I'} = \varphi_{\dot{z}} T$ and let

$$\tilde{V}_{I,I'} = \{\varphi_z^{I,I'} \mid z \in \tilde{W}_{I,I'}\}.$$

Let

$$\tilde{R} = \bigsqcup_{I,I' \subseteq S} W\tilde{V}_{I,I'}W.$$

Let $\tilde{R}(t)$ denote the generic 'monoid' in indeterminate t with multiplication analogous to $\tilde{\mathcal{M}}(t)$. Then we have,

Theorem 3.3. (i) *$\tilde{R}(t)$ is a generic monoid with $\tilde{R}(0) = \tilde{R}$, as in Theorem* 2.4.

(ii) *$\tilde{\mathcal{M}}(-1)$ is a regular monoid.*

(iii) *$\tilde{R}(-1)$ is a regular orthodox monoid, i.e. product of idempotents is idempotent.*

(iv) $\tilde{\mathcal{M}}(-1) = \bigsqcup_{r \in \tilde{R}(-1)} BrB$.

Proof. We only need to prove (iii). The idempotent set of $\tilde{R}(-1)$ is

$$E = \{x\, e_I\, y\, e_{I'} z \mid I \sim I',\ y \in \tilde{W}_{I,I'},\ z = y^{-1}x^{-1}\}.$$

If $e, f \in E$, then by the repeated use of (38)–(40), we see that

$$ef = x_0 e_{I_1} x_1 e_{I_2} x_2 e_{I_3} x_3\, e_{I_4} x_4$$

with $x_j \in \tilde{W}_{I_j, I_{j+1}}$, $j = 1,2,3,4$ and $x_4 = x_3^{-1} x_2^{-1} x_1^{-1} x_0^{-1}$. Then by (41),

$$ef = x_0 e_{I_1} x_1 x_2 x_3 e_{I_4} x_4 \in E.$$

This completes the proof. □

Example 3.4. Let $G = GL_2(\mathbb{F}_2)$. Then $\tilde{\mathcal{M}} = \mathcal{M} = M_2(\mathbb{F}_2)$. The multiplication Table 1 for the non-zero singular elements of $\tilde{\mathcal{M}}(t)$ is given below.

Table 1

	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$
$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$
$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$
$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$
$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$
$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$-t\begin{bmatrix}1&1\\0&0\end{bmatrix}$
$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$-t\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$
$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&1\end{bmatrix}$
$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$-t\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$-t\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$
$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$-t\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$-t\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$-t\begin{bmatrix}1&1\\1&1\end{bmatrix}$

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