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Omega-deformed Seiberg–Witten effective action from the M$_5$-brane

Neil Lambert [1], Domenico Orlando *, Susanne Reffert

Theory Group, Physics Department, Organisation européenne pour la recherche nucléaire (CERN), CH-1211 Geneva 23, Switzerland

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ABSTRACT

We obtain the leading order corrections to the effective action of an M$_5$-brane wrapping a Riemann surface in the eleven-dimensional supergravity Ω-background. The result can be identified with the first order ϵ-deformation of the Seiberg–Witten effective action of pure $SU(2)$ gauge theory. We also comment on the second order corrections and the generalization to arbitrary gauge group and matter content.

1. Introduction

Ever since the classic result of Seiberg and Witten (sw) [1], $\mathcal{N}=2$ gauge theories have occupied a prominent place in theoretical physics. The resulting low energy sw effective action is given in terms of a Riemann surface, the sw curve, which encodes all the perturbative and non-perturbative quantum effects of the gauge theory. While all the perturbative corrections had been known since [2–4], this solution gave a prediction for an infinite number of non-perturbative instanton corrections, the first few terms of which could be checked by explicit computation [5,6].

Not long afterwards, M-theory was developed as an eleven-dimensional non-perturbative completion of String Theory. In a striking paper Witten showed how the sw curve could be naturally obtained from the geometry of intersecting NS$_5$ and D$_4$-branes lifted to M-theory where they become a single M$_5$-brane [7]. Moreover the complete quantum sw effective action for $\mathcal{N}=2$ supersymmetric $SU(N)$ Yang–Mills theory was obtained in [8] from the classical dynamics of the M$_5$-brane.

An alternative method to compute the sw solution from first principles came with Nekrasov's seminal paper using the Ω-background [9]. This background deforms the gauge theory and allows for localization techniques to be used to compute all the instanton corrections and also reconstruct the curve and its associated quantities [10]. Since then the Ω-background has received a lot of interest, most recently in the context of the correspondence by Alday, Gaiotto and Tachikawa [11] and work related to it.

The so-called *fluxtrap* background [12,13] provides a string-theoretical construction of the Euclidean Ω-background determined by a two-form $\omega=\mathrm{d}U$. In particular the bosonic Abelian worldvolume action for D$_4$-branes suspended between NS$_5$-branes in this background was given in [14]. The generalization to non-Abelian fields is given by ($\mu,\nu=0,1,2,3$)

$$\begin{aligned}\mathscr{L}_{\mathrm{D}_4}=\frac{1}{g_4^2}\operatorname{Tr}\Big[&\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}\\&+\frac{1}{2}\Big(\mathbf{D}_\mu\boldsymbol{\varphi}+\frac{1}{2}\mathbf{F}_{\mu\lambda}\hat{U}^\lambda\Big)\Big(\mathbf{D}_\mu\bar{\boldsymbol{\varphi}}+\frac{1}{2}\mathbf{F}_{\mu\rho}\hat{U}^\rho\Big)\\&-\frac{1}{4}[\boldsymbol{\varphi},\bar{\boldsymbol{\varphi}}]^2+\frac{1}{8}\big(\hat{U}^\mu\mathbf{D}_\mu(\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}})\big)^2\Big],\end{aligned}\tag{1.1}$$

where a hat denotes the pullback to the brane and a bold-face indicates a non-Abelian field. This action agrees with the first order action obtained in [10]. The *fluxtrap* can be lifted to M-theory [14]. At order ϵ it is given by ($M,N=0,1,2,\dots,10$)

$$g_{MN}=\delta_{MN}+\mathcal{O}(\epsilon^2),\tag{1.2a}$$

$$G_4=(\mathrm{d}z+\mathrm{d}\bar{z})\wedge(\mathrm{d}s+\mathrm{d}\bar{s})\wedge\omega,\tag{1.2b}$$

where $s=x^6+\mathrm{i}x^{10}$, $z=x^8+\mathrm{i}x^9$, and

$$\omega=\epsilon_1\,\mathrm{d}x^0\wedge\mathrm{d}x^1+\epsilon_2\,\mathrm{d}x^2\wedge\mathrm{d}x^3+\epsilon_3\,\mathrm{d}x^4\wedge\mathrm{d}x^5.\tag{1.3}$$

The background has 8 Killing spinors if $\epsilon_1+\epsilon_2+\epsilon_3=0$, and 16 Killing spinors in the special case $\epsilon_1=-\epsilon_2$ and $\epsilon_3=0$.[2]

In this Letter we will derive the corrections to first order in ϵ to the Ω-deformed sw action. We do this by employing the M-theory lift of the fluxtrap background. As we will see, the classical M-theory calculation has the invaluable benefit of giving a quantum result in gauge theory since in this case, the result is independent of the effective coupling in the gauge theory. We embed the M$_5$-brane in the Ω-background and study the most supersymmetric configuration which to first order in ϵ is still of the form $\mathbb{R}_4\times\Sigma$ with an additional self-dual three-form. This is the ground

* Corresponding author.
E-mail address: domenico.orlando@cern.ch (D. Orlando).

[1] On leave from King's College London.

[2] The ϵ_3 component, although generically non-vanishing, will not play a role in this Letter as the M$_5$-brane will be held fixed in the x^4, x^5 plane.

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state of a six-dimensional theory on top of which we have fluctuations fulfilling some assumptions detailed in the following. These fluctuations obey *scalar* and *vector equations of motion* that arise from the six-dimensional theory, where the scalar equation encodes the fact that the M5-brane is a (generalized) *minimal surface* and the vector equation posits that the self-dual three-form on the brane is the (generalized) *pullback of the three-form field in the bulk*. To arrive at the four-dimensional gauge theory, we must integrate these equations over the Riemann surface Σ using an appropriate measure. The integration results in one vector equation and two scalar equations in four dimensions, which are the Euler–Lagrange equations for a four-dimensional action, which in the case $\epsilon = 0$ reproduces the undeformed sw action. We explicitly treat the case of $SU(2)$ without matter, however there is a natural generalization of our result to any gauge group and matter content.

The plan of this Letter is as follows. In Section 2 we describe the embedding of the M_5-brane, the six-dimensional equations of motion and their reduction to four dimensions. We also give an action that captures these equations of motion. This action can be extrapolated to second order in ϵ and generalized to arbitrary gauge group and matter content. In Section 3 we give our conclusions. We also provide an appendix that gives some technical steps in the evaluation of various non-holomorphic integrals over the Riemann surface that arise.

2. M_5-brane dynamics in the Ω-fluxtrap

The homogeneous embedding of the M_5*-brane.* Due to the fundamentally Euclidean nature of the fluxtrap background, we will be discussing the Euclidean version of sw-theory. For this reason, the self-duality condition for the three-form h_3 on the M_5-brane turns into

$$\mathrm{i} *_6 h_3 = h_3, \tag{2.1}$$

which we will refer to as *self-duality*.

The embedding of the M_5-brane in the fluxtrap background at order ϵ has already been discussed in [14], where it was found that the brane wraps a Riemann surface. Let us recall here the argument. As discussed in [7], the M-theory lift of a NS_5–D_4 system (extended respectively in $x^0, \ldots, x^3, x^8, x^9$ and $x^0, \ldots, x^3, x^6$) is a single M_5-brane extended in $x^0, \ldots, x^3$ and wrapping a two-cycle in x^6, x^8, x^9, x^{10}. We use static gauge and assume that the M_5-brane has coordinates x^μ, $\mu = 0, 1, 2, 3$ and $z = x^8 + \mathrm{i}x^9$. We also assume that the only non-vanishing scalar field is $s = x^6 + \mathrm{i}x^{10}$. The precise form of the embedding is found if we require this brane to preserve the same supersymmetries of the original IIA system. Given the Killing spinors η_0 of the bulk, the M_5-brane preserves those satisfying [15,16] ($m, n = 0, 1, 2, \ldots, 5$)

$$\begin{aligned} &\Pi_-^{M_5}\eta_0 = \frac{1}{2}(1 - \Gamma_{M_5})\eta_0 = 0, \\ &\Gamma_{M_5} = -\frac{\epsilon^{m_1\cdots m_6}\hat{\Gamma}_{m_1\cdots m_6}}{6!\sqrt{\hat{g}}}\left(1 - \frac{1}{3}\hat{\Gamma}^{n_1n_2n_3}h_{n_1n_2n_3}\right), \end{aligned} \tag{2.2}$$

where $\hat{\Gamma}$ and $\hat{g}$ are the gamma matrices and the metric, pulled back to the brane. Here h_3 is the self-dual three-form on the M_5-brane worldvolume which satisfies

$$\mathrm{d}H_3 = -\frac{1}{4}\hat{G}_4, \tag{2.3}$$

where $H_3 = h_3 + \mathcal{O}(h_3^3)$.

For $\epsilon = 0$ we have $h_3 = 0$ and the M_5-brane is described by a Riemann surface $\bar{\partial}s = 0$ [7]. Let us now consider the first order effect that arises when turning on ϵ. To this order we may simply take $H_3 = h_3$ but in principle s may pick up a non-holomorphic piece. However at $\mathcal{O}(\epsilon)$ the pullback only depends holomorphically on $s(z)$ since $\hat{\omega}$ is by itself of order ϵ:

$$\hat{G}_4 = -(\partial s - \bar{\partial}\bar{s})\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \hat{\omega} + \mathcal{O}(\epsilon^2). \tag{2.4}$$

Therefore we can take

$$h_3 = \frac{1}{4}\big(\bar{s} - \bar{z}\partial s + f(z)\big)\,\mathrm{d}z \wedge \hat{\omega}^- + \frac{1}{4}\big(s - z\bar{\partial}\bar{s} + \bar{f}(\bar{z})\big)\,\mathrm{d}\bar{z} \wedge \hat{\omega}^+, \tag{2.5}$$

where f is an arbitrary holomorphic function and we have decomposed the two-form $\hat{\omega}$ as

$$\begin{aligned} \hat{\omega} &= \frac{\epsilon_1 + \epsilon_2}{2}\big(\mathrm{d}x^0 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3\big) \\ &\quad + \frac{\epsilon_1 - \epsilon_2}{2}\big(\mathrm{d}x^0 \wedge \mathrm{d}x^1 - \mathrm{d}x^2 \wedge \mathrm{d}x^3\big) \\ &= \hat{\omega}^+ + \hat{\omega}^-. \end{aligned} \tag{2.6}$$

These are all the ingredients needed to write the supersymmetry condition,

$$\Pi_-^{M_5}\eta = \Pi_-^{M_5}\Pi_+^{NS_5}\Pi_+^{D_4}\eta_0 = 0, \tag{2.7}$$

where the projectors Π^{NS_5} and Π^{D_4} refer to the M_5-branes resulting from the lift of the NS_5-brane and D_4-brane introduced above such that $\eta = \Pi_+^{NS_5}\Pi_+^{D_4}\eta_0$ are the Killing spinors preserved by the branes. Since the two M_5-brane projectors commute, the full configuration preserves two supercharges in the generic case and four if $\epsilon_1 = -\epsilon_2$. An explicit calculation shows that the condition is satisfied at $\mathcal{O}(\epsilon)$ if

$$\begin{cases} \bar{\partial}s = 0, \\ f(z) = 0, \end{cases} \tag{2.8}$$

which completely fix the embedding of the M_5-brane and the self-dual field h_3.

Thus even at order $\mathcal{O}(\epsilon)$ the brane is embedded holomorphically in spacetime. For the simplest case corresponding to pure $SU(2)$ Yang–Mills, the precise form was found in [7] and is determined implicitly by

$$t^2 - 2B(z|u)t + \Lambda^4 = 0, \quad t = \Lambda^2 e^{-s/R}, \tag{2.9}$$

where $B(z|u) = \Lambda^4 z^2 - u$, Λ is a mass scale and R the radius of the x^{10}-direction. This embedding defines a Riemann surface Σ with modulus u,

$$\Sigma = \big\{(z, s)\ \big|\ s = s(z|u)\big\}. \tag{2.10}$$

It is useful to observe that

$$\frac{\partial s}{\partial u}\,\mathrm{d}z = -\frac{1}{2\Lambda^4 z}\frac{\partial s}{\partial z}\,\mathrm{d}z = \frac{R\,\mathrm{d}z}{\sqrt{Q(z|u)}} = R\lambda \tag{2.11}$$

is the unique holomorphic one-form on Σ where $Q(z|u) = B(z|u)^2 - \Lambda^4$. For most of this Letter we will simply set $R = \Lambda = 1$. They are in principle needed on dimensional grounds, since both s and z have dimensions of length whereas the modulus u is usually taken to have mass-dimension two. We will briefly reinstate them in the conclusions by simply rescaling z and s, when discussing the quantum nature of our result.

Equations of motion in 6d. Having found the embedding of the M_5-brane we want to describe the low energy dynamics of the fluctuations around the equilibrium. In fact, since we are interested in the effective four-dimensional theory living on $x^0, \dots, x^3$ which results from integrating the M_5 equations of motion over the Riemann surface Σ, we will assume that:

1. the geometry of the five-brane is still a fibration of a Riemann surface over $\mathbb{R}^4$;
2. for each point in $\mathbb{R}^4$ we have the same Riemann surface as above, but with a different value of the modulus u.

In other words, the modulus u of Σ is a function of the worldvolume coordinates and the embedding is still formally defined by the same equation, but now $s = s(z|u(x^\mu))$ so that the x^μ-dependence is entirely captured by

$$\partial_\mu s\big(z\big|u\big(x^\mu\big)\big) = \partial_\mu u \frac{\partial s}{\partial u}. \tag{2.12}$$

For ease of notation we will drop in the following the explicit dependence of s on $u(x^\mu)$ and write directly $s = s(z, x^\mu)$. Much of our discussion follows the undeformed case considered in detail in [8,17,18].

The dynamics can be obtained by evaluating the M_5-brane equations of motion. Here we will only focus on the bosonic fields. Covariant equations of motion for the M_5-brane were obtained in [15,16]. In general these are rather complicated equations, particularly with regard to the three-form. However in this Letter we only wish to work to linear order in ϵ and quadratic order in spatial derivatives ∂_μ. In particular we can take $H_3 = h_3$ and the equations of motion reduce to[3]

$$\big(\hat{g}^{mn} - 16h^{mpq}h^n{}_{pq}\big)\nabla_m\nabla_n X^I = -\frac{2}{3}\hat{G}^I{}_{mnp}h^{mnp}, \tag{2.13}$$

$$\mathrm{d}h_3 = -\frac{1}{4}\hat{G}_4, \tag{2.14}$$

where $I = 6, \dots, 10$ and the geometrical quantities are defined with respect to the pullback of the spacetime metric to the brane $\hat{g}_{mn}$.

As a first step we need to write the three-form field on the brane. In full generality, h_3 can be decomposed as

$$h_3 = -\frac{1}{4}\big(\hat{C}_3 + \mathrm{i} *_6 \hat{C}_3 - \Phi\big), \tag{2.15}$$

where $\hat{C}_3$ is the pullback of the three-form in the bulk, and Φ is a self-dual three-form that will encode the fluctuations of the four-dimensional gauge field.

Since we ultimately want to discuss the gauge theory living on the worldvolume coordinates $x^0, \dots, x^3$, we make the following self-dual ($\mathrm{i} *_6 \Phi = \Phi$) ansatz for Φ:

$$\begin{aligned}\Phi = {}& \frac{\kappa}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}z + \frac{\bar{\kappa}}{2}\widetilde{\mathcal{F}}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\bar{z} \\ & + \frac{1}{1+|\partial s|^2}\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\big(\partial^\tau s\bar{\partial}\bar{s}\kappa\mathcal{F}_{\sigma\tau} - \partial^\tau\bar{s}\partial s\bar{\kappa}\widetilde{\mathcal{F}}_{\sigma\tau}\big)\,\mathrm{d}x^\mu \\ & \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho.\end{aligned} \tag{2.16}$$

The two-form $\mathcal{F}$ is anti-self-dual in four dimensions, while $\widetilde{\mathcal{F}}$ is self-dual:

$$*_4\mathcal{F} = -\mathcal{F}, \qquad *_4\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}. \tag{2.17}$$

Here $*_4$ is the flat space Hodge star and $\kappa(z)$ is a holomorphic function given by [17]

$$\kappa = \frac{\mathrm{d}s}{\mathrm{d}a} = \left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^{-1}\lambda_z. \tag{2.18}$$

Here $\lambda = \lambda_z dz$ is the holomorphic one-form on Σ and a is the scalar field used in the Seiberg–Witten solution and related to λ by

$$\frac{\mathrm{d}a}{\mathrm{d}u} = \oint_A \lambda, \tag{2.19}$$

where A is the a-cycle of Σ. In the following, $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ will be related to the four-dimensional gauge field strength, thus justifying our ansatz.

We also need to choose a gauge for the three-form potential C_3 in the bulk:

$$C_3 = -\frac{1}{2}\big(\bar{s}\,\mathrm{d}v - \bar{v}\,\mathrm{d}s + s\,\mathrm{d}v - \bar{v}\,\mathrm{d}\bar{s}\big) \wedge \omega + \text{c.c.} \tag{2.20}$$

Its pullback on the Riemann surface $\{v = z, s = s(z, x^\mu)\}$ is given by

$$\begin{aligned}\hat{C}_3 = {}& -\frac{1}{2}\big(\bar{s}\,\mathrm{d}z - \bar{z}\partial s\,\mathrm{d}z - \bar{z}\partial_\mu s\,\mathrm{d}x^\mu + s\,\mathrm{d}z - \bar{z}\bar{\partial}\bar{s}\,\mathrm{d}\bar{z} - \bar{z}\partial_\mu\bar{s}\,\mathrm{d}x^\mu\big) \\ & \wedge \hat{\omega} + \text{c.c.}\end{aligned} \tag{2.21}$$

We are only interested in terms up to second order in the spacetime derivatives ∂_μ and in particular we observe that $\hat{\omega}$ is by itself of first order. It follows that the six-dimensional Hodge dual is given by

$$\begin{aligned}\mathrm{i} *_6 \hat{C}_3 = {}& \frac{1}{2}\big(\bar{s}\,\mathrm{d}z - \bar{z}\partial s\,\mathrm{d}z + s\,\mathrm{d}z + \bar{z}\bar{\partial}\bar{s}\,\mathrm{d}\bar{z} - s\,\mathrm{d}\bar{z} \\ & + z\bar{\partial}\bar{s}\,\mathrm{d}\bar{z} - \bar{s}\,\mathrm{d}\bar{z} - z\partial s\,\mathrm{d}z\big) \wedge {}^*\hat{\omega} \\ & + \frac{1}{2\cdot 3!}\big(1 + |\partial s|^2\big)\epsilon_{\mu\nu\lambda\rho}C^{\mu\nu\lambda}\,\mathrm{d}x^\rho \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ & + \frac{1}{1+|\partial s|^2}\epsilon_{\mu\nu\rho\sigma}\big(\partial^\tau s\bar{\partial}\bar{s}\hat{C}_{\sigma\tau z} - \partial^\tau\bar{s}\partial s\hat{C}_{\sigma\tau\bar{z}}\big)\,\mathrm{d}x^\mu \\ & \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho,\end{aligned} \tag{2.22}$$

where ${}^*\hat{\omega} = *_4\hat{\omega} = \hat{\omega}^+ - \hat{\omega}^-$.

The vector equation. Consider now the vector equation $\mathrm{d}h_3 = -\frac{1}{4}\hat{H}_4$. Given our expression for h_3, the equation becomes

$$\mathrm{d}\Phi = \mathrm{i}\mathrm{d} *_6 \hat{C}_3, \tag{2.23}$$

where we see explicitly the role of the bulk three-form as source for the gauge field on the brane. At this point it is useful to quickly discuss the issue of gauge covariance of the three-form equation. The bulk three-form is defined up to the differential of a two-form $C_3 \mapsto C_3' + \mathrm{d}B_2$. Under this shift the vector equation becomes

$$\mathrm{d}\Phi = \mathrm{i}\mathrm{d} *_6 \hat{C}_3' + \mathrm{i}\mathrm{d} *_6 \mathrm{d}\hat{B}_2, \tag{2.24}$$

which can be compensated for by an analogous shift in the fluctuations:

$$\Phi \mapsto \Phi' + \mathrm{d}\hat{B}_2 + \mathrm{i} *_6 \mathrm{d}\hat{B}_2. \tag{2.25}$$

Let us go back to our ansatz. The tensor Φ does not contribute to the $\mu\nu z\bar{z}$ component:

$$\mathrm{d}\Phi|_{\mu\nu z\bar{z}} \equiv 0 \tag{2.26}$$

so we only need to verify that

$$\mathrm{d} *_6 \hat{C}|_{\mu\nu z\bar{z}} = 0, \tag{2.27}$$

[3] Note that we have chosen the opposite sign to the *rhs* of the scalar equation as compared to what is given in [16]. This corresponds to a choice of brane or anti-brane.

which is satisfied up to terms of order $\mathcal{O}(\partial_\mu)^3$, taking into account the fact that $\hat{\omega}$ is by itself of order $\mathcal{O}(\partial_\mu)$. Similarly, also the $\mu\nu\lambda\rho$ component of the equation of motion is of higher order.

It is convenient to take the six-dimensional dual of the remaining terms and decompose them in coordinates:

$$*_6\mathrm{d}(\Phi - \mathrm{i} *_6 \hat{C}_3) = \frac{1}{2}E_{\mu z}\,\mathrm{d}x^\mu \wedge \mathrm{d}z + \frac{1}{2}E_{\mu\bar{z}}\,\mathrm{d}x^\mu \wedge \mathrm{d}\bar{z} = 0, \quad (2.28)$$

where explicitly

$$E_{\mu z} = \partial_\mu(\kappa\mathcal{F}_{\mu\nu} - \hat{C}_{\mu\nu z}) + \partial\left[\frac{\bar{\partial}\bar{s}\partial_\nu s}{1+|\partial s|^2}(\kappa\mathcal{F}_{\mu\nu} - \hat{C}_{\mu\nu z})\right] - \partial\left[\frac{\partial s\partial_\nu \bar{s}}{1+|\partial s|^2}(\bar{\kappa}\widetilde{\mathcal{F}}_{\mu\nu} - \hat{C}_{\mu\nu\bar{z}})\right], \quad (2.29a)$$

$$E_{\mu\bar{z}} = \partial_\mu(\bar{\kappa}\widetilde{\mathcal{F}}_{\mu\nu} - \hat{C}_{\mu\nu\bar{z}}) + \bar{\partial}\left[\frac{\partial s\partial_\nu \bar{s}}{1+|\partial s|^2}(\bar{\kappa}\widetilde{\mathcal{F}}_{\mu\nu} - \hat{C}_{\mu\nu\bar{z}})\right] - \bar{\partial}\left[\frac{\bar{\partial}\bar{s}\partial_\nu s}{1+|\partial s|^2}(\kappa\mathcal{F}_{\mu\nu} - \hat{C}_{\mu\nu z})\right]. \quad (2.29b)$$

Note that because of the epsilon tensors in the definition of $E_{\mu z}$, the equations only depend on $\hat{\omega}$ and not on ${}^*\hat{\omega}$.

To obtain the equations of motion of the vector zero-modes in four dimensions we need to reduce these equations on the Riemann surface. In order for the integral to be well-defined everywhere on Σ we have only two possible choices for the integrand, depending on the (unique) one-form λ or its complex conjugate:

$$\int_\Sigma *_6\mathrm{d}(\Phi - \mathrm{i}\mathrm{d} * \hat{C}_3) \wedge \bar{\lambda} = \mathrm{d}x^\mu \wedge \int_\Sigma E_{\mu z}\,\mathrm{d}z \wedge \bar{\lambda} = 0, \quad (2.30a)$$

$$\int_\Sigma *_6\mathrm{d}(\Phi - \mathrm{i}\mathrm{d} * \hat{C}_3) \wedge \lambda = \mathrm{d}x^\mu \wedge \int_\Sigma E_{\mu\bar{z}}\,\mathrm{d}\bar{z} \wedge \lambda = 0. \quad (2.30b)$$

The explicit integration is relatively straightforward using the techniques explained in Appendix A. The only non-vanishing integrals have been already evaluated in [8,18]:

$$I_0 = \int_\Sigma \lambda \wedge \bar{\lambda} = \frac{\mathrm{d}a}{\mathrm{d}u}(\tau - \bar{\tau})\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}}, \quad (2.31)$$

$$K = \int_\Sigma \bar{\partial}\left[\frac{\lambda_z\bar{\partial}\bar{s}}{1+|\partial s|^2}\right]\mathrm{d}\bar{z} \wedge \lambda = -\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^2\frac{\mathrm{d}\tau}{\mathrm{d}u}, \quad (2.32)$$

where one uses the following definitions:

$$a = \oint_A \lambda_{SW}, \qquad a_D = \oint_B \lambda_{SW},$$
$$\tau = \frac{\mathrm{d}a_D}{\mathrm{d}a}, \qquad \lambda = \frac{\partial\lambda_{SW}}{\partial u}, \quad (2.33)$$

along with the Riemann bi-linear identity

$$\int \lambda \wedge \bar{\lambda} = \oint_B \lambda \oint_A \bar{\lambda} - \oint_A \lambda \oint_B \bar{\lambda}. \quad (2.34)$$

The two integrals in Eq. (2.30) become

$$(\tau - \bar{\tau})(\partial_\mu\mathcal{F}_{\mu\nu} + \partial_\mu a\hat{\omega}_{\mu\nu}) + \partial_\mu\tau\mathcal{F}_{\mu\nu} - \partial_\mu\bar{\tau}\widetilde{\mathcal{F}}_{\mu\nu} = 0, \quad (2.35a)$$

$$(\tau - \bar{\tau})(\partial_\mu\widetilde{\mathcal{F}}_{\mu\nu} + \partial_\mu \bar{a}\hat{\omega}_{\mu\nu}) + \partial_\mu\tau\mathcal{F}_{\mu\nu} - \partial_\mu\bar{\tau}\widetilde{\mathcal{F}}_{\mu\nu} = 0. \quad (2.35b)$$

Taking the difference of the two equations we find

$$\partial_\mu(\mathcal{F}_{\mu\nu} - \widetilde{\mathcal{F}}_{\mu\nu}) = -\partial_\mu(a - \bar{a})\hat{\omega}_{\mu\nu}, \quad (2.36)$$

which is solved by writing

$$\begin{cases} \mathcal{F} = (1 - *)F - (a - \bar{a})\hat{\omega}^-, \\ \widetilde{\mathcal{F}} = (1 + *)F + (a - \bar{a})\hat{\omega}^+, \end{cases} \quad (2.37)$$

where F satisfies the standard Bianchi identity

$$\mathrm{d} * F = 0, \quad (2.38)$$

and can be written as the differential of a one-form $F = \mathrm{d}A$. In the following we will identify F with the four-dimensional gauge field and, in this sense, Eq. (2.36) represents the correction to the Bianchi equations introduced by the Ω-deformation. Substituting this condition into the first equation of (2.35), we derive the final form of the four-dimensional vector equations:

$$(\tau - \bar{\tau})\left[\partial_\mu F_{\mu\nu} + \frac{1}{2}\partial_\mu(a + \bar{a})\hat{\omega}_{\mu\nu} + \frac{1}{2}\partial_\mu(a - \bar{a}){}^*\hat{\omega}_{\mu\nu}\right] + \partial_\mu(\tau - \bar{\tau})\left[F_{\mu\nu} + \frac{1}{2}(a - \bar{a}){}^*\hat{\omega}_{\mu\nu}\right] - \partial_\mu(\tau + \bar{\tau})\left[{}^*F_{\mu\nu} + \frac{1}{2}(a - \bar{a})\hat{\omega}_{\mu\nu}\right] = 0, \quad (2.39)$$

where ${}^*F = *_4F$.

The scalar equation. Next we turn our attention to evaluating the scalar equation. The main new ingredient with respect to the calculation in the literature [17] is the presence of a *rhs* term in Eq. (2.13), which reads

$$-\frac{2}{3}\hat{G}^I{}_{mnp}h^{mnp} = \frac{2}{1+|\partial s|^2}\hat{\omega}^-_{\mu\nu}\mathcal{F}_{\mu\nu}\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^{-1}\lambda_z + \frac{2}{1+|\partial s|^2}\hat{\omega}^+_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\left(\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}}\right)^{-1}\bar{\lambda}_{\bar{z}}, \quad (2.40)$$

for both non-trivial cases $X^I = s$ and $X^I = \bar{s}$. The two corresponding scalar equations take the form

$$E = \partial_\mu\partial_\mu s - \partial\left[\frac{\partial_\rho s\partial_\rho\bar{s}\bar{\partial}\bar{s}}{1+|\partial s|^2}\right] - \frac{16\partial^2 s}{(1+|\partial s|^2)^2}h_{\mu\nu\bar{z}}h_{\mu\nu\bar{z}} - 2\hat{\omega}^-_{\mu\nu}\mathcal{F}_{\mu\nu}\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^{-1}\lambda_z + 2\hat{\omega}^+_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\left(\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}}\right)^{-1}\bar{\lambda}_{\bar{z}} = 0, \quad (2.41)$$

$$\bar{E} = \partial_\mu\partial_\mu \bar{s} - \bar{\partial}\left[\frac{\partial_\rho \bar{s}\partial_\rho s\partial s}{1+|\partial s|^2}\right] - \frac{16\bar{\partial}^2 \bar{s}}{(1+|\partial s|^2)^2}h_{\mu\nu z}h_{\mu\nu z} - 2\hat{\omega}^-_{\mu\nu}\mathcal{F}_{\mu\nu}\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^{-1}\lambda_z + 2\hat{\omega}^+_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\left(\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}}\right)^{-1}\bar{\lambda}_{\bar{z}} = 0. \quad (2.42)$$

In this case it is natural to integrate over the Riemann surface using the form $\mathrm{d}z \wedge \bar{\lambda}$ and obtain the four-dimensional scalar equations of motion as

$$\int_\Sigma E\,\mathrm{d}z \wedge \bar{\lambda} = \int_\Sigma \bar{E}\,\mathrm{d}\bar{z} \wedge \lambda = 0. \quad (2.43)$$

The details of the calculation are similar to those of the vector equation. The end result is

$$(\tau - \bar{\tau})\partial_\mu\partial_\mu a + \partial_\mu a\partial_\mu\tau + \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\bar{a}}\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu} - 2(\tau - \bar{\tau})\hat{\omega}_{\mu\nu}\mathcal{F}_{\mu\nu} + 2(L_1 - L_2)\left(\frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}}\right)^2\hat{\omega}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu} = 0, \quad (2.44)$$

$$(\tau - \bar{\tau})\partial_\mu\partial_\mu \bar{a} - \partial_\mu \bar{a}\partial_\mu\bar{\tau} - \frac{\mathrm{d}\tau}{\mathrm{d}a}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} - 2(\tau - \bar{\tau})\hat{\omega}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu} + 2(\bar{L}_1 - \bar{L}_2)\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^2\hat{\omega}_{\mu\nu}\mathcal{F}_{\mu\nu} = 0, \quad (2.45)$$

where L_1 and L_2 are the integrals

$$L_1 = -\int_\Sigma \partial\left(\frac{\partial s}{1+|\partial s|^2}\right)(\bar{s}+\bar{s}-z\bar{\partial}\bar{s}-\bar{z}\bar{\partial}\bar{s})\lambda_{\bar{z}}\,\mathrm{d}z\wedge\bar{\lambda}, \tag{2.46}$$

$$L_2 = \int_\Sigma \bar{\lambda}_{\bar{z}}\,\mathrm{d}z\wedge\bar{\lambda}. \tag{2.47}$$

The second integral can be evaluated straightforwardly in terms of u using the methods of Appendix A:

$$L_2 = \int_\Sigma \bar{\lambda}_{\bar{z}}^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = \pi\mathrm{i}\left(\frac{u-1}{|u-1|}-\frac{u+1}{|u+1|}\right). \tag{2.48}$$

The evaluation of L_1 is more involved but leads to $L_1 = L_2$ (see Appendix A).

The scalar equations take the final form

$$\begin{aligned}&(\tau-\bar{\tau})\partial_\mu\partial_\mu a+\partial_\mu a\partial_\mu\tau+2\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\bar{a}}\left(F_{\mu\nu}F_{\mu\nu}+F_{\mu\nu}{}^*F_{\mu\nu}\right)\\&\quad+4\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\bar{a}}(a-\bar{a})\hat{\omega}^+_{\mu\nu}F_{\mu\nu}-4(\tau-\bar{\tau})\hat{\omega}^-_{\mu\nu}F_{\mu\nu}=0,\end{aligned} \tag{2.49}$$

$$\begin{aligned}&(\tau-\bar{\tau})\partial_\mu\partial_\mu\bar{a}-\partial_\mu\bar{a}\partial_\mu\bar{\tau}-2\frac{\mathrm{d}\tau}{\mathrm{d}a}\left(F_{\mu\nu}F_{\mu\nu}-F_{\mu\nu}{}^*F_{\mu\nu}\right)\\&\quad+4\frac{\mathrm{d}\tau}{\mathrm{d}a}(a-\bar{a})\hat{\omega}^-_{\mu\nu}F_{\mu\nu}-4(\tau-\bar{\tau})\hat{\omega}^+_{\mu\nu}F_{\mu\nu}=0.\end{aligned} \tag{2.50}$$

The four-dimensional action. It is well known that the equations of motion for a generic M$_5$ embedding do not stem from a six-dimensional action. On the other hand our calculation results in the four-dimensional equations of motion for the Ω-deformation of the sw-theory, which we expect to have a Lagrangian description. In fact, a direct calculation shows that the vector equation (2.39) and the two scalar equations (2.49) and (2.50) are all derived from the variation of the following Lagrangian:

$$\begin{aligned}\mathrm{i}\mathscr{L} = &-(\tau-\bar{\tau})\left[\frac{1}{2}\partial_\mu a\partial_\mu\bar{a}+F_{\mu\nu}F_{\mu\nu}+(a-\bar{a})^*\hat{\omega}_{\mu\nu}F_{\mu\nu}\right.\\&\left.-2\partial_\mu(a+\bar{a})\hat{\omega}_{\mu\nu}A_\nu\right]+(\tau+\bar{\tau})\big[F_{\mu\nu}{}^*F_{\mu\nu}\\&+(a-\bar{a})\hat{\omega}_{\mu\nu}F_{\mu\nu}+2\partial_\mu(a-\bar{a})\hat{\omega}_{\mu\nu}A_\nu\big].\end{aligned} \tag{2.51}$$

This is the main result of this Letter and represents the Ω-deformation of the sw action. In this form the action is not manifestly gauge invariant. An equivalent, gauge invariant, form is given by

$$\begin{aligned}\mathrm{i}\mathscr{L} = &-(\tau-\bar{\tau})\left[\frac{1}{2}\partial_\mu a\partial_\mu\bar{a}+F_{\mu\nu}F_{\mu\nu}+(a-\bar{a})^*\hat{\omega}_{\mu\nu}F_{\mu\nu}\right.\\&\left.-2\partial_\mu(a+\bar{a})^*F_{\mu\nu}{}^*\hat{U}_\nu\right]+(\tau+\bar{\tau})\big[F_{\mu\nu}{}^*F_{\mu\nu}\\&+(a-\bar{a})\hat{\omega}_{\mu\nu}F_{\mu\nu}+2\partial_\mu(a-\bar{a})^*F_{\mu\nu}{}^*\hat{U}_\nu\big],\end{aligned} \tag{2.52}$$

where $\omega = \mathrm{d}U$ and ${}^*\omega = \mathrm{d}{}^*U$. Note that in a slight abuse of notation ${}^*U$ is a one-form and not the Hodge dual of U.

Let us consider some generalizations of our calculation. It is natural to write the action in a more supersymmetric form as a sum of squares:

$$\begin{aligned}\mathrm{i}\mathscr{L} = &-(\tau-\bar{\tau})\left[\frac{1}{2}\left(\partial_\mu a+\frac{2\bar{\tau}}{\tau-\bar{\tau}}{}^*F_{\mu\nu}{}^*\hat{U}_\nu\right)\right.\\&\times\left(\partial_\mu\bar{a}-\frac{2\tau}{\tau-\bar{\tau}}{}^*F_{\mu\nu}{}^*\hat{U}_\nu\right)+\left(F_{\mu\nu}+\frac{1}{2}(a-\bar{a})^*\hat{\omega}_{\mu\nu}\right)\\&\left.\times\left(F_{\mu\nu}+\frac{1}{2}(a-\bar{a})^*\hat{\omega}_{\mu\nu}\right)\right]\\&+(\tau+\bar{\tau})\left(F_{\mu\nu}+\frac{1}{2}(a-\bar{a})^*\hat{\omega}_{\mu\nu}\right)\\&\times\left({}^*F_{\mu\nu}+\frac{1}{2}(a-\bar{a})\hat{\omega}_{\mu\nu}\right).\end{aligned} \tag{2.53}$$

This therefore leads to a prediction for the $\mathcal{O}(\epsilon^2)$ terms. Note however that there could also be additional $\mathcal{O}(\epsilon^2)$ terms which are complete squares on their own, similar to the last term in (1.1).

Finally, although our calculations were only performed in the simplest case of an $SU(2)$ gauge group with one modulus, it is natural to propose that the generalization to arbitrary gauge group and matter content is given by

$$\begin{aligned}\mathrm{i}\mathscr{L} = &-(\tau_{ij}-\bar{\tau}_{ij})\left[\frac{1}{2}\left(\partial_\mu a^i+2\left(\frac{\bar{\tau}}{\tau-\bar{\tau}}\right)_{ik}{}^*F^k_{\mu\nu}{}^*\hat{U}_\nu\right)\right.\\&\times\left(\partial_\mu\bar{a}^j-2\left(\frac{\tau}{\tau-\bar{\tau}}\right)_{jl}{}^*F^l_{\mu\nu}{}^*\hat{U}_\nu\right)\\&\left.+\left(F^i_{\mu\nu}+\frac{1}{2}(a^i-\bar{a}^i)^*\hat{\omega}_{\mu\nu}\right)\left(F^j_{\mu\nu}+\frac{1}{2}(a^j-\bar{a}^j)^*\hat{\omega}_{\mu\nu}\right)\right]\\&+(\tau_{ij}+\bar{\tau}_{ij})\left(F^i_{\mu\nu}+\frac{1}{2}(a^i-\bar{a}^i)^*\hat{\omega}_{\mu\nu}\right)\\&\times\left({}^*F^j_{\mu\nu}+\frac{1}{2}(a^j-\bar{a}^j)\hat{\omega}_{\mu\nu}\right),\end{aligned} \tag{2.54}$$

where we have used a suitable form for the inverse of $(\tau-\bar{\tau})_{ij}$ which is taken to act from the left.

3. Conclusions

In this Letter we have computed the corrections to first order in ϵ to an M$_5$-brane wrapping a Riemann surface in the Ω-background of [12–14]. The result can be viewed as the leading correction to the Seiberg–Witten effective action of $\mathcal{N}=2$ super-Yang–Mills theory with an Ω-deformation.

The corrected effective action includes a shift in the gauge field strength as well as a sort of generalized covariant derivative for the scalar, including a non-minimal coupling to the gauge field. A similar generalized covariant derivative already appears in (1.1) and is reminiscent of the equivariant differential used in [9].

It is important to ask why the result we obtain, calculated as the classical motion of a single M$_5$-brane in M-theory, can capture quantum effects in four-dimensional gauge theory. To answer this we should restore the factors of R and Λ into the Riemann surface. This can be achieved by simply rescaling $\partial s\to\Lambda^2 R\partial s$, $\partial s/\partial a\to R\partial s/\partial a$ and $\partial s/\partial u\to R\partial s/\partial u$ along with their complex conjugates. However this replacement does not affect the final equations. On the other hand $R = g_s l_s$ can be related to the gauge coupling constant g_4 in the string theory picture. Thus the classical M-theory calculation in fact captures all orders of the four-dimensional gauge theory.

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Appendix A. Non-holomorphic integrals over Σ

Most of the integrals over the Riemann surface Σ that appear in this note can be evaluated using the same strategy that consists in reducing them to line integrals, as in [18]. As an example consider one of the integrals appearing in the vector equation:

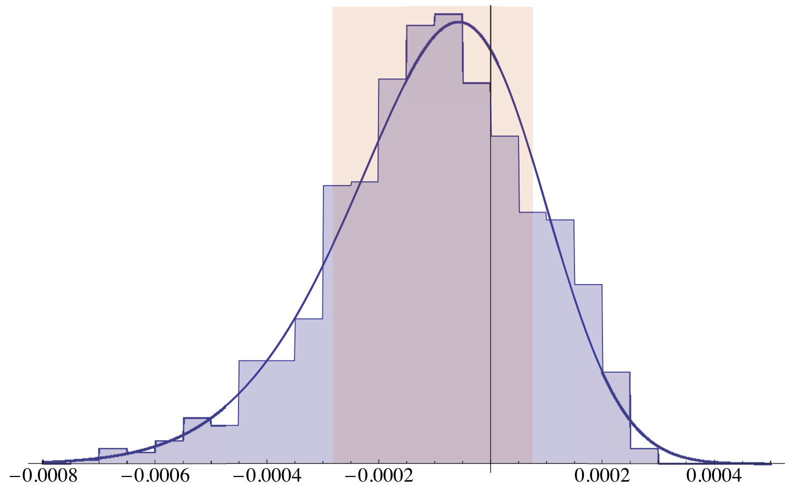

Fig. 1. Numerical integration of L_2. The histogram collects the frequency the values of $1-|L_1/L_2|$ obtained by integrating for 10^3 random values of u. The continuous line is a skew normal distribution with average $-1.0\times 10^{-4}\pm 1.7\times 10^{-4}$ (pink region). The result is consistent with $L_1=L_2$. We have also performed similar three-dimensional plots for the complex function $1-L_1/L_2$ which shows a clear peak around zero. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

$$I=\int_\Sigma \partial\left[\frac{\partial_\mu\bar{s}\partial s}{1+|\partial s|^2}\bar{z}\bar{\partial}\bar{s}\right]\mathrm{d}z\wedge\bar{\lambda}. \tag{A.1}$$

First we observe that $\bar{\lambda}$ is an anti-holomorphic one-form, so we can write

$$I=\int_\Sigma \mathrm{d}\left[\frac{\partial_\mu\bar{s}\partial s}{1+|\partial s|^2}\bar{z}\bar{\partial}\bar{s}\right]\wedge\bar{\lambda}. \tag{A.2}$$

From the explicit expression of $s(z)$ one finds that the integrand has singularities at the roots $\bar{e}_i$ of $Q(\bar{z})$:

$$\bar{e}_i=\pm\sqrt{\bar{u}\pm 1},\quad i=1,\dots,4. \tag{A.3}$$

For this reason we introduce a new surface Σ_δ by cutting holes of radius δ in Σ around e_i. Then I becomes an integral over the boundary $\partial\Sigma_\delta$:

$$I=\oint_{\partial\Sigma_\delta}\frac{\partial_\mu\bar{s}\bar{\partial}\bar{s}}{1+|\partial s|^2}\bar{z}\partial s\bar{\lambda}_{\bar{z}}\,\mathrm{d}\bar{z}. \tag{A.4}$$

Since we are interested in the behavior around e_i we can expand the integrand in powers of δ. Note that for $z=e_i+\delta$,

$$\frac{|\partial s|^2}{1+|\partial s|^2}=\frac{1}{1+1/|\partial s|^2}=\frac{1}{1+|Q|/(4|z|^2)}=1+\mathcal{O}(\delta). \tag{A.5}$$

Moreover, since $\bar{s}(\bar{z})$ depends on x^μ only via the modulus $\bar{u}$ (Eq. (2.12)), $\partial_\mu\bar{s}=\partial_\mu\bar{u}\bar{\lambda}_{\bar{z}}$, and the integral takes the form

$$I=\partial_\nu\bar{u}\sum_i\oint_{\gamma_i}\bar{e}_i\bar{\lambda}_{\bar{z}}^2\,\mathrm{d}\bar{z}+\mathcal{O}(\delta), \tag{A.6}$$

where γ_i is a circle of radius δ around e_i, and $\partial\Sigma_\delta=\bigcup_i\gamma_i$. From the explicit expression of $\bar{s}$ we find that

$$\bar{\lambda}_{\bar{z}}^2=\frac{1}{\bar{Q}(\bar{z})}, \tag{A.7}$$

so that each integral around γ_i can be evaluated using the residue theorem:

$$\oint_{\gamma_i}\frac{1}{\bar{Q}(\bar{z})}\mathrm{d}\bar{z}=-\frac{2\pi\mathrm{i}}{\prod_{j\neq i}(\bar{e}_i-\bar{e}_j)}, \tag{A.8}$$

and the whole integral is given by

$$I=-2\pi\mathrm{i}\partial_\mu\bar{u}\sum_{i=1}^4\frac{\bar{e}_i}{\prod_{j\neq i}(\bar{e}_i-\bar{e}_j)}. \tag{A.9}$$

By using the explicit values of e_i we finally find that I vanishes.

Let us now examine the L_1 integral that appeared in the scalar equation. First we integrate by parts:

$$\begin{aligned}L_1&=-\int_\Sigma \mathrm{d}\left(\frac{\partial s}{1+|\partial s|^2}\right)(s+\bar{s}-z\bar{\partial}\bar{s}-\bar{z}\bar{\partial}\bar{s})\lambda_{\bar{z}}\wedge\bar{\lambda}\\&=-\oint_{\partial\Sigma_\delta}\frac{\partial s(\bar{s}+\bar{s}-z\bar{\partial}\bar{s}-\bar{z}\bar{\partial}\bar{s})}{1+|\partial s|^2}\lambda_{\bar{z}}^2\,\mathrm{d}\bar{z}\\&\quad+\int_\Sigma\frac{(\partial s)^2-|\partial s|^2}{1+|\partial s|^2}\lambda_{\bar{z}}^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z}.\end{aligned} \tag{A.10}$$

Using similar techniques to the I integral above one finds that the boundary term is

$$\begin{aligned}&-\oint_{\partial\Sigma_\delta}\frac{\partial s(\bar{s}+\bar{s}-z\bar{\partial}\bar{s}-\bar{z}\bar{\partial}\bar{s})}{1+|\partial s|^2}\lambda_{\bar{z}}^2\,\mathrm{d}\bar{z}\\&\quad=-2\pi\mathrm{i}\sum_{i=1}^4\frac{e_i}{\prod_{j\neq i}(\bar{e}_i-\bar{e}_j)}\\&\quad=\pi\mathrm{i}\left(\frac{u-1}{|u-1|}-\frac{u+1}{|u+1|}\right)=L_2.\end{aligned} \tag{A.11}$$

Let us now look at the last term of (A.10). Rewriting the integrand in terms of Q we find

$$\begin{aligned}&\int_\Sigma\frac{(\partial s)^2-|\partial s|^2}{1+|\partial s|^2}\lambda_{\bar{z}}^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\\&\quad=\int_\Sigma\frac{|z|^2}{\frac{1}{4}|Q|+|z|^2}\left(\frac{z}{\bar{z}}-\sqrt{\frac{Q}{\bar{Q}}}\right)\frac{\mathrm{d}z}{\sqrt{Q}}\wedge\frac{\mathrm{d}\bar{z}}{\sqrt{\bar{Q}}}\\&\quad=\frac{1}{4}\int_\Sigma\frac{1}{1+|z'/z|^2}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}\end{aligned}$$

$$-\frac{1}{4}\int_{\Sigma}\frac{1}{1+|z/z'|^2}\frac{z'}{\bar{z}'}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}, \qquad \text{(A.12)}$$

where we changed variables to $\mathrm{d}y = 2\,\mathrm{d}z/\sqrt{Q}$ so that z is now a holomorphic function of y with $z' = \mathrm{d}z/\mathrm{d}y$. We will now show that both terms on the *rhs* vanish separately. Consider the first term on the *rhs* and expand in a power series of $|z'/z|$:

$$\int_{\Sigma}\frac{1}{1+|z'/z|^2}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y} = \sum_{n=0}^{\infty}\int_{\Sigma}(-1)^n\left|\frac{z'}{z}\right|^{2n}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}. \qquad \text{(A.13)}$$

Unfortunately the *rhs* here is not well-defined, even though the *lhs* is. To correct this we can introduce two-step regulator with parameters a and b which we will later set to zero. Thus we instead consider

$$\begin{aligned}
&\int_{\Sigma}\frac{\mathrm{e}^{-|z'/z|^2a^2}\mathrm{e}^{-b^2(|z|^2+1/|z|^2)}}{1+|z'/z|^2}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}\\
&\quad=\sum_{n=0}^{\infty}\int_{\Sigma}(-1)^n\left|\frac{z'}{z}\right|^{2n}\mathrm{e}^{-|z'/z|^2a^2}\mathrm{e}^{-b^2(|z|^2+1/|z|^2)}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\int_{\Sigma}\frac{z}{\bar{z}}\mathrm{e}^{-|z'/z|^2a^2}\mathrm{e}^{-b^2(|z|^2+1/|z|^2)}\,\mathrm{d}y_n\wedge\mathrm{d}\bar{y}_n,
\end{aligned} \qquad \text{(A.14)}$$

where we have changed variables again to $\mathrm{d}y_n = (z'/z)^n\,\mathrm{d}y$. Let us now set $a=0$ to deduce that

$$\begin{aligned}
&\int_{\Sigma}\frac{\mathrm{e}^{-b^2(|z|^2+1/|z|^2)}}{1+|z'/z|^2}\frac{z}{\bar{z}}\,\mathrm{d}y\wedge\mathrm{d}\bar{y}\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\int_{\Sigma}\frac{z}{\bar{z}}\mathrm{e}^{-b^2(|z|^2+1/|z|^2)}\,\mathrm{d}y_n\wedge\mathrm{d}\bar{y}_n.
\end{aligned} \qquad \text{(A.15)}$$

In each of the terms of the sum z is a holomorphic function of y_n and therefore $z(y_n)$ covers the whole complex plane (with the exception of one point) and hence the integral of the phases $z/\bar{z}$ must vanish since the b-regulator is independent of the phase. We can now set $b=0$ to see that each term in the sum vanishes and hence the first term on the *rhs* of (A.12) vanishes. Finally we can repeat a similar argument for the second term on the *rhs* of (A.12) only in this case the b-regulator should be taken to be $\mathrm{e}^{-b^2(|z'|^2+1/|z'|^2)}$. Thus we see that (A.12) vanishes and hence $L_1 = L_2$. The above proof that $L_1 = L_2$ is a little suspect since we required two regulators and needed to set $a=0$ first and then $b=0$. As a check we performed a numerical integration for random values of u which clearly supports our claim (see Fig. 1).

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