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Nonlinear boundary value problems of fractional functional integro-differential equations[☆]

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A B S T R A C T

In this paper, we consider the existence of generalized solutions for fractional functional integro-differential equations of mixed type with nonlinear boundary value conditions. By establishing a new comparison theorem and applying the monotone iterative technique, we show the existence of extremal generalized solutions.

1. Introduction

In this paper, we investigate the following nonlinear boundary value problem for fractional functional integro-differential equation:

$$\begin{cases} {}^cD^{\alpha}_{0+}x(t) = f(t, x(t), x(\theta(t)), Tx(t), Sx(t)), & t \in J = [0, T], \\ g(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where $f \in C(J \times R^4, R)$, $\theta \in C(J, J)$ and

$$(Tx)(t) = \int_0^t k(t, s)x(s)ds, \qquad (Sx)(t) = \int_0^T h(t, s)x(s)ds,$$

$k(t, s) \in C[D, R^+]$, $h(t, s) \in C[J \times J, R^+]$, $D = \{(t, s) \in R^2 | 0 \le s \le t \le T\}$, $R^+ = [0, +\infty]$, $k_0 = \max\{k(t, s)|(t, s) \in D\}$,and $h_0 = \max\{h(t, s)|(t, s) \in J \times J\}$, ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative of order $0 < \alpha \le 1$.

Differential equations with fractional order are generalizations of ordinary differential equations to non-integer order. These generalizations are not mere mathematical curiosities but rather they have interesting applications in many areas of science and engineering such as electrochemistry, control, porous media, electromagnetic, etc. (see [1–24]). For two noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator (see [3,4]), there has been a significant development in fractional differential equations in recent years.

Recently, many people have paid attention to the existence of solutions to nonlinear boundary value problems of fractional differential equations. In [19], Zhang and Su investigated the existence of a solution of the linear fractional differential equation with nonlinear boundary conditions:

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) - du(t) = h(t), & t \in (0, T], 0 < \alpha < 1, \\ g(u(0)) = u(T), \end{cases}$$

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where $d \geq 0$, $h \in C([0,T],R)$, and ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative. Using the method of upper and lower solutions in reverse order, some results on the existence of a solution are obtained for the above fractional boundary value problems.

And in [20], Zhang considered the existence of a solution of the nonlinear fractional differential equation with nonlinear boundary conditions:

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) = f(t,u(t)), & t \in (0,T], 0 < \alpha < 1, \\ g(u(0),u(T)) = 0, \end{cases}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative, $f \in C((0,T] \times R, R)$. Using the method of upper and lower solutions and its associated monotone iterative, some results on the existence of a solution are obtained for the above fractional boundary value problems.

Motivated by [19,20], we will investigate the existence of solutions of fractional functional integral-differential equation (1.1) by means of the method of lower and upper solutions combined with the monotone iterative technique in this paper.

This paper is organized as follows. In Section 2, we discuss the uniqueness of the generalized solutions to linear functional integro-differential equations and establish a new comparison principle. In Section 3, we obtain the existence of extremal generalized solutions for (1.1) by utilizing the monotone iterative technique and the method of lower and upper generalized solutions.

2. Linear problem and comparison principle

Let $C(J,R)$ be the Banach space of all continuous functions defined on J with the norm $\| x \| = \max\{| x(t) | : t \in J\}$.

For the readers' convenience, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [8,15].

Definition 2.1. Caputo's derivative for a function $f \in C^n([0,\infty),R)$ can be written as

$$ {}^cD^{s}_{0+}f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{s+1-n}}, \quad n = [s]+1 \tag{2.1}$$

where $[s]$ denotes the integer part of real number $s > 0$.

Definition 2.2. For $s > 0$, the integral

$$ I^{s}_{0+}f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}}dt, \tag{2.2}$$

is called the Riemann–Liouville fractional integral of order s.

Lemma 2.1. *Let $u \in C^m[0,1]$ and $q \in (m-1,m]$, $m \in N$. Then for $t \in [0,1]$,*

$$ I^{q\,c}D^{q}_{0+}u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}u^{(k)}(0). \tag{2.3}$$

By virtue of Lemma 2.1, we easily get the following

Lemma 2.2 (J,R). *Let $\sigma \in C(J,R)$, if $u \in C^1$ is a solution of the following fractional differential equations:*

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) - Mu(t) - M_1u(\theta(t)) - M_2Tu(t) - M_3Su(t) = \sigma(t), & t \in J, 0 < \alpha \leq 1, \\ u(0) = k, \quad k \in R, \end{cases} \tag{2.4}$$

then $u(t)$ is a solution of the following integral equation:

$$ u(t) = k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\sigma(s) + Mu(s) + M_1u(\theta(s)) + M_2Tu(s) + M_3Su(s)]ds, \quad t \in J, \tag{2.5}$$

where $M, M_1, M_2, M_3 \in R$ are constants.

Definition 2.3. We say that $u \in C(J,R)$ is a generalized solution of the initial value problem (2.4) of fractional differential equations if $u(t)$ is a solution of (2.5).

Remark. If $u \in C^1(J,R)$ is a solution of (2.4), we easily get $u(t)$ is a generalized solution of (2.4). However, by the following simple example, a generalized solution of (2.4) is not a solution of (2.4) in general.

Example. In (2.4), we let $\sigma(t) = a$ (a is a constant), $M = M_1 = M_2 = M_3 = 0$, $T = 1$, $\alpha = \frac{1}{3}$. According to (2.5), we get

$$u(t) = k + \frac{a}{\Gamma\left(\frac{1}{3}\right)} \int_0^t (t-s)^{-\frac{2}{3}} ds = k + \frac{3a}{\Gamma\left(\frac{1}{3}\right)} t^{\frac{1}{3}}, \quad \forall t \in [0, 1],$$

which implies that $u \notin C^1([0, 1], R)$. According to the definition of Caupto derivative, we could not define Caupto derivative for $u(t)$.

Lemma 2.3. *Assume that $M, M_1, M_2, M_3 \geq 0$ are constants and the following inequality holds*

$$\frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha+1)} T^\alpha < 1, \tag{2.6}$$

then (2.4) has a unique generalized solution.

Proof. We firstly define an operator $A : C(J, R) \to C(J, R)$ by

$$(Au)(t) = k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\sigma(s) + Mu(s) + M_1 u(\theta(s)) + M_2 Tu(s) + M_3 Su(s)] ds.$$

Then, we have

$$\begin{aligned}
\| (Au)(t) - (Av)(t) \|_C &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-s)^{\alpha-1} [M(u(s) - v(s)) + M_1(u(\theta(s) - v(\theta(s))))\right.\\
&\quad \left. + M_2(Tu(s) - Tv(s)) + M_3(Su(s) - Su(s))] ds \right\|_C \\
&\leq \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha)} \max_{t \in [0,T]} \int_0^t (t-s)^{\alpha-1} ds \, \| u - v \|_C \\
&\leq \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha+1)} T^\alpha \, \| u - v \|_C .
\end{aligned}$$

According to (2.6) and the Banach fixed point theorem, (2.4) has a unique generalized solution $u \in C(J, R)$. The proof is completed. □

Lemma 2.4 (*A Comparison Result*)**.** *Suppose that $M, M_1, M_2, M_3 \geq 0$ are constants and the inequality (2.6) holds. If $u \in C(J, R)$ and satisfies*

$$\begin{cases} u(t) \leq u(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [Mu(s) + M_1 u(\theta(s)) + M_2 Tu(s) + M_3 Su(s)] ds, & t \in J, \\ u(0) \leq 0, \end{cases} \tag{2.7}$$

then $u(t) \leq 0$ for all $t \in J$.

Proof. Suppose that the inequality $u(t) \leq 0, t \in J$ is not true. It means that there exists at least a $t^* \in J$ such that $u(t^*) > 0$. Without loss of generality, we assume $u(t^*) = \max\{u(t) : t \in J\} = \rho$, $\rho > 0$. We obtain that

$$\begin{aligned}
u(t) &\leq u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mu(s) + M_1 u(\theta(s)) + M_2 Tu(s) + M_3 Su(s)] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mu(s) + M_1 u(\theta(s)) + M_2 Tu(s) + M_3 Su(s)] ds \\
&\leq \rho \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&= \rho \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha+1)} t^\alpha \\
&\leq \rho \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha+1)} T^\alpha.
\end{aligned}$$

Let $t = t^*$, we have

$$\rho \leq \rho \frac{M + M_1 + (M_2 k_0 + M_3 h_0)T}{\Gamma(\alpha+1)} T^\alpha.$$

So

$$\frac{M + M_1 + (M_2k_0 + M_3h_0)T}{\Gamma(\alpha + 1)}T^{\alpha} \geq 1,$$

which is a contradiction. Hence $u(t) \leq 0$ for all $t \in J$. The proof is completed. □

3. Main results

In this section, we mainly prove the existence of extremal generalized solutions of problem (1.1) by the method of lower and upper generalized solutions and the monotone iterative technique.

To begin with, we need the following definitions.

Definition 3.1. A function $\alpha_0 \in C(J, R)$ is called a lower generalized solution of (1.1) if

$$\begin{cases} \alpha_0(t) \leq \alpha_0(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \alpha_0(s), \alpha_0(\theta(s)), T\alpha_0(s), S\alpha_0(s))ds, \quad t \in J, \\ g(\alpha_0(0), \alpha_0(T)) \leq 0. \end{cases}$$

Analogously, $\beta_0 \in C(J, R)$ is called an upper generalized solution of (1.1) if

$$\begin{cases} \beta_0(t) \geq \beta_0(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, \beta_0(s), \beta_0(\theta(s)), T\beta_0(s), S\beta_0(s))ds, \quad t \in J, \\ g(\beta_0(0), \beta_0(T)) \geq 0. \end{cases}$$

We need the following assumptions.

(H1) Functions α_0, β_0 are lower and upper generalized solutions of (1.1), respectively, such that $\alpha_0(t) \leq \beta_0(t)$, $t \in J$.

(H2) There exist constants $M, M_1, M_2, M_3 \geq 0$ such that

$$f(t, x, y, z, w) - f(t, \bar{x}, \bar{y}, \bar{z}, \bar{w}) \geq M(x - \bar{x}) + M_1(y - \bar{y}) + M_2(z - \bar{z}) + M_3(w - \bar{w}),$$

where $\alpha_0(t) \leq \bar{x} \leq x \leq \beta_0(t), \alpha_0(\theta(t)) \leq \bar{y} \leq y \leq \beta_0(\theta(t)), T\alpha_0(t) \leq \bar{z} \leq z \leq T\beta_0(t), S\alpha_0(t) \leq \bar{w} \leq w \leq S\beta_0(t), t \in J$.

(H3) There exist constants $L_1 > 0, L_2 \geq 0$, such that

$$g(x, y) - g(\bar{x}, \bar{y}) \leq L_1(x - \bar{x}) - L_2(y - \bar{y}),$$

where $\alpha_0(0) \leq \bar{x} \leq x \leq \beta_0(0)$ and $\alpha_0(T) \leq \bar{y} \leq y \leq \beta_0(T)$.

Let $[\alpha_0, \beta_0] = \{x \in C(J, R) : \alpha_0(t) \leq x(t) \leq \beta_0(t), t \in J\}$.

Now we are in the position to establish the main results of this paper.

Theorem 3.1. *Let inequality (2.6) and (H1)–(H3) hold. Assume that*

$$\begin{cases} y(t) = y(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} [f(s, \alpha_0(s), \alpha_0(\theta(s)), T\alpha_0(s), S\alpha_0(s)) - M(\alpha_0(s) - y(s)) \\ \quad - M_1(\alpha_0(\theta(s)) - y(\theta(s))) - M_2(T\alpha_0(s) - Ty(s)) - M_3(S\alpha_0(s) - Sy(s))]\, ds, \quad t \in J, \\ y(0) = \alpha_0(0) - \dfrac{1}{L_1}g(\alpha_0(0), \alpha_0(T)), \end{cases}$$

$$\begin{cases} z(t) = z(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} [f(s, \beta_0(s), \beta_0(\theta(s)), T\beta_0(s), S\beta_0(s)) - M(\beta_0(s) - z(s)) \\ \quad - M_1(\beta_0(\theta(s)) - z(\theta(s))) - M_2(T\beta_0(s) - Tz(s)) - M_3(S\beta_0(s) - Sz(s))]\, ds, \quad t \in J, \\ z(0) = \beta_0(0) - \dfrac{1}{L_1}g(\beta_0(0), \beta_0(T)). \end{cases}$$

Then $\alpha_0(t) \leq y(t) \leq z(t) \leq \beta_0(t)$, $t \in J$ and $y(t), z(t)$ are lower and upper generalized solutions of (1.1), respectively.

Proof. Let $p(t) = \alpha_0(t) - y(t)$, then

$$\begin{aligned} p(t) &\leq \alpha_0(0) - y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[M(\alpha_0(s) - y(s)) + M_1(\alpha_0(\theta(s)) - y(\theta(s))) \\ &\quad + M_2(T\alpha_0(s) - Ty(s)) + M_3(S\alpha_0(s) - Sy(s))]ds \\ &= p(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Mp(s) + M_1p(\theta(s)) + M_2Tp(s) + M_3Sp(s)]ds, \quad t \in J, \end{aligned} \tag{1}$$

$p(0) = \alpha_0(0) - y(0) = \frac{1}{L_1}g(\alpha_0(0), \alpha_0(T)) \leq 0.$

By Lemma 2.4, we get that $p(t) \le 0$, $t \in J$. That is, $\alpha_0(t) \le y(t)$. Similarly, we can prove that $z(t) \le \beta_0(t)$, $\forall t \in J$.
Next we verify that $y(t) \le z(t)$,$t \in J$. Let $p(t) = y(t) - z(t)$, then

$$\begin{aligned}
p(t) &= y(0) - z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \alpha_0(s), \alpha_0(\theta(s)), T\alpha_0(s), S\alpha_0(s)) - M\alpha_0(s) \\
&\quad - M_1\alpha_0(\theta(s)) - M_2 T\alpha_0(s) - M_3 S\alpha_0(s) - f(s, \alpha_0(s), \alpha_0(\theta(s)), T\alpha_0(s), S\alpha_0(s)) \\
&\quad + M\alpha_0(s) + M_1\alpha_0(\theta(s)) + M_2 T\alpha_0(s) + M_3 S\alpha_0(s) + M(y(s) - z(s)) \\
&\quad + M_1(y(\theta(s)) - z(\theta(s))) + M_2(Ty(s) - Tz(s)) + M_3(Sy(s) - Sz(s))]ds \\
&\le p(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mp(s) + M_1 p(\theta(s)) + M_2 Tp(s) + M_3 Sp(s)]ds, \quad t \in J, \\
p(0) &= y(0) - z(0) \\
&= \alpha_0(0) - \frac{1}{L_1} g(\alpha_0(0), \alpha_0(T)) - \beta_0(0) + \frac{1}{L_1} g(\beta_0(0), \beta_0(T)) \\
&\le \alpha_0(0) - \beta_0(0) + \frac{1}{L_1}[L_1(\beta_0(0) - \alpha_0(0)) - L_2(\beta_0(T) - \alpha_0(T))] \\
&= -\frac{L_2}{L_1}(\beta_0(T) - \alpha_0(T)) \\
&\le 0.
\end{aligned}$$

By Lemma 2.4, we get that $p(t) \le 0$ on J. That is $y(t) \le z(t)$. So $\alpha_0(t) \le y(t) \le z(t) \le \beta_0(t)$, $t \in J$.
In the following,we need to prove $y(t)$ is a lower generalized solution of (1.1).

$$\begin{aligned}
y(t) &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \alpha_0(s), \alpha_0(\theta(s)), T\alpha_0(s), S\alpha_0(s)) - M(\alpha_0(s) - y(s)) \\
&\quad - M_1(\alpha_0(\theta(s)) - y(\theta(s))) - M_2(T\alpha_0(s) - Ty(s)) - M_3(S\alpha_0(s) - Sy(s))]ds \\
&\le y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), y(\theta(s)), Ty(s), Sy(s))ds, \ t \in J, \\
g(y(0), y(T)) &\le g(\alpha_0(0), \alpha_0(T)) + L_1(y(0) - \alpha_0(0)) - L_2(y(T) - \alpha_0(T)) \\
&\le g(\alpha_0(0), \alpha_0(T)) + L_1(y(0) - \alpha_0(0)) \\
&= g(\alpha_0(0), \alpha_0(T)) - g(\alpha_0(0), \alpha_0(T)) \\
&= 0.
\end{aligned}$$

So $y(t)$ is a lower generalized solution of (1.1). Similarly, we can prove that $z(t)$ is an upper generalized solution of (1.1). The proof is completed. □

Theorem 3.2. *Let inequality* (2.6) *and* (H1)–(H3) *hold. Then there exist monotone sequences* $\{\alpha_n\}$, $\{\beta_n\} \subset [\alpha_0, \beta_0]$ *which converge uniformly to the extremal generalized solutions of* (1.1) *in* $[\alpha_0, \beta_0]$, *respectively.*

Proof. For $n = 1, 2, \ldots$, we suppose that

$$\begin{cases}
{}^cD_{0+}^{\alpha}\alpha_n(t) - M\alpha_n(t) - M_1\alpha_n(\theta(t)) - M_2 T\alpha_n(t) - M_3 S\alpha_n(t) \\
\quad = f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t)), T\alpha_{n-1}(t), S\alpha_{n-1}(t)) - M\alpha_{n-1}(t) - M_1\alpha_{n-1}(\theta(t)) \\
\qquad - M_2 T\alpha_{n-1}(t) - M_3 S\alpha_{n-1}(t), \quad t \in J, \\
\alpha_n(0) = \alpha_{n-1}(0) - \frac{1}{L_1} g(\alpha_{n-1}(0), \alpha_{n-1}(T)),
\end{cases} \tag{3.1}$$

and

$$\begin{cases}
{}^cD_{0+}^{\alpha}\beta_n(t) - M\beta_n(t) - M_1\beta_n(\theta(t)) - M_2 T\beta_n(t) - M_3 S\beta_n(t) \\
\quad = f(t, \beta_{n-1}(t), \beta_{n-1}(\theta(t)), T\beta_{n-1}(t), S\beta_{n-1}(t)) - M\beta_{n-1}(t) - M_1\beta_{n-1}(\theta(t)) \\
\qquad - M_2 T\beta_{n-1}(t) - M_3 S\beta_{n-1}(t), \quad t \in J, \\
\beta_n(0) = \beta_{n-1}(0) - \frac{1}{L_1} g(\beta_{n-1}(0), \beta_{n-1}(T)).
\end{cases} \tag{3.2}$$

Obviously, by Lemma 2.4, Eqs. (3.1) and (3.2) have the following generalized solutions, respectively.

$$\begin{cases}
\alpha_n(t) = \alpha_n(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \alpha_{n-1}(s), \alpha_{n-1}(\theta(s)), T\alpha_{n-1}(s), S\alpha_{n-1}(s)) - M(\alpha_{n-1}(s) - \alpha_n(s)) \\
\quad - M_1(\alpha_{n-1}(\theta(s)) - \alpha_n(\theta(s))) - M_2(T\alpha_{n-1}(s) - T\alpha_n(s)) - M_3(S\alpha_{n-1}(s) - S\alpha_n(s))]\, ds, \quad t \in J, \\
\alpha_n(0) = \alpha_{n-1}(0) - \frac{1}{L_1} g(\alpha_{n-1}(0), \alpha_{n-1}(T)),
\end{cases}$$

$$\begin{cases} \beta_n(t) = \beta_n(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} [f(s, \beta_{n-1}(s), \beta_{n-1}(\theta(s)), T\beta_{n-1}(s), S\beta_{n-1}(s)) - M(\beta_{n-1}(s) - \beta_n(s)) \\ \quad - M_1(\beta_{n-1}(\theta(s)) - \beta_n(\theta(s))) - M_2(T\beta_{n-1}(s) - T\beta_n(s)) - M_3(S\beta_{n-1}(s) - S\beta_n(s))]\, ds, \quad t \in J, \\ \beta_i(0) = \beta_{n-1}(0) - \dfrac{1}{L_1} g(\beta_{n-1}(0), \beta_{n-1}(T)). \end{cases}$$

In view of Theorem 3.1, we have that

$$\alpha_0 \le \alpha_1 \le \cdots \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0.$$

Obviously the sequences $\{\alpha_n\}, \{\beta_n\}$ are uniformly bounded and equicontinuous, one can employ the standard arguments, namely the Ascoli–Arzela criterion to conclude that the sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly on [0,T] with $\lim_{n\to\infty} \alpha_n = x_*$, $\lim_{n\to\infty} \beta_n = x^*$ uniformly on J. Moreover, x_*, x^* are generalized solutions of (1.1) in $[\alpha_0, \beta_0]$.

To prove that x_*, x^* are extremal generalized solutions of (1.1), let $x \in [\alpha_0, \beta_0]$ be any generalized solution of (1.1). That is,

$$\begin{cases} x(t) = x(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} f(s, x(s), x(\theta(s)), Tx(s), Sx(s))ds, \quad t \in J, \\ g(x(0), x(T)) = 0. \end{cases}$$

Suppose that there exists a positive integer n such that $\alpha_n(t) \le x(t) \le \beta_n(t)$ on J. Let $p(t) = \alpha_{n+1}(t) - x(t)$, we have

$$\begin{aligned} p(t) &= \alpha_{n+1}(0) - x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \alpha_n(s), \alpha_n(\theta(s)), T\alpha_n(s), S\alpha_n(s)) - M\alpha_n(s) \\ &\quad - M_1\alpha_n(\theta(s)) - M_2T\alpha_n(s) - M_3S\alpha_n(s) - f(s, x(s), x(\theta(s)), Tx(s), Sx(s)) \\ &\quad + Mx(s) + M_1x(\theta(s)) + M_2Tx(s) + M_3Sx(s) + M(\alpha_{n+1}(s) - x(s)) \\ &\quad + M_1(\alpha_{n+1}(\theta(s)) - x(\theta(s))) + M_2(T\alpha_{n+1}(s) - Tx(s)) + M_3(S\alpha_{n+1}(s) - Sx(s))]ds \\ &\le p(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mp(s) + M_1p(\theta(s)) + M_2Tp(s) + M_3Sp(s)]ds, \quad t \in J, \\ p(0) &= \alpha_{n+1}(0) - x(0) \\ &= \alpha_n(0) - \frac{1}{L_1} g(\alpha_n(0), \alpha_n(T)) - x(0) \\ &\le -\frac{L_2}{L_1}(x(T) - \alpha_n(T)) - \frac{1}{L_1} g(x(0), x(T)) \\ &\le 0. \end{aligned}$$

By Lemma 2.4, we know that $p(t) \le 0$ on J, i.e. $\alpha_{n+1}(t) \le x(t)$ on J. Similarly we obtain that $x(t) \le \beta_{n+1}(t)$ on J. Since $\alpha_0(t) \le x(t) \le \beta_0(t)$ on J, by induction we get that $\alpha_n(t) \le x(t) \le \beta_n(t)$ on J for every n. Therefore, $x_*(t) \le x(t) \le x^*(t)$ on J by taking $n \to \infty$. Thus, we complete this proof. □

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