MATHEMATICS Proceedings A 86 (1), March 28, 1983

Some results on Hankel invariant distribution spaces

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Communicated by Prof. J. Korevaar at the meeting of June 21, 1982

ABSTRACT

Three Hankel invariant test function spaces and the associated generalized function spaces are introduced. The elements of the respective test function spaces are described both in functional analytic and in classical analytic terms. It is shown that one of the test function spaces equals the space $\boldsymbol{H}_\mu$ of Zemanian.

A.M.S. Subject Classification 46F12, 46F05, 46F10.

INTRODUCTION

Formally the Hankel transform of order α is defined by

$$(0.1)\qquad (\mathbb{H}_\alpha f)(x) = \int_0^\infty f(y)\sqrt{xy}\, J_\alpha(xy)dy.$$

Here J_α is the Bessel function of the first kind and order α. We consider the case $\alpha > -1$ in this paper. The Hankel transform $\mathbb{H}_\alpha$ is treated as a linear operator in $L_2(0,\infty)$. $\mathbb{H}_\alpha$ can be extended to a unitary operator on $L_2(0,\infty)$. It can be shown that for each $f \in L_2(0,\infty)$ the integral (0.1) converges in L_2-sense.

In this paper we construct three Hankel invariant test function spaces. (In fact infinitely many can be constructed.) The constructions are based on the two theories of generalized functions given in [G] and [E]. Since these theories may not be known to the reader we review them in Section 1.

One of the authors (SJLVE) was supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

The distribution theories [G] and [E] are described in functional analytic terms. Such a description contributes to the generality of the theories but it does not make them directly applicable in analysis. Therefore we devote three sections to the characterization in classical analytic terms of the elements in the respective test function spaces.

Besides the usual aspects of distribution theory such as the definition of the test function space, the definition of the generalized function space, their topological structures and the pairing, [G] and [E] also contain a detailed characterization of continuous linear mappings on these spaces, the introduction of topological tensor product spaces and four Kernel theorems. Since the Hankel invariant test function space $\boldsymbol{H}_\mu$ of Zemanian [Z] equals one of our test function spaces, all results of [G] and [E] carry over to this space. The present paper is partly an excerpt of a university report [E1]. Many technicalities in that report are not reproduced here, but we indicate the main lines of the argument.

(1) *The general theory*

In the first part of this section we review the distribution theory in [G]; in the second part the distribution theory in [E].

In a Hilbert space $\mathbf{X}$ consider the evolution equation

$$(1.1) \qquad \frac{du}{dt} = -\mathscr{A}u$$

where $\mathscr{A}$ is a positive, self-adjoint operator which is unbounded in order that the semigroup $(e^{-t\mathscr{A}})_{t\geq 0}$ is smoothing. A solution u of (1.1) is called a trajectory if u satisfies

$$(1.2.\text{i}) \qquad \forall_{t>0} : u(t) \in \mathbf{X}$$

$$(1.2.\text{ii}) \qquad \forall_{t>0}\forall_{\tau>0} : e^{-\tau\mathscr{A}}u(t) = u(t+\tau).$$

We emphasize that only $t>0$ is considered, and $\lim_{t\downarrow 0} u(t)$ does not necessarily exist in $\mathbf{X}$-sense. The complex vector space of all trajectories is denoted by $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$. The elements of $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$ are called generalized functions. The test function space $\boldsymbol{S}_{\mathbf{X},\mathscr{A}}$ is defined to be the dense linear subspace of $\mathbf{X}$ consisting of smooth elements of the form $e^{-t\mathscr{A}}h$, where $h \in \mathbf{X}$ and $t>0$; we have $\boldsymbol{S}_{\mathbf{X},\mathscr{A}} = \bigcup_{t>0} e^{-t\mathscr{A}}(\mathbf{X})$. The densely defined inverse of $e^{-t\mathscr{A}}$ is denoted by $e^{t\mathscr{A}}$. For each $\varphi \in \boldsymbol{S}_{\mathbf{X},\mathscr{A}}$, there exists $\tau>0$ such that $e^{\tau\mathscr{A}}\varphi$ makes sense. The pairing between $\boldsymbol{S}_{\mathbf{X},\mathscr{A}}$ and $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$ is defined by

$$(1.3) \qquad \langle \varphi, F \rangle = (e^{\tau\mathscr{A}}\varphi, F(\tau)), \ \varphi \in \boldsymbol{S}_{\mathbf{X},\mathscr{A}}, \ F \in \boldsymbol{T}_{\mathbf{X},\mathscr{A}}.$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathbf{X}$. Definition (1.3) makes sense for $\tau>0$ sufficiently small and due to the trajectory property (1.2.ii) it does not depend on the specific choice of τ. For a detailed discussion of the spaces $\boldsymbol{S}_{\mathbf{X},\mathscr{A}}$ and $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$ we refer to [G].

In [E], we start with the evolution equation

$$(1.4) \qquad \frac{d\varphi}{dt} = \mathscr{A}\varphi$$

where $\mathscr{A}$ is a positive self-adjoint operator in a Hilbert space $\mathbf{X}$. A solution φ of (1.4) is called a trajectory, if it satisfies

(1.5.i) $\forall_{t\in\mathbb{C}}: \varphi(t)\in\mathbf{X}$

(1.5.ii) $\forall_{t\in\mathbb{C}}\forall_{\tau\in\mathbb{C}}: \varphi(t+\tau)=e^{\tau\mathscr{A}}\varphi(t).$

The complex vector space of all trajectories is denoted by $\tau(\mathbf{X},\mathscr{A})$. Each trajectory φ is uniquely determined by its value $\varphi(0)$, where $\varphi(0)\in D((e^{\mathscr{A}})^{\infty})=$ $=\bigcap_{k=1}^{\infty} D(e^{k\mathscr{A}})$. The space $\tau(\mathbf{X},\mathscr{A})$ is the test function space in this theory. The generalized function space $\sigma(\mathbf{X},\mathscr{A})$ consists of elements F for which there exists $t>0$ such that $e^{-t\mathscr{A}}F\in\mathbf{X}$. We have $\sigma(\mathbf{X},\mathscr{A})=\bigcup_{t>0}(\mathbf{X}_t)$, where $\mathbf{X}_t$ is the completion of X with respect to the norm $\|\cdot\|_t$,

$$\|f\|_t=\|e^{-t\mathscr{A}}f\|,\ f\in\mathbf{X}.$$

Thus for all $F\in\sigma(\mathbf{X},\mathscr{A})$ there exists $t>0$ and $h\in\mathbf{X}$ such that $F=e^{t\mathscr{A}}(h)$. The pairing between $\tau(\mathbf{X},\mathscr{A})$ and $\sigma(\mathbf{X},\mathscr{A})$ is defined by

(1.6) $\langle\varphi,F\rangle=(\varphi(\tau),\ e^{-\tau\mathscr{A}}F),\ \varphi\in\tau(\mathbf{X},\mathscr{A}),\ F\in\sigma(\mathbf{X},\mathscr{A}).$

Here $(\cdot,\cdot)$ denotes the inner product of $\mathbf{X}$. Definition (1.6) makes sense for $\tau>0$ sufficiently large and due to the trajectory property (1.5.ii) it does not depend on the specific choice of τ. For a detailed discussion of this theory we refer to [E].

(2) *Introduction of Hankel invariant test function spaces and generalized function spaces*

Throughout the whole paper we take $\alpha\in\mathbb{R}$, $\alpha>-1$, fixed. The following equality can be derived from [MOS], p. 244:

(2.1) $$\boldsymbol{L}_n^{(\alpha)}(x)=(-1)^n\int_0^{\infty}\boldsymbol{L}_n^{(\alpha)}(y)\sqrt{xy}\,J_\alpha(xy)dy.$$

Here

$$\boldsymbol{L}_n^{(\alpha)}(x)=\left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac12}x^{\alpha+\frac12}e^{-x^2/2}L_n^{(\alpha)}(x^2),\quad x>0$$

and $L_n^{(\alpha)}$ is the n-th generalized Laguerre polynomial of type α,

$$L_n^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{n!}\left(\frac{d}{dx}\right)^n(e^{-x}x^{n+\alpha}),\ x>0.$$

The functions $\boldsymbol{L}_n^{(\alpha)}$ are eigenfunctions of the operator

(2.2) $$\mathscr{A}_\alpha: -\frac{d^2}{dx^2}+x^2+\frac{\alpha^2-\frac14}{x^2}-2\alpha,$$

and their respective eigenvalues are $4n+2$, $n=0,1,2,\ldots$. The operator $\mathscr{A}_\alpha$ is positive and self-adjoint in $\mathbf{X}=\boldsymbol{L}_2(0,\infty)$ and its eigenfunctions $\boldsymbol{L}_n^{(\alpha)}$ establish a complete orthonormal basis in $\mathbf{X}$. By (2.1), the Hankel transform of $\boldsymbol{L}_n^{(\alpha)}$ is equal to $(-1)^n \mathcal{L}_n^{(\alpha)}$. So the following definition of the Hankel transform would seem natural.

(2.3) DEFINITION

$$\mathbb{H}_\alpha f = \sum_{n=0}^{\infty} (-1)^n (f, \mathcal{L}_n^{(\alpha)}) \mathcal{L}_n^{(\alpha)}, \quad f \in \mathbf{X}.$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathbf{X}$.

It is obvious that $\mathbb{H}_\alpha$ is a self-adjoint, unitary operator in $\mathbf{X}$. The relation with the classical Hankel integral transform is expressed by the following Plancherel-type formula.

(2.4) THEOREM.
Let $f \in \mathbf{X}$. Then for all $x>0$

$$(\mathbb{H}_\alpha f)(x) = \underset{R\to\infty}{\text{l.i.m.}} \int_0^R f(y)\sqrt{xy}\, J_\alpha(xy)dy$$

i.e.

$$\int_0^\infty |(\mathbb{H}_\alpha f)(x) - \int_0^R f(y)\sqrt{xy}\, J_\alpha(xy)dy|^2 dx \to 0 \text{ as } R\to\infty.$$

PROOF. The proof follows from the validity of the formula (0.1) for very well behaved functions and the fact that $\mathbb{H}_\alpha$ is continuous on $\mathbf{X}$. Cf. [E1], section 1. □

Since the operator $\mathcal{A}_\alpha$ is positive and self-adjoint, the test function space $S_{\mathbf{X},\mathcal{A}_\alpha}$ is well-defined and so are the test function spaces $\tau(\mathbf{X}, \log \mathcal{A}_\alpha)$ and $\tau(\mathbf{X}, \mathcal{A}_\alpha)$. We give a short functional analytic characterization.

(2.5) *Characterization of test function spaces*

(a) $f \in S_{\mathbf{X},\mathcal{A}_\alpha} \Leftrightarrow \exists_{\tau>0} : (f, \mathcal{L}_n^{(\alpha)}) = O(e^{-n\tau})$

(b) $f \in \tau(\mathbf{X}, \log \mathcal{A}_\alpha) \Leftrightarrow \forall_{k\in\mathbb{N}} : (f, \mathcal{L}_n^{(\alpha)}) = O((n+1)^{-k})$

(c) $f \in \tau(\mathbf{X}, \mathcal{A}_\alpha) \Leftrightarrow \forall_{t>0} : (f, \mathcal{L}_n^{(\alpha)}) = O(e^{-nt})$.

PROOF. We shall show equivalence (a). The proof of (b) and (c) runs similarly. So let $f \in S_{\mathbf{X},\mathcal{A}_\alpha}$. Then there is $\tau>0$ and $w\in\mathbf{X}$ such that $f = e^{-\tau\mathcal{A}_\alpha}w$. Hence

$$|(f, \mathcal{L}_n^{(\alpha)})| = e^{-(4n+2)\tau} |(w, \mathcal{L}_n^{(\alpha)})| \le \|w\| e^{-4n\tau}.$$

On the other hand, suppose $f \in \mathbf{X}$ satisfies

$$|(f, \mathcal{L}_n^{(\alpha)})| \le K e^{-n\tau}, \; n \in \mathbb{N},$$

for some $K>0$, $\tau>0$. Then for $t=\frac{1}{8}\tau$

$$\|e^{t\mathscr{A}_\alpha}f\|^2 = e^{\frac{1}{2}\tau}\sum_{n=0}^{\infty} e^{n\tau}|(f,\boldsymbol{L}_n^{(\alpha)})|^2$$

$$\leq K^2 e^{\frac{1}{2}\tau}\sum_{n=0}^{\infty} e^{-n\tau}<\infty.$$

This implies $f\in \boldsymbol{S}_{\mathbf{X},\mathscr{A}_\alpha}$. □

The Hankel transform $\mathbb{H}_\alpha$ is well-defined on these spaces. We have

(2.6) THEOREM.

$\mathbb{H}_\alpha$ is a continuous bijection on $\boldsymbol{S}_{\mathbf{X},\mathscr{A}_\alpha}$. The same holds true for the spaces $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ and $\tau(\mathbf{X}, \mathscr{A}_\alpha)$.

PROOF. The proof is very simple. If for $f\in\mathbf{X}$, the order estimate (2.5.a) is satisfied by $(f,\boldsymbol{L}_n^{(\alpha)})$ then $(\mathbb{H}_\alpha f,\boldsymbol{L}_n^{(\alpha)})=(-1)^n(f,\boldsymbol{L}_n^{(\alpha)})$ satisfies the same one. So $\mathbb{H}_\alpha$ is a continuous injection on $\boldsymbol{S}_{\mathbf{X},\mathscr{A}_\alpha}$. $\mathbb{H}_\alpha$ is surjective because $\mathbb{H}_\alpha^2 f=f$. The proofs for the other spaces run similarly. □

We shall also characterize the spaces of generalized functions $\boldsymbol{T}_{\mathbf{X},\mathscr{A}_\alpha}$, $\sigma(\mathbf{X}, \log \mathscr{A}_\alpha)$ and $\sigma(\mathbf{X}, \mathscr{A}_\alpha)$. From (2.5) one has the following.

(2.7) *Characterization of generalized function spaces*

(a) $F\in \boldsymbol{T}_{\mathbf{X},\mathscr{A}_\alpha} \Leftrightarrow \forall_{t>0} : \langle \boldsymbol{L}_n^{(\alpha)},F\rangle = \boldsymbol{O}(e^{nt})$

(b) $F\in \sigma(\mathbf{X}, \log \mathscr{A}_\alpha) \Leftrightarrow \exists_{k\in\mathbb{N}} : \langle \boldsymbol{L}_n^{(\alpha)},F\rangle = \boldsymbol{O}(n^k)$

(c) $F\in \sigma(\mathbf{X}, \mathscr{A}_\alpha) \Leftrightarrow \exists_{\tau>0} : \langle \boldsymbol{L}_n^{(\alpha)},F\rangle = \boldsymbol{O}(e^{n\tau})$.

As a corollary of Theorem (2.6) we have

(2.8) COROLLARY.

The Hankel transform $\mathbb{H}_\alpha$ can be extended to a continuous bijection on each of the spaces $\boldsymbol{T}_{\mathbf{X},\mathscr{A}_\alpha}$, $\sigma(\mathbf{X}, \log \mathscr{A}_\alpha)$ and $\sigma(\mathbf{X}, \mathscr{A}_\alpha)$.

(3) *Analytic characterization of the elements of* $\boldsymbol{S}_{\mathbf{X},\mathscr{A}_\alpha}$

We start with the following equality

$$(3.1)\quad \begin{cases} \displaystyle\sum_{n=0}^{\infty} e^{-(4n+2)t}\boldsymbol{L}_n^{(\alpha)}(x)\boldsymbol{L}_n^{(\alpha)}(y) = \\ \displaystyle = \frac{e^{-2t}(xy)^{\frac{1}{2}}}{\sinh 2t}\exp\left[-\tfrac{1}{2}\frac{\cosh 2t}{\sinh 2t}(x^2+y^2)\right]\boldsymbol{I}_\alpha(xy/\sinh 2t) \end{cases}$$

where $\boldsymbol{I}_\alpha$ is the modified Bessel function of the first kind and of the order α. Formula (3.1) follows from [MOS], p. 242. It is an expression for the Hilbert-Schmidt kernel of $e^{-t\mathscr{A}_\alpha}$, $t>0$, which belongs to $\boldsymbol{L}_2(\mathbb{R}^+\times\mathbb{R}^+)$.

For fixed z, the $\boldsymbol{L}_n^{(\alpha)}(z)$ satisfy the following inequality

$$|\boldsymbol{L}_n^{(\alpha)}(z)| \le K e^{\delta\sqrt{n}}, \; n \in \mathbb{N} \cup \{0\}, \tag{3.2}$$

where $K>0$ and $\delta>0$. This inequality follows from a straightforward estimate of the $\boldsymbol{L}_n^{(\alpha)}(z^2)$ for large n. Cf [E1], section 4. So for each $t>0$ the series

$$\sum_{n=0}^{\infty} e^{-(4n+2)t} \boldsymbol{L}_n^{(\alpha)}(z) \boldsymbol{L}_n^{(\alpha)}(w)$$

converges uniformly on compacta in $\mathbb{C}^2$, and by (3.1)

$$\left\{\begin{aligned} &\sum_{n=0}^{\infty} e^{-(4n+2)t} \boldsymbol{L}_n^{(\alpha)}(z) \boldsymbol{L}_n^{(\alpha)}(w) = \\ &= \frac{e^{-2\alpha t}(zw)^{\frac12}}{\sinh 2t} \exp\left[-\tfrac12 \frac{\cosh 2t}{\sinh 2t}(z^2+w^2)\right] \boldsymbol{I}_\alpha\left(\frac{zw}{\sinh 2t}\right). \end{aligned}\right. \tag{3.3}$$

If we take $w=\bar{z}$ in (3.3) and keep in mind that $\boldsymbol{L}_n^{(\alpha)}(\bar{z}) = \overline{\boldsymbol{L}_n^{(\alpha)}(z)}$, we derive for $z=x+iy$, $|\arg z|<\pi$,

$$\left\{\begin{aligned} &\sum_{n=0}^{\infty} e^{-(4n+2)t} |\boldsymbol{L}_n^{(\alpha)}(x+iy)|^2 = \\ &= \frac{e^{-2\alpha t}(x^2+y^2)^{\frac12}}{\sinh 2t} \exp\left[-\frac{\cosh 2t}{\sinh 2t}(x^2-y^2)\right] \boldsymbol{I}_\alpha\left(\frac{x^2+y^2}{\sinh 2t}\right). \end{aligned}\right. \tag{3.4}$$

Now let $g \in \mathbf{X}$, and put $f = e^{-t\mathscr{A}_\alpha} g$, $t>0$. Then

$$|f(x+iy)| = |\sum_{n=0}^{\infty} e^{-(4n+2)t} (g, \boldsymbol{L}_n^{(\alpha)}) \boldsymbol{L}_n^{(\alpha)}(x+iy)| \le$$

$$\le \|g\| \{\sum_{n=0}^{\infty} e^{-(4n+2)2t} |\boldsymbol{L}_n^{(\alpha)}(x+iy)|^2\}^{\frac12} =$$

$$= \|g\| \frac{e^{-2\alpha t}}{(\sinh 4t)^{\frac12}} \exp\left[-\tfrac12 \frac{\cosh 4t}{\sinh 4t}(x^2-y^2)\right]\left((x^2+y^2)^{\frac12} \boldsymbol{I}_\alpha\left(\frac{x^2+y^2}{\sinh 4t}\right)\right)^{\frac12}$$

where $|\arg(x+iy)|<\pi$. Since we have the inequality

$$((x^2+y^2)^{-\alpha} \boldsymbol{I}_\alpha((x^2+y^2)/\sinh 4t))^{\frac12} \le K_t \exp(\tfrac12(x^2+y^2)/\sinh 4t),$$

we get for all $z=x+iy$

$$|z^{-(\alpha+\frac12)} f(z)| \le K_t' \exp\left(-\tfrac12 \frac{\sinh 2t}{\cosh 2t} x^2 + \tfrac12 \frac{\cosh 2t}{\sinh 2t} y^2\right). \tag{3.5}$$

Moreover, with the aid of (3.3) we can write for all z, $|\arg z|<\pi$,

$$f(z) = \int_0^{\infty} g(y)\left(\frac{e^{-2\alpha t}}{\sinh 2t}(zy)^{\frac12} \exp\left[-\tfrac12 \frac{\cosh 2t}{\sinh 2t}(z^2+y^2)\right] \boldsymbol{I}_\alpha\left(\frac{zy}{\sinh 2t}\right)\right) dy. \tag{3.6}$$

So it is obvious that $z \mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is an entire analytic and an even function of $z \in \mathbb{C}$. Thus we proved

(3.7) LEMMA.

Let $g \in \mathbf{X}$ and $t>0$. Put $f = e^{-t\mathscr{A}_\alpha}g$. Then

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is an entire analytic, even function.

(ii) There are A, $0<A<1$, $B>1$ and $C>0$ such that

$$|z^{-(\alpha+\frac{1}{2})}f(z)| \leq C \exp\left(-\tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2\right)$$

where $z = x+iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

One can also prove the converse of Lemma (3.7).

The proof is rather technical. We only point out the main arguments. In the Hilbert space $\mathbf{X}_\alpha = \boldsymbol{L}_2(\mathbb{R}^+, x^{2\alpha+1}dx)$ we introduce the positive,, selfadjoint operator $\tilde{\mathscr{A}}_\alpha$

$$(3.8) \qquad \tilde{\mathscr{A}}_\alpha = -\frac{d^2}{dx^2} + x^2 - \frac{2\alpha+1}{x}\frac{d}{dx} - 2\alpha.$$

For a function f satisfying (3.7.i) and (3.7.ii) the function g defined by

$$(3.9) \qquad g(z) = z^{-(\alpha+\frac{1}{2})}f(z); \; z \in \mathbb{C}$$

is even and entire analytic. It is not difficult to verify that $g \in \boldsymbol{S}_{\mathbf{X}_\alpha, \tilde{\mathscr{A}}_\alpha}$ iff $f \in \boldsymbol{S}_{\mathbf{X}, \mathscr{A}_\alpha}$. We want to prove that g defined by (3.9) is an element of $\boldsymbol{S}_{\mathbf{X}_\alpha, \tilde{\mathscr{A}}_\alpha}$. Put

$$\mathscr{H} = \tilde{\mathscr{A}}_{-\frac{1}{2}} = -\frac{d^2}{dx^2} + x^2.$$

Then by [B], Theorem 10.1 it follows that $g \in \boldsymbol{S}_{\mathbf{X}, \mathscr{H}}$. So there exists $t>0$ such that $e^{t\mathscr{H}}g \in \mathbf{X}$. By a rather technical proof it was shown in [E1], section 4, that given $t>0$ and τ, $0<\tau<t$, there exists $r_0>0$ such that the series

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} e^{\tau\mathscr{H}} \tilde{\mathscr{A}}_\alpha^n e^{-t\mathscr{H}}, \; |r| \leq r_0$$

converges uniformly in the Banach algebra of bounded operators on $\mathbf{X}$. We conclude, that for τ, $0<\tau<t$, fixed, there is $r>0$ such that

$$e^{r\tilde{\mathscr{A}}_\alpha}g = (e^{r\tilde{\mathscr{A}}_\alpha}e^{-t\mathscr{H}})(e^{t\mathscr{H}}g) \in e^{-\tau\mathscr{H}}(\mathbf{X}).$$

But then it follows from [B], Theorem 6.3 that $e^{r\tilde{\mathscr{A}}_\alpha}g$ satisfies

$$|(e^{r\tilde{\mathscr{A}}_\alpha}g)(x+iy)| \leq C_1 \exp\left(-\tfrac{1}{2}A_1x^2 + \tfrac{1}{2}B_1y^2\right), \; x,y \in \mathbb{R}$$

for some $A_1>0$, $B_1>0$ and $C_1>0$. Hence $e^{r\tilde{\mathscr{A}}_\alpha} g \in \mathbf{X}_\alpha$. So we get $g \in \boldsymbol{S}_{\mathbf{X}_\alpha, \tilde{\mathscr{A}}_\alpha}$ and therefore $f \in \boldsymbol{S}_{\mathbf{X}, \mathscr{A}_\alpha}$.

Thus we have derived

(3.10) THEOREM

$f \in \boldsymbol{S}_{\mathbf{X}, \mathscr{A}_\alpha}$ if and only if the function $z \mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is even and belongs to the space $\boldsymbol{S}_{\frac{1}{2}}^{\frac{1}{2}}$.

REMARK. $\boldsymbol{S}^{\frac{1}{2}}_{\frac{1}{2}}$ is an $\boldsymbol{S}$-space of Gelfand and Shilov ([GS], ch. IV). We note that the function space $S^{\frac{1}{2}}_{\frac{1}{2}}$ equals $\boldsymbol{S}_{L_2(\mathbb{R}),\mathscr{H}}$ where

$$\mathscr{H} = -\frac{d^2}{dx^2} + x^2,$$

see [G].

(4) *Analytic characterization of the elements of* $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$

The strong topology in $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ is generated by the seminorms $p_n^{(\alpha)}$

(4.1) $$p_n^{(\alpha)}(f) = \| \mathscr{A}_\alpha^n f \|, \; f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha),\; n \in \mathbb{N}.$$

It can be shown that $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ is a Frêchet space with this topology (see [E]). Let $f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$, and put

$$g(x) = x^{-(\alpha+\frac{1}{2})} f(x),\; x > 0.$$

Then with the same notation as in section 3

$$g \in \tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha}).$$

Define the operators $\mathscr{R}$ and $\mathscr{Q}$ on $\tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha})$ by

(4.2) $$\mathscr{R}f(x) = \frac{1}{x} f'(x),\; x > 0,$$

(4.3) $$\mathscr{Q}f(x) = x f(x),\; x > 0.$$

Then one can prove that for each $i, j \in \mathbb{N}$, there exists $d > 0$ and $k \in \mathbb{N}$ such that

(4.4) $$\| \mathscr{Q}^i \mathscr{R}^j h \|_\alpha \leq d \| \tilde{\mathscr{A}_\alpha}^k h \|_\alpha.$$

Cf [E1], section 3. Here $\| \cdot \|_\alpha$ denotes the norm on $\mathbf{X}_\alpha$. Hence the seminorms $q_{ij}^{(\alpha)} : h \mapsto \| \mathscr{Q}^i \mathscr{R}^j h \|_\alpha$ are continuous on $\tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha})$. Further, since

$$\tilde{\mathscr{A}_\alpha} = -\mathscr{R}\mathscr{Q}^2 - 2\alpha\mathscr{R} + \mathscr{Q}^2 - 2\alpha$$

and since

$$\mathscr{R}\mathscr{Q}^2 - \mathscr{Q}^2\mathscr{R} = 2$$

it follows that there exist constants $c_{ij} \geq 0$ such that,

(4.5) $$\| \tilde{\mathscr{A}_\alpha}^k h \|_\alpha \leq \sum_{i,j=1,1}^{2k,2k} c_{ij} q_{ij}^{(\alpha)}(h),\; h \in \tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha}).$$

So the $q_{ij}^{(\alpha)}$ generate the strong topology in $\tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha})$. With the aid of Sobolev's embedding theorem, and some straightforward estimates, it can be shown that $\| \cdot \|_\alpha$ in the definition of the $q_{ij}^{(\alpha)}$, can be replaced by the supremum norm. Hence the seminorms $p_{ij}^{(\alpha)}$, $i, j \in \mathbb{N}$,

$$p_{ij}^{(\alpha)} : h \mapsto \sup_{x>0} |(\mathscr{Q}^i \mathscr{R}^j h)(x)|,\; h \in \tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha}),$$

are continuous and induce the strong topology on $\tau(\mathbf{X}_\alpha, \log \tilde{\mathscr{A}_\alpha})$.

Going back to our original space $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ we have shown that the strong topology on $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ is also generated by the seminorms $\gamma_{ij}^{(\alpha)}$:

$$(4.6) \qquad \gamma_{ij}^{(\alpha)}(f) = \sup_{x \geq 0} |x^i (x^{-1} \mathscr{D})^j x^{-(\alpha+\frac{1}{2})} f(x)|, \; i,j \in \mathbb{N} \cup \{0\},$$

$f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$. Here $\mathscr{D}$ is the operator d/dx.

For every $f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ we have

$$f(x) = \sum_{n=0}^{\infty} (f, \boldsymbol{L}_n^{(\alpha)}) \boldsymbol{L}_n^{(\alpha)}(x), \; x > 0.$$

Since the functions $x \mapsto x^{-(\alpha+\frac{1}{2})} \boldsymbol{L}_n^{(\alpha)}(x)$ are functions of x^2 it is obvious that $x \mapsto x^{-(\alpha+\frac{1}{2})} f(x)$ is a function of x^2 on $\mathbb{R}$.

(4.7) THEOREM.

Each element $f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ satisfies

$$f(x) = x^{\alpha+\frac{1}{2}} \varphi(x^2), \; x > 0,$$

for some $\varphi \in \boldsymbol{S}$, i.e. Schwartz's space of functions of rapid decrease.

PROOF. Let $f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$. Then g defined by

$$g(x) = x^{-(\alpha+\frac{1}{2})} f(x), \; x > 0,$$

is a function of x^2. Thus $g(x) = h(x^2)$, $x \in \mathbb{R}$, for some function h on $[0, \infty)$. For all $i,j \in \mathbb{N} \cup \{0\}$ we have

$$\sup_{x \in \mathbb{R}} \left|(x^2)^i \left(\frac{1}{x} \mathscr{D}_x\right)^j h(x^2)\right| < \infty$$

with the new variable $\xi = x^2$

$$\sup_{\xi \geq 0} |\xi^i (\mathscr{D}_\xi)^j h(\xi)| < \infty, \; i,j = 0, 1, 2, \ldots.$$

Since in $\xi = 0$ all derivatives on the right of h exist, there is an infinitely differentiable function h_1 on $\mathbb{R}$ of bounded support with $(\mathscr{D}_x^m h_1)(0) = (\mathscr{D}_x^m h)(0)$ for all $m \in \mathbb{N}$.

Define φ on $\mathbb{R}$ by

$$\varphi(x) = \begin{cases} h(x), & x \geq 0 \\ h_1(x), & x < 0 \end{cases}$$

Then $\varphi \in \boldsymbol{S}$ and $f(x) = x^{(\alpha+\frac{1}{2})} \varphi(x^2)$, $x > 0$. □

As a corollary of Theorem (4.2) we have

(4.8) COROLLARY.

$f \in \tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ if and only if the even extension of $x \mapsto x^{-(\alpha+\frac{1}{2})} f(x)$ belongs to Schwartz's space $\boldsymbol{S}$.

For the sake of completeness, note that $\boldsymbol{S}=\tau(\boldsymbol{L}_2(\mathbb{R}), \log \mathscr{H})$ with

$$\mathscr{H}=\frac{-d^2}{dx^2}+x^2+1$$

(see [E]).

It follows that $f\in X$ is in $\tau(\mathbf{X}, \log \mathscr{A}_\alpha)$ if and only if $\gamma_{ij}^{(\alpha)}(f)$ is finite for all $i,j=0,1,2,\ldots$. Comparing this result with the definition of the space $\mathscr{H}_\mu$ in [Z] we have

$$\mathscr{H}_\mu=\tau(\mathbf{X}, \log \mathscr{A}_\mu)$$

both set-theoretically and topologically.

(5) *Analytic characterization of the elements of* $\tau(\mathbf{X}, \mathscr{A}_\alpha)$

For convenience we introduce the function classes $\boldsymbol{S}_{A,B}^{(\alpha)}$.

(5.1) DEFINITION.

$f\in \boldsymbol{S}_{A,B}^{(\alpha)}$ if and only if

(i) $z\mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is entire analytic and even.

(ii) There is $C>0$ such that for all $x,y\in\mathbb{R}$

$$|(x+iy)^{-(\alpha+\frac{1}{2})}f(x+iy)|\leq C\exp\left(-\tfrac{1}{2}Ax^2+\tfrac{1}{2}By^2\right).$$

The following inclusions hold true

$$(5.2)\qquad e^{-t\mathscr{A}_\alpha}(\mathbf{X})\subset \boldsymbol{S}_{A,B}^{(\alpha)}\subset e^{-t'\mathscr{A}_\alpha}(\mathbf{X})$$

where $0<A<1$ and $B>1$ and $t,t'>0$ depend on the choice of A,B. Since

$$(5.3)\qquad \tau(\mathbf{X}, \mathscr{A}_\alpha)=\bigcap_{t>0} e^{-t\mathscr{A}_\alpha}(\mathbf{X})$$

(see [E]), it follows that

$$(5.4)\qquad \tau(\mathbf{X}, \mathscr{A}_\alpha)=\bigcap_{\substack{0<A<1,\\ B>1}} \boldsymbol{S}_{A,B}^{(\alpha)}.$$

In other words

(5.5) THEOREM.

$f\in\tau(\mathbf{X}, \mathscr{A}_\alpha)$ if and only if

(i) $z\mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is even and entire.

(ii) for each A, $0<A<1$ and B, $B>1$ there exists $C>0$ such that for all $x,y\in\mathbb{R}$

$$|(x+iy)^{-(\alpha+\frac{1}{2})}f(x+iy)|\leq C\exp\left(-\tfrac{1}{2}Ax^2+\tfrac{1}{2}By^2\right).$$

As a corollary of Theorem 5.5 we have

(5.6) COROLLARY.

$f\in\tau(\mathbf{X}, \mathscr{A}_\alpha)\Leftrightarrow z\mapsto z^{-(\alpha+\frac{1}{2})}f(z)$ is even and belongs to $\tau(\boldsymbol{L}_2(\mathbb{R}), \mathscr{H})$. Here

$$\mathscr{H}=-\frac{d^2}{dx^2}+x^2.$$

PROOF. See [E], ch. VIII.

REMARK: The dual space $\sigma(L_2(\mathbb{R}), \mathscr{H})$ of $\tau(L_2(\mathbb{R}), \mathscr{H})$ is the Hermite pansion space introduced by Korevaar [K].

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