Uniqueness of Solutions of the Cauchy Problem for Linear Partial Differential Equations with Characteristics of Variable Multiplicity*

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Received January 20, 1976

INTRODUCTION

The first results in the study of uniqueness in the Cauchy problem for partial differential equations with multiple characteristics were given by Hörmander [5], Mizohata [11], and Calderón [1]. The multiple characteristics, in all of these cases, were nonreal and the multiplicity was constant and at most two. For the case where the multiplicity of the nonreal characteristics is more than two, or where the multiplicity of the real characteristics is more than one, results under some conditions on the lower-order terms were given by Matsumoto [9], Watanabe [15], and Zeman [17]. For the case where the multiplicity is variable, Pederson [13] proved a uniqueness theorem for elliptic operators having at most double characteristics, assuming that the characteristics are smooth enough.

In this paper we prove uniqueness for linear partial differential operators which may have characteristics of variable multiplicity. The real characteristics are allowed to become complex under certain circumstances; the characteristics are required to be sufficiently smooth. If the multiplicity of the real characteristics (which may become complex, as above) is greater than one or if multiplicity of the nonreal characteristics is greater than two, we also require an additional assumption on the manner in which the real characteristics cross each other and a condition which restricts the lower-order terms. This condition on the lower-order terms can be shown to reduce to the condition on the lower terms presented in Zeman [17] if the multiplicity is constant.

* This work has been partially supported by N.S.F. under Grant No. MPS75-06687, while author was at the University of Wisconsin-Madison.

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Notation

First, recall the problem. Let

$$P(x, t, D_x, D_t) := P_m + P_{m-1} + \cdots$$

be a linear partial differential operator of order $m$ and the $P_i$ are homogeneous of order $i$ in $(x, t)$

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad t \in \mathbb{R}^1.$$ 

Let $P_m(x, t, \xi, \tau)$ be the leading symbol of $P$ where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$.

Assume that the hyperplane $t = 0$ is not characteristic at the origin, i.e., $P_m(0, 0, 0, 1) \neq 0$. The Cauchy problem is to find a solution $v$ of $Pv = f$ in a neighborhood of $t = 0$ with given (say homogeneous) Cauchy data on the plane $t = 0$: $D_j v |_{t=0} = 0, j = 0, 1, \ldots, m - 1$. For an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

$$D_x := (1/\partial) \partial_x,$$

$$D_t := (1/\partial) \partial_t.$$ 

$L_x^\gamma$ is the class of pseudodifferential operators on order $\gamma$ in the $x$-variables. See Kohn and Nirenberg [7] and Friedrichs [4] for more details.

By $D_r$, $r$ any nonnegative integer, we mean an arbitrary homogeneous operator of order $r$, which is a partial differential operator in $t$ and a pseudodifferential operator in $x$.

$$D_m := \sum_{|\alpha| = m} D^\alpha;$$

$(u, v)$ is the $L_2$ scalar product of $u$ and $v$. $\| u \|$ is the corresponding $L_2$ norm of $u$.

$$\| u \|^2 = \int_T \| u \|^2 e^{k(t-T)} dt$$

where $\| \cdot \|$ is the $L_2$ norm in the $x$-variables. $H_m$ is the Hilbert space with norm $\| u \|^2_m = \int (1 + |\xi|^2)^2 \hat{u}(\xi)^2 d\xi$ where $\hat{u}$ is the Fourier transform of $u$. $C'$ is the space of distributions with compact support.

Since $t = 0$ is noncharacteristic at the origin with respect to $P$ we may assume that the coefficient of $D_t^m$ in $P_m$ is 1.

It is convenient to make a local transformation of variable so that the surface $t = 0$ becomes transformed to a convex surface $S: t = \delta \sum_{j=1}^n (x_j)^2$ where $\delta > 0$ is constant. The condition that we require depends on the roots $\tau$ of $P_m(x, t, \xi, \tau)$ in these new variables. It is clear from the proof that in these
new variables the operator $P$ need not be a partial differential operator; it may be an operator of the form

$$P(x, t, D_x, D_t) = D_t^m + \sum_{j=1}^{m} R_j(x, t, D_x) D_t^{m-j}$$

where the $R_j$ are pseudodifferential operators in the $x$-variable of order $j$, varying smoothly in $t$.

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We now describe the conditions which are sufficient for the uniqueness of the Cauchy problem.

The first condition deals with the characteristic roots $\lambda_j(x, t, \xi)$ of $P_m(x, t, \xi, \tau)$. We allow $h_j(x, t, \xi)$ to belong to the following classes:

Class (A): For $t \geq 0$ and $|\xi| = 1$, $\lambda_j = a_j + ib_j$ satisfies for all $(x, t, \xi)$, $0 \leq t \leq T$, for some fixed $T$ which is designated later, one of the following:

(i) $b_j > 0$,

(ii) $b_j, \leq \sum_{k=1}^{n} (a_{j,k} b_{j,k} - a_{j,k} b_{j,k})$.

Here $\epsilon$ is a fixed positive constant.

Class (B): $\lambda_j(x, t, \xi)$ is nonreal for all $(x, t, \xi)$, $0 \leq t \leq T$, and $|\xi| = 1$.

Notation. Let $\partial_t = D_t - \lambda_j(x, t, D_x)$. Then we say that $\partial_t \in (A)$ and $\partial_t \in (B)$, respectively, if $\lambda_j(x, t, \xi)$ belongs to class (A) and $\lambda_j(x, t, \xi)$ belongs to class (B), respectively.

Remark 1. The condition that the characteristic roots belong either to class (A) or class (B) is found first in Nirenberg [12]. Some condition such as this which restricts the manner in which real roots can become complex seems to be needed for the uniqueness of the Cauchy problem. Theorem 8.9.2 (Hörmander [6]) contains an example in $\mathbb{R}^2$ of the form

$$(\partial \psi/\partial t) + ia(x, t)(\partial \psi/\partial x) = 0 \quad \text{with} \quad \psi \equiv 0 \quad \text{for} \quad t < 0,$$

but $\psi \neq 0$ in any neighborhood of the origin; the function $a$ is a real $C^\infty$ function which changes sign infinitely often near the origin.

A condition similar to that found in class (A)(ii) is given by Hörmander [6]. He describes the operators which satisfy this condition as principally normal. This condition is extended by Menikoff [10]. Another condition similar to (ii) is given by Kumano-go [8].
The next condition that we require is designed to ensure that we may
smoothly factor $P_m$ in the form

$$P_m(x, t, \xi, \tau) = \prod_{i=1}^{m} (\tau - \lambda_i(x, t, \xi)),$$

and also that we may commute the factors $[\tau - \lambda_i(x, t, \xi)]$. With these require-
ments in mind, we ask that

**Condition (I).** The roots $\lambda_i(x, t, \xi)$, $1 \leqslant i \leqslant m$, are of class $C^{m-1}$ in $x$, $t$,
and $\xi$.

**Remark.** Some kind of smoothness condition of this sort seems to be needed
to ensure uniqueness. Plis [14] has given an example of a fourth-order equation
with real $C^\infty$ coefficients which has nontrivial solutions which vanish in a half-
plane. An examination of Plis's counterexample shows that the characteristic
roots have unbounded partial derivatives near the initial surface.

Next, we formulate the conditions on the lower-order terms.

For each fixed $(x, t, \xi)$, let

$$r(x, t, \xi) = \max \text{ multiplicity of the characteristic roots } \lambda_i \text{ belonging to class (A)},$$

and let

$$s(x, t, \xi) = \max \text{ multiplicity of the characteristic roots } \lambda_i \text{ belonging to class (A) or to class (B)}.$$  

Also let

$$r = \max_{(x, t, t \geq 0, |\xi| = 1)} r(x, t, \xi)$$

and

$$s = \max_{(x, t, t \geq 0, |\xi| = 1)} s(x, t, \xi).$$

Finally $q = \max\{r, [(s \div 1)/2]\}$, where $[K]$ denotes the integral part of $K$.

If $r > 1$ or $s > 2$ (i.e., $q > 1$), some condition on the lower-order terms
is needed. Examples of nonuniqueness of the Cauchy problem have been

Another example of nonuniqueness is given by Plis [14]. He presents a
smooth elliptic equation with characteristics of multiplicity at least 4 which
has a nonunique solution to the Cauchy problem. Interestingly, Watanabe
[15] shows that certain fairly general elliptic operators with triple characteristic
roots whose lower-order terms have Lipschitz continuous coefficients do give
uniqueness in the Cauchy problem. Both Plis and Watanabe deal with constant
multiplicity.
Although these examples show that some conditions on the lower-order terms are necessary for uniqueness to hold, it is still unclear what conditions are optimum. Matsumoto [9] has presented conditions on lower-order terms for operators with multiple characteristic roots which are substantially different from the ones presented in either this paper or the one preceding it, dealing with constant multiplicity (Zeman [17]). Matsumoto also deals with constant multiplicity.

Before we formulate the conditions that we require the lower-order terms to satisfy, let us consider the following module $S$ over $L_2^n$, the ring of pseudo-differential operators in the $x$-variable of order zero. It is associated with the operator $\Pi_{\alpha} = \partial_1 \cdots \partial_m$.

$S$ is generated by "monomial" operators which are formed as follows: We first describe the operators of order $m - 1$ which generate $S_{(m-1)}$. Suppose $\Pi_m = \partial_1 \cdots \partial_m$ with $\partial_i$ belonging either to (A) or to (B). If $\partial_i \in (A)$, form the operators $\Pi_m/\partial_i$ by omitting one factor from $\Pi_m$ at a time. Call the module generated by these operators $S^{(A)}_{(m-1)}$. If $\partial_j \in (B)$, form $\partial_j \Pi_m/\partial_i$ where $\partial_i$ can belong to (A) or (B) and call the module generated by these operators $S^{(B)}_{(m-1)}$. $S_{(m-1)}$ is the module generated by the operators which generate $S^{(A)}_{(m-1)}$ or $S^{(B)}_{(m-1)}$.

The module $S^{(A)}_{(m-2)}$ is formed in a similar way to $S^{(B)}_{(m-1)}$: We cancel one factor $\partial_i \in (A)$ at a time from the monomial operators in $S_{(m-1)}$. $S^{(A)}_{(m-2)}$ is formed by cancelling $\partial_i \partial_j$ and replacing by a $D$ from the operators in $S_{(m-1)}$ if $\partial_j \in (B)$. Here, as in the formation of $S^{(B)}_{(m-1)}$, $\partial_i$ can belong either to (A) or to (B). $S_{(m-2)}$ is the module generated by the operators which generate either $S^{(A)}_{(m-2)}$ or $S^{(B)}_{(m-2)}$. We go on in this manner to form $S_{(m-3)}$, $S_{(m-4)}$, ... . Finally $S$ is the module generated by all the operators which generate any of the $S^{(A)}_{(m-i)}$, $i \leq 1$.

Remark. It is clear that if $S^{(A)}_{(m-j)}$ is not empty, then $S^{(A)}_{(m-j)} \subseteq S^{(B)}_{(m-j)}$. Hence, in such cases $S_{(m-j)} = S^{(B)}_{(m-j)}$.

Next, we reformulate $P$ as follows: $P = \Pi_m + P_{m-1} + P_{m-2} + \cdots$ where $P_{m-j}$ is an operator of order $m - j$ (not necessarily homogeneous).

Now we are ready to state the condition which we require the lower-order terms to satisfy.

Condition (V). Suppose the multiplicity of the characteristic roots of $P$ may vary. Suppose $P = P_m + P_{m-1} + \cdots$ with $P_m$ := leading part of $\Pi_{\alpha}$ where $\partial_i \in (A)$ or $\partial_i \in (B)$.

Then the lower-order terms are to satisfy

$$P'_{m-j} \in S_{(m-j)}, \quad 0 \leq j \leq q - 1,$$

where $q = \max\{r, [(s + 1)/2]\}$ where $r$ and $s$ have been defined earlier.

Remark. Unfortunately, this condition is difficult to verify for an arbitrary operator. However, if the multiplicity turns out to be constant, then Condition
(V) simplifies to Condition (C), the algebraic condition introduced by Zeman [17], a condition on lower-order terms for operators with characteristics of constant multiplicity.

More specifically, we have

**Lemma 2.1.** Suppose \( P = P_m + P_{m-1} + \cdots \) and suppose the characteristic roots of \( P \) are of constant multiplicity with \( P_m = \) leading part of \( \prod \tilde{\partial}_i^{m_i} \prod \tilde{\tilde{\partial}}_j^{n_i} \), where \( \tilde{\partial}_i \in (A) \) and \( \tilde{\tilde{\partial}}_j \in (B) \). Then \( P_m \in S_{(m, \kappa)} \) if and only if \( P_{(m-k)} = 0 \) \( \mod(\prod \tilde{\partial}_i^{m_i-k} \prod \tilde{\tilde{\partial}}_j^{n_i-2k}) \) with the convention that

\[
m_i - k = 0 \quad \text{if} \quad m_i - k \leq 0
\]

and

\[
n_i - 2k = 0 \quad \text{if} \quad n_i - 2k \leq 0.
\]

**Proof.** See Zeman [16].

Finally, we require the following additional condition concerning the crossings of the characteristic roots belonging to class (A)

\[
[\tilde{\partial}_i, \tilde{\tilde{\partial}}_j] = a\tilde{\partial}_i + b\tilde{\tilde{\partial}}_j + N \quad \text{for some } a, b, N \in L_{x,0} \] if \( \tilde{\partial}_i, \tilde{\tilde{\partial}}_j \in (A). \) (A)

**Remark 1.** Although condition (A) is a rather strong condition, it is satisfied by a fairly wide class of operators. Simple examples of operators which satisfy the condition are

(a) Partial differential operators with constant coefficients.

(b) Elliptic operators (since they have no real characteristics).

(c) The operators \((D-tD_x)(D_x + xD_t)\) and \((D_t - xD_x)(D_t - x^2D_x)\) in a neighborhood of the origin.

Simple examples of operators which do not satisfy condition (A) are \((D_t + tD_x)(D_t + xD_x)\) and \((D_t - tD_x)(D_t + tD_x)\).

**Remark 2.** Condition (A) is automatically satisfied by operators whose characteristic roots do not cross each other. (This is proved later. See the Corollary to Lemma 4.2.) Hence, all of the conditions presented here, in the case where the multiplicity is variable, reduce to the conditions required in Zeman [17] where the multiplicity is constant. Thus Theorem 1 is a generalization of the results presented in Zeman [17].

The main results of this paper are

**Theorem 1.** Suppose \( t = 0 \) is noncharacteristic at the origin with respect to the operator \( P = P_m + P_{m-1} + \cdots \). Suppose \( P_m \) satisfies Condition (I). Then
$P_m$ can be written smoothly as the leading part of $\prod_{i=1}^{m} \partial_i$. Suppose $\partial_i \in (A)$ or $\partial_i \in (B)$. If $q > 1$, we require, in addition, that the operator $P$ satisfy Condition (*) and Condition (V). Then, if $u \in H_{[\omega]}^{\infty}(\Omega')$ where $\Omega' = \{(x, t): 0 \leq t \leq T\}$ such that $u \equiv 0$ for $t < 0$ and $Pu = 0$ for $t < T$, then $u \equiv 0$ for $t < T$.

The essential tool in uniqueness proofs to date has been a weighted $L_2$ inequality analogous to an $L_1$ inequality used by Carleman [2]. Our version of Carleman's inequality is given by

**Theorem 2.** Suppose $t = 0$ is noncharacteristic at the origin with respect to $P = P_m + P_{m-1} + \cdots$. Suppose $P_m$ satisfies Condition (I). Then we may factor $P_m$ smoothly into $P_m = \text{leading part of } \prod_{i=1}^{m} \partial_i$. Suppose $\partial_i \in (A)$ or $\partial_i \in (B)$. If $q > 1$, we require, in addition, that the operator satisfy Condition (*) and Condition (V). Then there are constant $C_1, C_2$ independent of $u$ such that for $T, k^{-1}$ sufficiently small, the following inequalities hold

$$
\sum_{|\alpha| \leq m-\rho} k^{m-|\alpha|} \| D^\alpha u \|^2 \leq C_1 \| Pu \|^2 \quad \text{for } u \in C_0^{\infty}(\Omega'),
$$

where

$$
\Omega' = \{(x, t): 0 \leq t \leq T\} \quad \text{if } r = s
$$

and

$$
\sum_{|\alpha| \leq m-s} k^{m-|\alpha|} \| D^\alpha u \|^2 + \sum_{m-s \leq |\alpha| \leq m-r} (1 + kT^2)^{m-|\alpha|-s} k^{m-|\alpha|} \| D^\alpha u \|^2
\leq C_2 \| Pu \|^2,
$$

for $u \in C_0^{\infty}(\Omega')$ if $s > r$.

**Remark 1.** If $q \leq 1$, Theorem 1 improves the result presented in [17, Theorem 1]. It allows the nonreal characteristic roots to cross each other. It is also an improvement of the result of Pederson [13], who only deals with elliptic operators. Simple real characteristic roots are allowed here.

**Remark 2.** The theorem is proved for $u \in C_0^{\infty}(\Omega')$ where $\Omega' = \{(x, t): 0 \leq t \leq T\}$, a strip. Hence, it is not necessary to restrict $\Omega'$ to $\{(x, t): 0 \leq t \leq T, 0 \leq |x| \leq r\}$ a small neighborhood of the origin, as is required in Nirenberg [12] and Zeman [17].

We first show how the proof of Theorem 1 follows from Theorem 2.

**Proof of Theorem 1.** Assuming that Theorem 2 holds, the following inequality holds:

$$
k^m \int_0^T \| u \|^2 e^{k(t \tau)} \tau \, dt \leq c \int_0^T \| Pu \|^2 e^{k(t \tau)} \tau \, dt \quad \text{for } u \in C_0^{\infty}(\Omega').
$$

(3.3)
It is clear that (3.3) must also be valid for all $u \in \mathcal{C}^\infty(\Omega) \cap H_{(u)}$ since such functions $u$ can be approximated in $H_{(u)}$ by functions in $C_0^\infty(\Omega)$ with supports in a fixed bounded set. Fix $T_1$ and $T_2$ such that $0 < T_2 < T_1 < T$ and let $\zeta(t)$ be a nonnegative $C^\infty$ function defined in $t \geq 0$ such that $\zeta(t) = 1$ for $t \leq T_1$ and $\zeta(t) = 0$ for $t > T_1$. If $v$ is the solution of $Pv = 0$ then for $T$ small we may apply (3.3) to $u = \zeta v$ and infer that

$$\int_0^{T_1} \| \psi \|_2^2 e^{k(T)} \, dt < \text{left-hand side of (3.3)}$$

$$\leq C' k^{-m} \int_{T_1}^{T} P(\zeta(v))_2^2 e^{k(t-T)} \, dt$$

$$\leq C' k^{-m} \int_{T_1}^{T} e^{k(t-T)} \, dt,$$

where $C'$ is a constant depending on $T$, which we keep fixed, but independent of $K$. Thus, in particular,

$$e^{k(T-T_2)} \int_0^{T_2} \| \psi \|_2^2 \, dt < \int_0^{T_2} \| \psi \|_2^2 e^{k(t-T)} \, dt$$

$$< C' k^{-m} T e^{k(T-T_1)}.$$

Letting $k \to \infty$, we see this is impossible unless $v = 0$ for $t < T_2$, where $T_2 < T$ can be chosen arbitrary. Hence the theorem is proved.

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Technical Lemmas

Before we prove Theorem 2, we need the following technical lemmas.

**Lemma 4.1.** For any operator $\Pi_m = \partial_1 \cdots \partial_m$ of order $m$ with $\partial_i \in (A)$ or $\partial_i \in (B)$, let $\hat{\Pi}_m$ be an operator obtained from $\Pi_m$ by an arbitrary permutation of the factors $\partial_i$. Suppose also that if $\partial_i, \partial_j \in (A)$, then $[\partial_i, \partial_j] = a\partial_i + b\partial_j + N$ for some $a, b, N \in L_2$. Then $h = \Pi_m - \hat{\Pi}_m$ is an operator of order $m - 1$, belonging to $S$.

**Proof.** It suffices to carry out the proof for the special permutation

$$s = \partial^{t_1} \partial_1 \partial_2 \cdots \partial_j \partial_i \cdots \partial_m$$

$$\hat{s} = \partial^{t_m} \partial_1 \partial_2 \cdots \partial_j \partial_i \cdots \partial_m$$

$$t = n = m.$$
because an arbitrary permutation can be achieved by a finite number of this kind. We have

\[ s - \tilde{s} = D'\partial_1 \cdots \partial_{i-1}[\partial_i \partial_j - \partial_j \partial_i] \partial_{j+1} \cdots \partial_k. \]

There are two cases to consider:

(a) both \( \partial_i \) and \( \partial_j \) belong to (A),
(b) at least one of the \( \partial_i \), \( \partial_j \) belongs to (B).

Suppose (a) is true; then \([\partial_i, \partial_j] = a\partial_i \mp b\partial_j + N\). Thus

\[ s - \tilde{s} = D'\partial_1 \cdots \partial_{i-1}(a\partial_i) \partial_{j+1} \cdots \partial_n + D'\partial_1 \cdots \partial_{i-1}(b\partial_j) \partial_{j+1} \cdots \partial_n \]

To complete the proof for this case we need only show that if

\[ s = \partial_1 \partial_2 \cdots \partial_t \in S \]

then any element

\[ \sigma = \partial_1 \partial_2 \cdots \partial_k f \partial_{k+1} \cdots \partial_t \]

also belongs to \( S \) for any \( f \in L_x^0 \). Let

\[ v = \partial_{k+1} \partial_{k+2} \cdots \partial_t[u] \]

then \( \sigma[u] = \partial_1 \cdots \partial_{k-1} \partial_k f[v] \). Now \( \tilde{D} f[v] = f(D)[v] \) for any \( \tilde{D} \) with \( g = \tilde{D}[f] \in L_x^0 \), if \( f \in L_x^0 \).

In particular this holds if \( \tilde{D} = \partial_k \) and so

\[ \sigma = \partial_1 \partial_2 \cdots \partial_{k-1} f \partial_k \partial_{k+1} \cdots \partial_t + \partial_1 \partial_2 \cdots \partial_{k-1} g \partial_{k-1} \cdots \partial_t, \]

with \( f, g \in L_x^0 \).

These two terms are of the same form as \( \sigma \) except that \( f \) has moved leftward in the first term and the second term contains one less \( \partial_t \) than \( \sigma \). Clearly if we continue in this way we would get \( \sigma = f \cdot s \) a linear combination of terms in \( S \) of order less than the order of operator \( s \).

Hence \( s - \tilde{s} \in S \).

Now suppose (b) is true, then as before,

\[ s - \tilde{s} = D'\partial_1 \cdots \partial_{i-1}[\partial_i \partial_j - \partial_j \partial_i] \partial_{j+1} \cdots \partial_m. \]

Now \([\partial_i, \partial_j] = D_1 + N\) for some homogeneous operator \( D_1 \) and for some \( N \in L_x^0 \). Thus

\[ s - \tilde{s} = \partial_1 \cdots \partial_{i-1} \partial_j \partial_{j+1} \cdots \partial_m + \partial_1 \cdots \partial_{i-1} N \partial_{j-1} \cdots \partial_m. \]
Also as before we want to show that \( s - \hat{s} = D_{e_1} e_1 \cdots D_{e_{i-1}} e_{i-1} \cdots e_m + \) lower-order terms belonging to \( S = \sigma + \) lower-order terms where \( \sigma \) belongs to \( S_{(i-1)} \). Now \( [e_{i-1}, D_1] = D_2 - N_2 \) for some \( D_2 \) and some \( N_2 \in L_x^0 \). Hence
\[
s - \hat{s} = D_{e_1} e_1 \cdots D_{e_{i-2}} e_{i-2} D_{e_{i-1}} e_{i-1} \cdots e_m + D_{e_1} e_1 \cdots D_{e_{i-2}} e_{i-2} N_2 e_{i-1} \cdots e_m.
\]

If we now argue exactly as in case (a), we prove that Lemma 4.1 is true.

**Remark.** This lemma shows that if Condition (\(-\)) is assumed, then Condition (V) is invariant under an arbitrary permutation of \( e_i \).

**Lemma 4.2.** Suppose the multiplicity of the roots \( \lambda_i(x, t, \xi), 1 \leq i \leq k \), is equal to \( l \), with \( k > l \). Let \( \hat{e}_i = D_i - \lambda_i(x, t, D_x) \). Then for any \( a(x, t, D_x) \in L_x^0 \) and \( b(x, t, D_x) \in L_x^1 \), we can find \( c_1, \ldots, c_k, \lambda \in L_x^0 \) such that
\[
c_1 \hat{e}_1 + \cdots + c_k \hat{e}_k = aD_x - b(x, t, D_x) + \lambda.
\]

**Proof.** To find \( c_1, \ldots, c_k \) we have to solve the following system of pseudo-differential equations (\( N \) turns out to be some lower-order term which shows up when we solve the system)
\[
\sum_{i=1}^k c_i(D_i - \lambda_i(x, t, D_x)) = aD_x - b(x, t, D_x).
\]

This implies that \( \sum_{i=1}^k c_i(x, t, D_x) = a(x, t, D_x) \) and
\[
\sum_{i=1}^k c_i(x, t, D_x) \lambda_i(x, t, D_x) = b(x, t, D_x).
\]

Modulo lower-order terms which belong to \( L_x^0 \), this system can be solved for \( c_i \) if the symbol matrix
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1(x, t, \xi) & \lambda_2(x, t, \xi) & \cdots & \lambda_k(x, t, \xi)
\end{bmatrix}
\]
is of rank 2 at any point \( (x, t, \xi), \| \xi \| = 1 \); we then apply an ellipticity argument to solve the system. This matrix is of rank 2 if for any fixed \( (x, t, \xi), \| \xi \| = 1 \),
\[
\begin{bmatrix}
1 & 1 \\
\lambda_i(x, t, \xi) & \lambda_j(x, t, \xi)
\end{bmatrix}
\]
are not all singular for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \). That is, \( \lambda_i(x, t, \xi) - \lambda_j(x, t, \xi) \neq 0 \) for some \( i \) and \( j \). But, since there are \( k \) of these \( 2 \times 2 \) matrices and by definition of multiplicity, at most \( l \) are singular at any point \( (x, t, \xi), \| \xi \| = 1 \), we see that \( k - l \) are not singular.
Corollary. Suppose $\lambda_1(x, t, \xi)$ and $\lambda_2(x, t, \xi)$ are unequal for all $(x, t, \xi)$, $|\xi| = 1$. Let $\partial_i = D_t - \lambda_i(x, t, D_x)$ for $i = 1, 2$. Then $[\partial_1, \partial_2] = a\partial_1 + b\partial_2 + N$ for some $a, b, N \in L^p_x$.

Proof. $[\partial_1, \partial_2] = D + N$ for some $D$ and for some $N \in L^p_x$. By Lemma 4.2, $D = a\partial_1 + b\partial_2 + N$ for some $a, b, N \in L^p_x$. Hence

$$[\partial_1, \partial_2] = a\partial_1 + b\partial_2 + N$$

where $a, b, N = N_1 + N_2 \in L^p_x$.

In this section, we state the two lemmas basic to the proofs of the Carleman estimates found in Theorem 2. They are slight extensions of lemmas of Calderón [1] and may be found in Nirenberg [12].

Lemma 5.1. Suppose $\partial_i \in (A)$ or $\partial_i \in (B)$. Then for $T$ and $k^{-1}$ sufficiently small, the following inequality holds

$$\|u\|^2 \leq (c|k|)\|\partial_i u\|^2$$

for $u \in C_0^\infty(\Omega')$, $\Omega' = \{(x, t): 0 < t \leq T\}$ with $c$ independent of $k$, $T$, and $u$.

Lemma 5.2. Suppose $\partial_i \in (B)$, then for $T$ and $k^{-1}$ sufficiently small, the following inequality holds

$$(\|\Lambda u\|^2 + \|D_i u\|^2) \leq c(1 + kT^2)\|\partial_i u\|^2$$

for $u \in C_0^\infty(\Omega')$ where $\Omega' = \{(x, t): 0 < t \leq T\}$ with $c$ independent of $k$, $T$, and $u$, and where $\Lambda$ is the pseudodifferential operator in the $x$-variables with symbol $$(1 + |\xi|^2)^{1/2}$$

Remark. Lemma 5.2 implies that

$$\sum_{|\alpha| = 1} \|D^\alpha u\|^2 \leq c(1 + kT^2)\|\partial_i u\|^2$$

for $\partial_i \in (B)$.

Before we state the next lemma, we examine the module $S$ more closely. This examination reveals that the sequence $S_{(m-1)}$, $S_{(m-2)}$, ..., stops at some $S_{(c)}$. If $\Pi_m$ is composed only of $\partial_i$ belonging to (A), then $\gamma = 0$ and the identity operator is the only monomial belonging to $S_{(c)}$. If however $\Pi_m$ is composed of some $\partial_i \in (B)$, then $\gamma \neq 0$ and the only monomial belonging to $S_{(c, \gamma)}$ is the operator $\bar{D}^\nu (\bar{D}^\nu \equiv 1$ if $\gamma = 0$).
Lemma 5.3. Let $S$ be the module generated by $\Pi_m = \partial_1 \cdots \partial_m$. Suppose $[\partial_i, \partial_j] = a \partial_i + b \partial_j + N$ for some $a, b, N \in \mathbb{L}_x$ if $\partial_i, \partial_j \in (A)$; then

$$\sum_{j=1}^{n-1} \left[ \frac{k}{1 + kT^2} \right]^{n-j} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)} u \|^2 \leq c \left[ \frac{k}{1 + kT^2} \right]^{n-a} \sum_{s_{(\alpha)} \in S_{(\alpha)}} \| s_{(\alpha)} u \|^2$$

for $\alpha < m$.

Proof. Suppose $s_{(\nu+1)}$ is some monomial in $S_{(\nu+1)}$, then by the way $S_{(\nu)}$ is constructed we have

$$D_s_{(\nu+1)} = \partial_i \partial_j \partial^{\nu} + \text{some lower terms in } S_{(\nu)},$$

if $\partial_j \in (B)$ and $\partial_i \in (A)$ or (B) by Lemma 4.1. This works, of course, only if $s_{(\nu+1)}$ contains some $\partial_j \in (B)$. If not, then $s_{(\nu+1)} = \partial_i \partial^{\nu} + \text{lower-order terms in } S_{(\nu)}$ with $\partial_i \in (A)$. In either case, by Lemma 5.1 and 5.2, we have

$$\left[ \frac{k}{1 + kT^2} \right]^{\nu} \| D^{\nu} u \|^2 \leq c \| S_{(\nu+1)} u \| + \text{some lower terms in } S_{(\nu)} \|^{2},$$

which implies that

$$\left[ \frac{k}{1 + kT^2} \right]^{\nu} \| D^{\nu} u \|^2 \leq c \sum_{s_{(\nu+1)} \in S_{(\nu+1)}} \| S_{(\nu+1)} u \|^{2} + C \| D^{\nu} u \|^2.$$ 

More generally, if $s_{(\beta)} \in S_{(\beta)}$ denote any monomial, then as above we know that there exists some $s_{(\beta+1)} \in S_{(\beta+1)}$ such that

$$\left[ \frac{k}{1 + kT^2} \right]^{\beta} \| s_{(\beta)} u \|^2 \leq c \| s_{(\beta+1)} u \|^{2} + \text{some lower-order terms in } S_{(\beta)} \|^{2},$$

which implies that

$$\left[ \frac{k}{1 + kT^2} \right]^{\beta} \| s_{(\beta)} u \|^2 \leq c \sum_{s_{(\beta+1)} \in S_{(\beta+1)}} \| s_{(\beta+1)} u \|^2 + c \sum_{s_{(\beta)} \in S_{(\beta)}} \| s_{(\beta)} u \|^2.$$ 

This means that

$$\left[ \frac{k}{1 + kT^2} \right]^{\nu-\beta} \| D^{\nu} u \|^2 \leq c \left[ \frac{k}{1 + kT^2} \right]^{\nu-\beta} \sum_{s_{(\nu+1)} \in S_{(\nu+1)}} \| s_{(\nu+1)} u \|^2$$

$$+ c \left[ \frac{k}{1 + kT^2} \right]^{\nu-\beta} \| D^{\nu} u \|^2$$

$$\leq c \left[ \frac{k}{1 + kT^2} \right]^{\nu-\beta} \sum_{s_{(\nu+2)} \in S_{(\nu+2)}} \| s_{(\nu+2)} u \|^2$$

$$+ c \left[ \frac{k}{1 + kT^2} \right]^{\nu-\beta} \sum_{s_{(\nu+1)} \in S_{(\nu+1)}} \| s_{(\nu+1)} u \|^2$$

$$\leq \cdots$$
Hence,
\[
\sum_{j=y}^{a-1} \left[ \frac{k}{1 + kT^2} \right]^{m-j} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2 \leq c \left[ \frac{k}{1 + kT^2} \right]^{m-j} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2 + c \sum_{j=y}^{a-1} \left[ \frac{k}{1 + kT^2} \right]^{m-j-1} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2.
\]

Now since
\[
\left[ \frac{k}{1 + kT^2} \right]^{m-j} > c \left[ \frac{k}{1 + kT^2} \right]^{m-j+1}
\]
for some constant c if \( k^{-1} \) and \( T \) are small enough, we can absorb
\[
c \sum_{j=y}^{a-1} \left[ \frac{k}{1 + kT^2} \right]^{m-j+1} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2
\]
into the left-hand side and get
\[
\sum_{j=y}^{a-1} \left[ \frac{k}{1 + kT^2} \right]^{m-j} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2 \leq c \left[ \frac{k}{1 + kT^2} \right]^{m-a} \sum_{s_{(a)} \in S_{(a)}} \| s_{(a)}u \|^2.
\]

**Corollary.** Since \( s_{(m)} = \Pi_m \) we have
\[
\sum_{j=y}^{m-1} \left[ \frac{k}{1 + kT^2} \right]^{m-j} \sum_{s_{(j)} \in S_{(j)}} \| s_{(j)}u \|^2 \leq c \| \Pi_m u \|^2.
\]

**Proof of Theorem 2.** First suppose \( r = s \). We show that
\[
\sum_{|\alpha| \leq m-r} k^{m-|\alpha|} \| D^\alpha u \|^2 \leq c \| Pu \|^2.
\]
Let $|\alpha| = m - j - r$, $0 \leq j \leq m - r$. It suffices to show that $k^{m-|\alpha|} \|D^\alpha u\|^2 \leq c \|P u\|^2$ for each $\alpha$.

$$P = P^j_m + P^j_{m-1} + \cdots$$

$$= \Pi^j_m + \Pi^j_{m-1} + \cdots,$$

where $P^j_m \in S_{(m-j)}$ and $\Pi^j_m = \partial_1 \cdots \partial_m$, by Condition (V). By Lemma 5.3 we can handle any $P^j_m \in S_{(m-j)}$ and any lower-order terms resulting from any permutation of the $\partial_i$; since by Lemma 4.1, these terms belong to $S$. Hence without loss of generality we may assume that $[\partial_i, \partial_j] = 0$ for all $\partial_i \in (A)$ or $(B)$.

It also suffices to show that

$$k^{m-|\alpha|} \|D^\alpha u\|^2 \leq c \|\partial_1 \cdots \partial_m u\|^2$$

for

$$0 \leq |\alpha| = m - j - r, \quad 0 \leq j \leq m - r,$$

i.e.,

$$k^{j+r} \|D^{m-(j+r)} u\|^2 \leq c \|\partial_1 \cdots \partial_m u\|^2, \quad 0 \leq j \leq m - r, \quad (6.1)$$

where the maximum multiplicity of the characteristic roots is $r$.

The proof is by induction on $m$. By repeated use of Lemma 5.1 we have

$$k^{j+r} \|u\|^2 \leq c \|\partial_1 \cdots \partial_{j+r} u\|^2.$$

Hence (6.1) holds for $m = j + r$. Suppose (6.1) is true for $m > j + r$; we prove it also holds for $m + 1$; i.e., suppose

$$k^{j+r} \|D^{m-(j+r)} u\|^2 \leq c \|\partial_1 \cdots \partial_m u\|^2, \quad (6.2)$$

then

$$k^{j+r} \|D^{m+1-(j+r)} u\|^2 \leq c \|\partial_1 \cdots \partial_{m+1} u\|^2. \quad (6.3)$$

(6.2) implies that

$$k^{j+r} \|D^{m-(j+r)} u\|^2 \leq c \left\| \prod_{i=1}^{m} \partial'_i u \right\|^2$$

for any product of $m$ operators $\partial'_i$ such that the maximum multiplicity of the characteristic roots of the operator $\prod \partial'_i$ is $r$. Thus

$$k^{j+r} \|D^{m-(j+r)} (\partial_l u)\|^2 \leq c \left\| \left( \prod_{i=1}^{m-1} \partial_i \right) (\partial_l u) \right\|^2 \quad 1 \leq l \leq m + 1.$$

Since $D^{m-(j+r)} \partial_l = \partial_l D^{m-(j+r)} + c \partial'_l D^{m-(j+r)}$ where $c \partial'_l D^{m-(j+r)}$ is some operator of order $m - (j + r)$, we can show that

$$k^{j+r} \|\partial_l D^{m-(j+r)} u\|^2 \leq c \left\| \left( \prod_{i=1}^{m-1} \partial_i \right) (\partial_l u) \right\|^2 \quad (6.4)$$

for $1 \leq l \leq m + 1$. 
Since we assumed that we could commute the $\partial_i$ we have

$$k^{i+r} \left\| \partial_i D^{m-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2 \quad \text{for} \quad 1 \leq l \leq m \mid 1.$$  

Hence

$$k^{i+1} \sum_{l=1}^{m+1} \left\| c_l \partial_i D^{m-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2$$  

for any $c_l \in L_{Z^0}$.

By Lemma 4.2, since $m + 1 > r$, we can find $c_i, N \in L_{Z^0}$ such that

$$\sum_{l=1}^{m+1} c_l \partial_i = \tilde{D} + N$$

for any operator of order 1. We then have

$$-c_i k^{i+r} \left\| D^{m-(j+r)} u \right\|^2 + k^{i+r} \left\| D(D^{m-(j+r)} u) \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2.$$

By (6.4)

$$k^{i+r+1} \left\| D^{m-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2.$$

Hence

$$(k^{i+r+1} - c_i k^{i+r}) \left\| D^{m-(j+r)} u \right\|^2 + k^{i+r} \left\| D^{m+1-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2.$$

For large enough $k$, $k^{i+r+1} - c_i k^{i+r} > 0$, and so

$$k^{i+r} \left\| D^{m+1-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_{m+1} u \right\|^2$$

and so (6.1) is proved for all $m$ if $r = s$.

Now suppose $s > r$. Exactly the same argument shows that

$$\sum_{|\alpha| \leq m-s} k^{m-|\alpha|} \left\| D^\alpha u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_m u \right\|^2.$$

If $m - s \leq |\alpha| \leq m - r$, we show that

$$(1 + kT^2)^{m-|\alpha|-s} k^{m-|\alpha|} \left\| D^\alpha u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_m u \right\|^2.$$

Let $|\alpha| = m - j - r$, $0 \leq j \leq s - r$. It suffices to show that

$$(1 + kT^2)^{(j+r)-s} k^{j+r} \left\| D^{m-(j+r)} u \right\|^2 \leq c \left\| \partial_1 \ldots \partial_m u \right\|^2, \quad (6.5)$$

where $0 \leq j \leq s - r$.

The proof is by induction on $m$. First, we prove (6.5) for $m = s$.

Since $s > r$, then if $\partial_1 \ldots \partial_{r+1} \ldots \partial_s$ satisfies the conditions of the theorem, then $r$ of the $\partial_i$ belong to (A) and the rest belong to (B). Since we can commute the $\partial_i$ we may assume that $\partial_1, \ldots, \partial_r \in (A)$ and $\partial_{r+1}, \ldots, \partial_s \in (B)$.  

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We have, as before,

$$k^{j+r} \| u \|^2 \leq c \| \partial_1 \cdots \partial_{j+r} u \|^2$$  \hspace{1cm} (6.6)$$

by Lemma 5.1. Since $\partial_{j+r+1}, \ldots, \partial_s \in (B)$, then

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{s-(j+r)} u \|^2 \leq c \| \partial_{j+r+1} \cdots \partial_s u \|^2$$  \hspace{1cm} (6.7)$$

by repeated use of Lemma 5.2. Combining (6.6) and (6.7) we get

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{s-(j+r)} u \|^2 \leq c k^{j+r} \| \partial_{j+r+1} \cdots \partial_s u \|^2$$

Thus (6.5) holds for $m = s$. Suppose (6.5) is true for $m > s$. We show that it holds for $m + 1$, i.e., we prove that

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{m-(j+r)} u \|^2 \leq c \| \partial_1 \cdots \partial_{m+1} u \|^2$$

implies that

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{m-(j+r)} u \|^2 \leq c \| \partial_1 \cdots \partial_{m+1} u \|^2.$$

As in the case $r = s$, we can show that

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{m-(j+r)} (\partial_1 u) \|^2 \leq c \left\| \prod_{i=1}^{m+1} \partial_i \right\| \left( \partial_1 u \right) \|^2.$$

We can also show as in the case $r = s$ that

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| \partial_l D^{m-(j+r)} u \|^2 \leq c \| \partial_1 \cdots \partial_{m+1} u \|^2$$

for $1 \leq l \leq m + 1$; this implies that

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| \sum_{l=1}^{m+1} c_l \partial_l (D^{m-(j+r)} u) \|^2 \leq c \| \partial_1 \cdots \partial_{m+1} u \|^2 \quad \text{for any} \quad c_l \in L^0.$$  \hspace{1cm} (6.10)$$

Applying Lemma 4.2, as before, we get

$$\left[1/(1 + kT^2)^{s-(j+r)}\right] \| D^{m+1-(j+r)} u \|^2 \leq c \| \partial_1 \cdots \partial_{m+1} u \|^2,$$

and since $\left[1/(1 + kT^2)^{s-(j+r)}\right] = (1 + kT^2)^{(j+r)-s}$ we have proved (6.5) for all $m \geq s$.

What remains to complete the proof of the theorem is to show that it is sufficient to require that

$$P_{m-j}' \in S(m-j) \quad \text{for} \quad j \text{ only up to } q - 1.$$
This is done if we show that the coefficient of \( \| D^\alpha u \| \) can grow as large as we want as \( k \to \infty \) for \( \alpha \leq m - q \). Since the coefficients of \( \| D^\alpha u \| \) get larger as \( \alpha \) gets smaller, it suffices to show this for \( \alpha = m - q \).

If \( r = s \), then \( q = r \) and the coefficient of \( \| D^{m-q} u \| \) is \( k^q \). Now suppose \( s > r \). First suppose \( [(s + 1)/2] > r \); then, \( m - s \leq m - [(s + 1)/2] < m - r \) and the coefficients of \( \| D^\alpha u \| \) for \( \alpha = m - q \) is \( (1 + kT^2)^{m-q} k^{m-\alpha} = (1 + kT^2)^{m-r} k^q \). Now

\[
(1 + kT^2)^{q-s} k^q = [k/(1 + kT^2)](1 + kT^2)^{q-s-1} k^q-1
\]

\[
\geq [k/(1 + kT^2)] k^{q-s-1} k^{q-1}
\]

\[
\geq [k/(1 + kT^2)] k^{q-s}
\]

if \( q - s + 1 \leq 0 \); i.e., if \( q \leq s - 1 \). But this is always true for \( s > 1 \) if \( [(s + 1)/2] > r \). Since \( k^{q-s} \to \infty \) as \( k \to \infty \) for \( q = s/2 \), then \( k^q \to \infty \) as \( k \to \infty \) for \( q = [(s + 1)/2] \) if \( s > 1 \). If \( s = 1 \) then the coefficient of \( \| D^\alpha u \| \) is \( (1 + kT^2)^{q-1} k^q \) and for \( q = 1 \), \( k^q \to \infty \) as \( k \to \infty \). What if \( s > r \) but \( r > [(s + 1)/2] \)? Then

\[
|\alpha| \leq m - r = m - [(s + 1)/2].
\]

Since we have

\[
\sum_{m-s\leq|\alpha|\leq m-r} (1 + kT^2)^{m-|\alpha|-s} k^{m-|\alpha|} \| D^\alpha u \|^2 \leq c \| Pu \|^2
\]

\( |\alpha| \) can go only as high as \( m - r \). The coefficient in the case \( |\alpha| = m - r \) is

\[
(1 + kT^2)^{m-(m-r)-s} k^{m-(m-r)} = (1 + kT^2)^{r-s} k^r \geq (1 + kT^2)^{2r-s}.
\]

Since \( 2r - s \geq 0 \) if \( r > [(s + 1)/2] \) we have

\[
(1 + kT^2)^{2r-s} \to \infty \text{ as } k \to \infty.
\]

ACKNOWLEDGMENT

I wish to express my gratitude to Professor Louis Nirenberg for his constant encouragement and assistance.

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