Proof of the oval conjecture for planar partition functions

Nils Rosehr

Mathematisches Institut, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany

Received 22 March 2005; accepted 26 July 2005
Available online 7 November 2005

Abstract

We prove that the translation plane and the shift plane defined by a planar partition function form an oval pair of projective planes, in the sense that the planes share a line pencil and any line of either plane not in this pencil forms an oval in the other plane. This is achieved by building upon substantial work of Betten–Löwen and by using Rabier’s fibration theorem, which allows one to conclude — without the assumption of properness — that certain local diffeomorphisms are covering maps.

A function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called planar if the map \( f_d : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto f(x + d) - f(x) \) is bijective for all \( d \in \mathbb{R}^n \setminus \{0\} \). It is called a partial spread function if it is continuously differentiable, the derivative \( Df : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n) \) is injective and the set

\[
S_f := \{(\xi, Df(x) \cdot \xi) : \xi \in \mathbb{R}^n \} : x \in \mathbb{R}^n \}
\]

of \( n \)-dimensional subspaces of \( \mathbb{R}^n \times \mathbb{R}^n \) is a partial spread, i.e. any two subspaces intersect in the trivial subspace; if \( S_f \cup \{0\} \times \mathbb{R}^n \) is a spread, i.e. in addition all of \( \mathbb{R}^n \times \mathbb{R}^n \) is covered, then \( f \) is called a partition function.

The relevance of these definitions lies in the fact that planar functions and partition functions define projective planes, namely shift planes and translation planes, respectively. It had been noticed for a long time that the two projective planes defined by any of the known planar partition functions form an oval pair, which means that they share a line pencil through a point \( p \) and any line not in this pencil of either plane together with \( p \) forms an oval in the other plane; see \([1, 3.2]\) for a precise definition. Recently, Dieter Betten and Rainer Löwen proved this conjecture under

E-mail address: rosehr@mathematik.uni-wuerzburg.de.
the additional assumption that the planar partition function is a proper map. Unfortunately, there is a gap in the proof concerning the properness assumption, and a somewhat technical a priori stronger assumption is needed in order to make the ingenious main part of the proof go through. The purpose of this note is to prove the conjecture without any additional assumption. For more details and some other as-yet unsolved problems in this context, see [1] and [2, 74.17].

In the proof of our fundamental lemma we use a special case of Rabier’s fibration theorem, which we discuss now; see [4, Section 2]. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and set

$$
\nu(F) := \inf\{|y^* \circ F| : \text{ for functionals } y^* : \mathbb{R}^m \to \mathbb{R} \text{ with } |y^*| = 1\}.
$$

Note that $\nu(F) > 0$ if and only if $F$ is surjective; so $\nu(F)$ can be seen as a measure for the “degree of surjectivity” of $F$. Furthermore, if $F$ is invertible, as is the case in our application, we have the easier form $\nu(F) = 1/|F^{-1}| = \min |F(S_{n-1})|$; see [4, Lemma 2.1]. A continuously differentiable map $f : U \to V$ for open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ is called a strong submersion if there is no sequence $(x_k)$ in $U$ such that $f(x_k)$ converges in $V$ and $\nu(df(x_k))$ converges to 0. Note that any strong submersion is a submersion, which can be seen by substituting constant sequences.

The exponential map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a strong submersion, because $\nu(D\exp(z)) = \exp(\Re(z))$. So, although $\exp$ is not a proper map, the following theorem can be applied to establish that $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map. For a proof, see [4, Corollary 4.2 and Remark 4.2].

**Fact** (Rabier [4]). Let $U$ be an open non-empty subset, and let $V$ be an open connected subset of $\mathbb{R}^n$. Furthermore, let $f : U \to V$ be a strong submersion; assume that $f$ is a local diffeomorphism and that there is no sequence $(x_k)$ in $U$ which converges to a point in $\partial U$ and for which $f(x_k)$ converges in $V$. Then $f : U \to V$ is a covering map.

Part (a) of the following lemma is the main result of this note. We will call maps $f : X \to X$ and $g : Y \to Y$ between topological spaces topologically equivalent if there are homeomorphisms $\varphi, \psi : X \to Y$ such that $\varphi \circ f = g \circ \psi$.

**Lemma.** (a) For every partial spread function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and every $x_0 \in \mathbb{R}^2$, the map $f - Df(x_0)$ is topologically equivalent to the map $\mathbb{C} \to \mathbb{C}, z \mapsto z^k$ for some $k \in \mathbb{N}$, and in particular the map is proper.

(b) For every planar continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and every $x_0 \in \mathbb{R}^2$, the map $f - Df(x_0)$ is topologically equivalent to the map $\mathbb{C} \to \mathbb{C}, z \mapsto z^2$ of complex squaring, and in particular the map is proper.

**Proof.** (a) By passing from $f$ to $x \mapsto f(x + x_0) - f(x_0) - Df(x_0)(x)$ we may assume that $f(0) = 0$ and $Df(0) = 0$. We will apply the Fact to

$$
f^* : U \to V, x \mapsto f(x) \quad \text{ for } U := \mathbb{R}^2 \setminus f^{-1}(0) \quad \text{ and } \quad V := \mathbb{R}^2 \setminus \{0\}.
$$

As the graph of $Df(x)$ is complementary to the graph of $Df(0) = 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$, we have that $Df(x)$ is invertible. Thus the restriction $f^*$ is a local diffeomorphism. Let $(x_k)$ be a sequence in $U$ such that $\nu(Df(x_k))$ converges to 0. Because $f$ is a partial spread function, we have that, for any open neighbourhood $W$ of 0 in $\mathbb{R}^2$, the map

$$
W \times (\mathbb{R}^2 \setminus \{0\}) \to (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \{(0, 0)\}, \quad (x, \xi) \mapsto (\xi, Df(x)(\xi))
$$

is a continuous injection; so by domain invariance its image is an open neighbourhood of $((\mathbb{R}^2 \times \{0\}) \setminus \{(0, 0)\}$. We want to show that $(x_k)$ converges to 0; so we may assume that $(x_k)$
converges in the one-point compactification of $\mathbb{R}^2$ and need to show that this limit is 0. Now $v(Df(x_k)) \to 0$ means that there is a sequence $(\xi_k)$ in $\mathbb{R}^2$ with $|\xi_k| = 1$ for all $k$ such that $(\xi_k, Df(x_k)(\xi_k))$ converges — possibly after passing to subsequences — to some $(\xi, 0) \neq (0, 0)$. As $W$ is arbitrary, this implies that $(x_k)$ converges to $0 \in \mathbb{R}^2$ and consequently $f(x_k)$ converges to $0 \notin V$. Thus $f^*$ is a strong submersion and the Fact establishes that $f^* : U \to V$ is a covering map.

Next we determine $U$. As the covering map induces an embedding of the fundamental group of $U$ into that of $V = \mathbb{R}^2 \setminus \{0\}$, which is infinite cyclic, the group $\pi_1(U)$ is abelian. Since the restriction of $f$ to $\mathbb{R}^2 \setminus \{0\}$ is a local diffeomorphism, $f^{-1}(0) \setminus \{0\}$ is discrete and therefore empty: indeed, if $f^{-1}(0)$ contains at least two elements, then (because $f^{-1}(0)$ is countable and there are uncountably many directions) there are two open stripes and two open half-planes (bounded by lines which are all parallel) such that the union of these four sets is $\mathbb{R}^2$ and such that each stripe contains precisely one element from $f^{-1}(0)$ and no other set contains this element; then the intersection of neighbouring sets and $U$ is simply connected, and a three-fold application of the theorem of Seiffert–Van Kampen implies that $\pi_1(U)$ contains the free group with two generators and is therefore not abelian. Thus we have $U = \mathbb{R} \setminus \{0\}$, and $f^*$ is a $k$-fold covering map for $k \in \mathbb{N}$, because all infinite subgroups of the infinite cyclic group have finite index; see [3, 2.3.9]. By the classification of covering spaces (see, for example, [3, 2.5.3]) the map $f^*$ is topologically equivalent via homeomorphisms $\varphi, \psi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ to the map $g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto z^k$. The homeomorphisms $\varphi, \psi$ extend to the sphere $\mathbb{R}^2 \cup \{\infty\}$ (as this is the Freudenthal compactification). Thus both extension maps fix 0 and $\infty$ or both exchange these two points, since we have $f(0) = 0$. So we may assume that both maps fix 0, as the inversion map $z \mapsto z^{-1}$ commutes with $g$. This shows that $f$ is topologically equivalent to $\mathbb{C} \to \mathbb{C}, z \mapsto z^k$. In particular, $f$ is a proper map as a composition of proper maps.

(b) Every planar continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a partial spread function, because the invertibility of $Df(x + d) - Df(x) = Df_d(x)$ for every $d \in \mathbb{R}^2 \setminus \{0\}$ implies that $Df$ defines a partial spread. Now we proceed as in Steps (5) to (11) of the proof of [1, 3.4] in order to obtain that $k$ as in (a) equals 2; note, however, that in Step (11) it is only necessary that $f$ is a partial spread function. $\Box$

Step (2) in the proof of [1, 3.4] is not applicable, because at this stage of the proof it is not known that the map $f$ transformed according to Step (1) is proper. By replacing Step (2) with the first paragraph of the proof of the Lemma — or simply by using part (a) of the Lemma — we obtain the following generalization.

**Theorem.** The translation plane and the shift plane defined by a planar partition function form an oval pair of projective planes.

**References**


