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Products in categories of relations

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Abstract

The relational product construction is often consider as an abstract version of cartesian products. The existence of those products is strongly connected with the representability of that category. In this paper we investigate a canonical weakening of the notion of a relational product. Unlike the strong version, any (small) category of relations can be embedded into a suitable category providing all weak relational products. Furthermore, we provide several examples, and we study the categorical properties of the new construction.

1. Introduction

The cartesian product is an important construction on sets. In particular, cartesian products are fundamental in the study of datatypes, programming languages and most kinds of logics. In the abstract theory of categories one usually considers categorical products as an abstract version of cartesian products. Allegories, as an abstract theory of relations, are involutive so that products and co-products coincide. This structure is a biproduct and corresponds to the disjoint union of sets and not to the set of all pairs. Therefore, one considers either categorical products in the subcategory of mappings or relational products. If an allegory has relational products it is representable, i.e. there is an embedding into a power of the category **Rel** of sets and relations. However, not every allegory is representable and yet it is still desirable to have some notion of products. On the other hand, by embedding the given allegory into a matrix algebra other operations that are usually required, such as sums and powers, i.e. the counterparts of disjoint unions and powersets, can be created.

This paper is an extended version and a continuation of [11]. We are going to define a canonical weakening of the concept of a relational product, the weak relational product. This will be done within the theory of allegories – a categorical model of relations. We will investigate certain properties of the new construction and compare them to those of relational products. In particular, we are interested in the following properties and results, which are not necessarily independent.

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Property	Relational Product	Weak Relational Product
Product in MAP($\mathcal{R}$)	+	+
$\mathcal{R}$ representable	+	-
Unsharpness property	-	+
Embedding property	-	+
Equational theory	+	-/+*

$*$ can be defined by equations in division allegories where all partial identities split

Fig. 1. Properties of relational products.

The given construction may establish a categorical product in the subcategory of mappings. If valid, this property ensures that the corresponding concept is suitable as an abstract version of cartesian products of sets. Therefore, it is essential for any notion of products.

It is desirable that the existence of all products does not necessarily imply that the allegory is representable.

The unsharpness property is a violation of an equality in terms of relational products. It was claimed [2] that this property may be important to model certain behavior of concurrent processes.

It might be possible to embed a given allegory into another allegory providing all products of a certain kind. With this property we refer to whether this can always be done.

Since the theory of allegories and several of its extensions are/can be defined as an equational theory it is interesting whether a given concept of products can also be expressed by equations.

In Fig. 1 we have summarized the validity of the properties above within the concepts of relational and weak relational products. In addition to those properties we are going to prove several (weak) versions of propositions well-known for relational products. Last but not least, we will present several examples either visualizing the concepts or verifying that certain properties are not true.

2. Relational preliminaries

Throughout this paper, we use the following notation. To indicate that a morphism R of a category $\mathcal{R}$ has source A and target B we write $R : A \to B$. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$ and the composition of a morphism $R : A \to B$ followed by a morphism $S : B \to C$ by $R; S$. Last but not least, the identity morphism on A is denoted by $\mathbb{I}_A$.

We recall briefly some fundamentals on allegories [4] and relational constructions within them. For further details we refer to [4,7–10]. Furthermore, we assume that the reader is familiar with the basic notions from category theory such as products and co-products. For unexplained material we refer to [1].

Definition 1. An allegory $\mathcal{R}$ is a category satisfying the following:

(1) For all objects A and B the collection $\mathcal{R}[A, B]$ is a meet semi-lattice, whose elements are called relations. Meet and the induced ordering are denoted by $\sqcap$ and $\sqsubseteq$, respectively.
(2) There is a monotone operation $\smile$ (called converse) such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q; R)^\smile = R^\smile; Q^\smile$ and $(Q^\smile)^\smile = Q$.
(3) For all relations $Q : A \to B$, $R, S : B \to C$ we have $Q; (R \sqcap S) \sqsubseteq Q; R \sqcap Q; S$.
(4) For all relations $Q : A \to B$, $R : B \to C$ and $S : A \to C$ the modular law $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$ holds.
An allegory is called a distributive allegory if
(5) The collection $\mathcal{R}[A, B]$ is a distributive lattice with a least element. Join and the least element are denoted by $\sqcup$ and $\perp\!\!\!\perp_{AB}$, respectively.
(6) For all relations $Q : A \to B$ and objects C we have $Q; \perp\!\!\!\perp_{BC} = \perp\!\!\!\perp_{AC}$.
(7) For all relations $Q : A \to B$, $R, S : B \to C$ we have $Q; (R \sqcup S) = Q; R \sqcup Q; S$.
A distributive allegory is called locally complete iff each $\mathcal{R}[A, B]$ is a complete lattice. Finally, a division allegory is a distributive allegory with a binary operation/satisfying the following:

(8) For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $Q : A \to B$ the following holds: $Q; R \sqsubseteq S \iff Q \sqsubseteq S/R$.

If $\mathcal{R}[A, B]$ has a greatest element it is denoted by $\top\!\!\top_{AB}$.

Notice that allegories and distributive allegories are defined by equations. The same can be done for division allegories [4].

The left residual can be used to define another residual operation $Q\backslash S := (S^{\smile}/Q^{\smile})^{\smile}$, called the right residual of S and Q. A symmetric version, called the symmetric quotient, of the residuals may be defined by

$$\mathrm{syq}(Q, R) := (Q\backslash R) \sqcap (Q^{\smile}/R^{\smile}).$$

For further properties of relations in allegories we refer to [4,9,10].

An important class of relations is given by mappings.

Definition 2. Let $Q : A \to B$ be a relation. Then we call

(1) Q univalent (or functional) iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,
(2) Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$,
(3) Q a mapping (or a map) iff Q is univalent and total,
(4) Q injective iff $Q^{\smile}$ is univalent,
(5) Q surjective iff $Q^{\smile}$ is total.

In the next lemma we have summarized some properties of the residuals and the symmetric quotient. Proofs can be found in [4,5,9,10].

Lemma 1. *Let $\mathcal{R}$ be a division allegory and $Q : A \to B$, $R : A \to C$, $S : A \to D$ be relations, and $f : D \to A$ be a mapping. Then we have*

(1) $Q; (Q\backslash R) \sqsubseteq R$,
(2) $f; \mathrm{syq}(Q, R) = \mathrm{syq}(Q; f^{\smile}, R)$,
(3) $\mathrm{syq}(Q, R)^{\smile} = \mathrm{syq}(R, Q)$,
(4) $\mathrm{syq}(Q, R); \mathrm{syq}(R, S) \sqsubseteq \mathrm{syq}(Q, S)$,
(5) *if* $\mathrm{syq}(Q, R)$ *is total then equality holds in (4),*
(6) *if* $\mathrm{syq}(Q, R)$ *is surjective then* $Q; \mathrm{syq}(Q, R) = R$.

In the next lemma we have collected several properties of univalent relations used in this paper. A proof can be found in [9,10].

Lemma 2. *Let $\mathcal{R}$ be an allegory so that $\top\!\!\top_{AA}$ exists, $Q : A \to B$ be univalent, $P : A \to B$, $R, S : B \to C$, $T : C \to B$ and $U : C \to A$. Then we have*

(1) $Q; (R \sqcap S) = Q; R \sqcap Q; S$,
(2) $(T; Q^{\smile} \sqcap U); Q = T \sqcap U; Q$,
(3) $\top\!\!\top_{AA}(Q^{\smile} \sqcap P) \sqcap Q^{\smile} = Q^{\smile} \sqcap P$.

The collection of all mappings of a division allegory $\mathcal{R}$ constitutes a subcategory and is denoted by MAP($\mathcal{R}$).

The subcategory of mappings may provide categorical products for certain pairs of objects. As mentioned in the introduction any useful concept of products should establish a categorical product in MAP($\mathcal{R}$). Notice that $\mathcal{R}$ itself may have categorical products. But, contrary to the products in MAP($\mathcal{R}$), those products are not suitable to provide an abstract description of ordered pairs. Any allegory is involutive (i.e. isomorphic to its opposite category) by the converse operation $^{\smile}$, which implies that products and co-products coincide. It can be shown that they establish biproducts, and that they are related to the relational sums defined below, which constitutes the abstract counterpart of a disjoint union [4,12].

Definition 3. Let $\{A_i \mid i \in I\}$ be a set of objects of a locally complete distributive allegory indexed by some set I. An object $\sum_{i\in I} A_i$, together with relations $\iota_j \in \mathcal{R}[A_j, \sum_{i\in I} A_i]$ for all $j \in I$, is called a relational sum of $\{A_i \mid i \in I\}$ iff for all $i, j \in I$ with $i \neq j$ the following holds

$$\iota_i;\iota_i^{\smile} = \mathbb{I}_{A_i}, \qquad \iota_i;\iota_j^{\smile} = \perp\!\!\!\perp_{A_iA_j}, \qquad \bigsqcup_{i\in I}(\iota_i^{\smile};\iota_i) = \mathbb{I}_{\sum_{i\in I} A_i}$$

$\mathcal{R}$ has (binary) relational sums iff for every (pair) set of objects the relational sum does exist.

The relational sum is a categorical product and co-product, and hence, unique up to isomorphism. In **Rel** the relational sum is given by the disjoint union of sets and the corresponding injection functions.

Definition 4. Let $Q : A \to A$ be a symmetric idempotent relation, i.e., $Q^{\smile} = Q$ and $Q; Q = Q$. An object B together with a relation $R : B \to A$ is called a splitting of Q (or R splits Q) iff $R; R^{\smile} = \mathbb{I}_B$ and $R^{\smile}; R = Q$.

In **Rel** the splitting of Q is given by the set of equivalence classes (note that it is not assumed that Q is reflexive, so the union of the equivalence classes is in general just a subset of A), and R relates each equivalence class to its elements. A splitting is unique up to isomorphism.

The last construction we want to introduce is the abstract counterpart of a power set – the relational power.

Definition 5. Let $\mathcal{R}$ be a division allegory. An object $\mathcal{P}(A)$, together with a relation $\varepsilon : A \to \mathcal{P}(A)$ is called a relational power of A iff

$$\text{syq}(\varepsilon,\varepsilon) \sqsubseteq \mathbb{I}_{\mathcal{P}(A)} \quad \text{and} \quad \text{syq}(R,\varepsilon) \text{ is total}$$

for all relation $R : B \to A$. If the relational power does exist for any object then $\mathcal{R}$ is called a power allegory.

Notice that $\text{syq}(\varepsilon,\varepsilon) = \mathbb{I}_{\mathcal{P}(A)}$, and that $\text{syq}(R,\varepsilon)$ is, in fact, a mapping. In **Rel** the relation $e_A := \text{syq}(\mathbb{I}_A,\varepsilon) : A \to \mathcal{P}(A)$ maps each element to the singleton set containing that element. This relation is an injective mapping (cf. [10]).

Definition 6. An allegory is called systemic complete iff it is a power allegory that has relational sums, and in which all symmetric idempotent relations split.

The univalent part $\text{unp}(R)$ of a relation R was introduced in [9] in the context of (heterogeneous) relation algebras, i.e. division allegories where the order structure is a complete atomic Boolean algebra.

Definition 7. Let $\mathcal{R}$ be a division allegory, and let be $R : A \to B$ in $\mathcal{R}$. The univalent part of R is defined by $\text{unp}(R) := R \sqcap (R^{\smile}\backslash\mathbb{I}_B)$.

A proof of the following lemma can be found in [9,11]. Notice that the proof provided in [9] uses complements, which are not available in an arbitrary division allegory.

Lemma 3. *Let $\mathcal{R}$ be a division allegory and $R : A \to B$. Then we have*
(1) $\text{unp}(R)$ *is univalent and included in* R,
(2) $\text{unp}(\text{unp}(R)) = \text{unp}(R)$,
(3) R *is univalent iff* $\text{unp}(R) = R$.

3. Relational products

We are now going to define the two version of relational products in allegories (cf. [3,6,9,12]).

Definition 8 (*Weak relational product*). Let $\mathcal{R}$ be an allegory, and A and B be objects of $\mathcal{R}$. An object $A \times B$ together with two relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a weak relational product iff

(P1) $\pi^{\smile}; \pi \sqsubseteq \mathbb{I}_A$,
(P2) $\rho^{\smile}; \rho \sqsubseteq \mathbb{I}_B$,
(P3) $\pi; \pi^{\smile} \sqcap \rho; \rho^{\smile} = \mathbb{I}_{A\times B}$,
(P4) $f^{\smile}; g \sqsubseteq \pi^{\smile}; \rho$ for all mappings $f : C \to A$ *and* $g : C \to B$.

$\mathcal{R}$ is called a weak pairing allegory iff a weak relational product for each pair of objects exists.

A (strong) relational product satisfies (P1)–(P3) and requires that $\pi^{\smile}; \rho$ is the greatest element in $\mathcal{R}[A, B]$, i.e., (P4s) $\pi^{\smile}; \rho = \top\!\!\!\top_{AB}$.

A relational product is a categorical product in the subcategory of mappings but not necessarily vice versa. The property (P4s) may not be valid. This equation states that the greatest relation is tabular [4]. Obviously, (P4s) implies (P4) so that each relational product is a weak relational product.

Example. Consider the concrete allegory with one object $A = \{0, 1, 2\}$ and the four relations $\perp\!\!\!\perp_{AA} := \emptyset$, $\mathbb{I}_A$, $\overline{\mathbb{I}}_A := \{(0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1)\}$ and $\top\!\!\!\top_{AA} := A \times A$. It is easy to verify that this structure establishes indeed an allegory with exactly one map $\mathbb{I}_A$. It is well-known that the matrices with entries of a (complete) allegory form an allegory. Mappings in our example are matrices with exactly one entry $\mathbb{I}_A$ in each row and $\perp\!\!\!\perp_A$ otherwise. The pair

$$\pi := \begin{pmatrix} \mathbb{I}_A & \perp\!\!\!\perp_A \\ \mathbb{I}_A & \perp\!\!\!\perp_A \\ \perp\!\!\!\perp_A & \mathbb{I}_A \\ \perp\!\!\!\perp_A & \mathbb{I}_A \end{pmatrix} \qquad \rho := \begin{pmatrix} \mathbb{I}_A & \perp\!\!\!\perp_A \\ \perp\!\!\!\perp_A & \mathbb{I}_A \\ \mathbb{I}_A & \perp\!\!\!\perp_A \\ \perp\!\!\!\perp_A & \mathbb{I}_A \end{pmatrix}$$

establishes a weak relational product. Notice that if we replace $\mathbb{I}_A$ by $\mathbf{1}$ and $\perp\!\!\!\perp_0$ by $\mathbf{0}$, where $\{\mathbf{0},\mathbf{1}\}$ denotes the Boolean algebra with two elements $\mathbf{0}\leq\mathbf{1}$, we obtain the well-known matrix representation of the projections (cf. [9]).

If an allegory has weak relational products one may consider the structure $\mathrm{tab}(\mathcal{R})$ of all relations less than or equal to $\pi^{\smile}; \rho$. In this structure every relation is obviously contained in tabular relation (called pre-tabular in [4]). Unfortunately, $\mathrm{tab}(\mathcal{R})$ is not necessarily an allegory. Consider the allegory given by the object A from the previous example and the object $B = \{a\}$ with relations $\perp\!\!\!\perp_{BB}$, $\mathbb{I}_B = \top\!\!\!\top_{BB}$, $\perp\!\!\!\perp_{AB}$ and $\top\!\!\!\top_{AB}$. In this structure we have exactly 3 mappings $\mathbb{I}_A$, $\top\!\!\!\top_{BA}$ and $\mathbb{I}_B$. It is easy to verify that $A \times A = A \times B = A$ and $B \times B = B$. The relations $\pi^{\smile}; \rho$ are given by $\mathbb{I}_A$, $\top\!\!\!\top_{AB}$, $\top\!\!\!\top_{BA}$ and $\mathbb{I}_B$. The structure $\mathrm{tab}(\mathcal{R})$ is not closed under composition since we have $\top\!\!\!\top_{AB}; \top\!\!\!\top_{BA} = \top\!\!\!\top_{AA} \sqsupset \mathbb{I}_A$.

Theorem 1. *Let $\mathcal{R}$ be an allegory with weak relational products. If $\rho; \pi^{\smile}$ is in $\mathrm{tab}(\mathcal{R})[A \times B, B \times C]$ for all object A, B and C, then $\mathrm{tab}(\mathcal{R})$ is a sub-allegory of $\mathcal{R}$.*

Proof. It is sufficient to show that the relations in $\mathrm{tab}(\mathcal{R})$ under composition since all operations are inherited from $\mathcal{R}$. Let us denote by π_i, ρ_i the projections of the weak relational products $A \times B$, $B \times C$, $(A \times B) \times (B \times C)$ and $A \times C$, respectively. Assume that $Q : A \to B$, $: B \to C$ with $Q \sqsubseteq \pi_1^{\smile}; \rho_1$ and $R \sqsubseteq \pi_2^{\smile}; \rho_2$. By the assumption we get $\rho_1; \pi_2^{\smile} \sqsubseteq \pi_3^{\smile}; \rho_3$ so that we conclude $Q; R \sqsubseteq \pi_1^{\smile}; \rho_1; \pi_2^{\smile}; \rho_2 \sqsubseteq \pi_1^{\smile}; \pi_3^{\smile}; \rho_3; \rho_2$. Since $\pi_3; \pi_1$ and $\rho_3; \rho_1$ are mappings we obtain $\pi_1^{\smile}; \pi_3^{\smile}; \rho_3; \rho_2 \sqsubseteq \pi_4^{\smile}; \rho_4$, and, hence, $Q; R \in \mathrm{tab}(\mathcal{R})[A, C]$. □

One important property of relational products is that one can transform any relation into the abstract counterpart of a set of pairs, i.e. by a vector or a left ideal element $\top\!\!\!\top_{AA}; v = v$. We want to investigate whether this is also possible for weak relational products. Consider the two operations

$$\tau(R) := \top\!\!\!\top_{AA}; (\pi^{\smile} \sqcap R; \rho^{\smile}) \quad \text{and} \quad \sigma(v) := (\top\!\!\!\top_{AA}; v \sqcap \pi^{\smile}); \rho$$

τ maps relations to vectors and σ vectors to relations.

Lemma 4. *Let $\mathcal{R}$ be a weak pairing allegory with greatest elements, $R : A \to B$, $v : A \to A \times B$ be a vector, $Q : C \to A$ univalent, and $S : C \to A \times B$. Then we have*

(1) $Q; \pi^{\smile} \sqcap (Q; \pi^{\smile} \sqcap S); \rho; \rho^{\smile} = Q; \pi^{\smile} \sqcap S$,
(2) $\tau(\sigma(v)) = v$,
(3) $\sigma(\tau(R)) \sqsubseteq R$ *with* '=' *if* $R \sqsubseteq \pi^{\smile}; \rho$.

A proof may be found in [11].

3.1. Product in MAP($\mathcal{R}$)

In this section we first want to show that an allegory $\mathcal{R}$ has weak relational products if and only if MAP($\mathcal{R}$) has products and the relation $\pi; \pi^{\smile} \sqcap \rho; \rho^{\smile}$ splits. We start with the following theorem.

Theorem 2. *Let $\mathcal{R}$ be an allegory. Then a weak relational product $(A \times B, \pi, \rho)$ is a categorical product of A and B in* MAP($\mathcal{R}$).

Proof. Let $(A \times B, \pi, \rho)$ be a weak relational product. By P1, P2 and P3 the relations π and ρ are mappings, and hence in MAP($\mathcal{R}$). Let $f : C \to A$ and $g : C \to B$ be mappings. Then we have

$$\begin{aligned} (f; \pi^{\smile} \sqcap g; \rho^{\smile}); \rho &= f; \pi^{\smile}; \rho \sqcap g && \text{Lemma 2(2)} \\ &= g \end{aligned}$$

where the last equality follows from

$$\begin{aligned} g &\sqsubseteq f; f^{\smile}; g && f \text{ is total} \\ &\sqsubseteq f; \pi^{\smile}; \rho && \text{Axiom P4} \end{aligned}$$

The equality $(f; \pi^{\smile} \sqcap g; \rho^{\smile}); \pi = f$ is shown analogously. Furthermore, the following computation shows that $f; \pi^{\smile} \sqcap g; \rho^{\smile}$ is a mapping, and hence an element of MAP($\mathcal{R}$)

$$\begin{aligned} (f; \pi^{\smile} \sqcap g; \rho^{\smile})^{\smile}; (f; \pi^{\smile} \sqcap g; \rho^{\smile}) &= (\pi; f^{\smile} \sqcap \rho; g^{\smile}); (f; \pi^{\smile} \sqcap g; \rho^{\smile}) \\ &\sqsubseteq \pi; f^{\smile}; f; \pi^{\smile} \sqcap \rho; g^{\smile}; g; \rho^{\smile} \end{aligned}$$

$$\begin{aligned} &\sqsubseteq \pi; \pi^{\smile} \sqcap \rho; \rho^{\smile} && f \text{ and } g \text{ are univalent} \\ &= \mathbb{I}_{A \times B} && \text{Axiom P3} \end{aligned}$$

$$\begin{aligned} &(f; \pi^{\smile} \sqcap g; \rho^{\smile}); (f; \pi^{\smile} \sqcap g; \rho^{\smile})^{\smile} \\ &= (f; \pi^{\smile} \sqcap g; \rho^{\smile}); (\pi; f^{\smile} \sqcap \rho; g^{\smile}) \\ &= (f; \pi^{\smile} \sqcap g; \rho^{\smile}); \pi; f^{\smile} \sqcap (f; \pi^{\smile} \sqcap g; \rho^{\smile}); \rho; g^{\smile} && \text{Lemma 2(1)} \\ &= f; f^{\smile} \sqcap g; g^{\smile} && \text{previous computation} \\ &\sqsupseteq \mathbb{I}_C && f \text{ and } g \text{ are total} \end{aligned}$$

Last but not least, let h be a mapping with $h; \pi = f$ and $h; \rho = g$. Then we conclude

$$\begin{aligned} f; \pi^{\smile} \sqcap g; \rho^{\smile} &= h; \pi; \pi^{\smile} \sqcap h; \rho; \rho^{\smile} \\ &= h; (\pi; \pi^{\smile} \sqcap \rho; \rho^{\smile}) && \text{Lemma 2(1)} \\ &= h && \text{Axiom P3} \end{aligned}$$

This completes the proof. □

Notice, that we have also shown that for a weak relational product the product morphism induced by the mappings $f : C \to A$ and $g : C \to B$, i.e. the unique mapping $h : C \to A \times B$ satisfying $h; \pi = f$ and $h; \rho = g$, is actually given by the relation $f; \pi^{\smile} \sqcap g; \rho^{\smile}$.

The converse implication is not necessarily valid. But we are able to prove the following:

Lemma 5. *Let $\mathcal{R}$ be an allegory. Then a categorical product $(A \times B, \pi, \rho)$ of A and B in* MAP($\mathcal{R}$) *fulfils the axioms* P1, P2, P4 *and the inclusion $\sqsupseteq$ of* P3.

Proof. Suppose $(A \times B, \pi, \rho)$ is a categorical product of A and B in MAP($\mathcal{R}$). Axioms P1, P2 and the inclusion $\sqsupseteq$ of P3 are trivial since π and ρ are mappings.

Now, let $f : C \to A$ and $g : C \to B$ be mappings. Then there is a unique mapping $h : C \to A \times B$ with $h; \pi = f$ and $h; \rho = g$. We conclude

$$\begin{aligned} g &= h; \rho \\ &\sqsubseteq h; (\pi; \pi^{\smile} \sqcap \rho; \rho^{\smile}); \rho && \text{inclusion } \sqsupseteq \text{ of P3} \\ &\sqsubseteq h; \pi; \pi^{\smile}; \rho \\ &= f; \pi^{\smile}; \rho \end{aligned}$$

which implies $f^{\smile}; g \sqsubseteq f^{\smile}; f; \pi^{\smile}; \rho \sqsubseteq \pi^{\smile}; \rho$ since f is univalent. □

Example. Consider an allegory $\mathcal{R}$ with two objects $A := \{a, b\}$ and $B := \{(a, a), (a, b), (b, a), (b, b), c\}$, i.e., B is the cartesian product of A with itself plus an additional c. Let $\text{MAP}(\mathcal{R})[B, B] := \{\mathbb{I}_B, g_1, g_2, g_3\}$, $\text{MAP}(\mathcal{R})[B, A] := \{p_1, p_2\}$ and $\text{MAP}(\mathcal{R})[A, A] := \{\mathbb{I}_A\}$ with the matrix representation of g_1, g_2, g_3 and p_1, p_2 given below. The rows and columns of a matrix correspond to elements of the source and target, i.e, they correspond to elements of A or B. They are listed from left to right and top to bottom in the same order they are listed in the sets above. An entry **0** denotes that the two elements are not in relation, and a **1** denotes that the elements are in relation. For example, the **1** in the third row and fourth column of g_1 indicates that g_1 maps (b, a) to (b, b).

$$\underbrace{\begin{pmatrix} 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{pmatrix}}_{g_1} \quad \underbrace{\begin{pmatrix} 1&0&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{pmatrix}}_{g_2} \quad \underbrace{\begin{pmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}}_{g_3} \quad \underbrace{\begin{pmatrix} 1&0\\ 1&0\\ 0&1\\ 0&1\\ 1&0 \end{pmatrix}}_{p_1} \quad \underbrace{\begin{pmatrix} 1&0\\ 0&1\\ 1&0\\ 0&1\\ 1&0 \end{pmatrix}}_{p_2}$$

The allegory $\mathcal{R}$ is now defined to be the closure of the three set above using the operation converse, intersection and composition of concrete relations. This allegory has the following cardinalities:

$$|\mathcal{R}[B, B]| = 30, \quad |\mathcal{R}[B, A]| = 7, \quad |\mathcal{R}[A, A]| = 2$$

The subcategory of mappings is given by the sets of mapping defined above plus $\text{MAP}(\mathcal{R})[A, B] = \{i\}$ with

$$i = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix}$$

By simply testing its defining property the triple (B, p_1, p_2) establishes a categorical product of A with itself in the subcategory of mappings. For example, the pairing of p_2 and p_1 is g_3. B is not a weak relational product since

$$\begin{aligned} p_1; p_1^{\smile} \sqcap p_2; p_2^{\smile} &= \begin{pmatrix} 1&0\\ 1&0\\ 0&1\\ 0&1\\ 1&0 \end{pmatrix}; \begin{pmatrix} 1&1&0&0&1\\ 0&0&1&1&0 \end{pmatrix} \sqcap \begin{pmatrix} 1&0\\ 0&1\\ 1&0\\ 0&1\\ 1&0 \end{pmatrix}; \begin{pmatrix} 1&0&1&0&1\\ 0&1&0&1&0 \end{pmatrix} \\ &= \begin{pmatrix} 1&1&0&0&1\\ 1&1&0&0&1\\ 0&0&1&1&0\\ 0&0&1&1&0\\ 1&1&0&0&1 \end{pmatrix} \sqcap \begin{pmatrix} 1&0&1&0&1\\ 0&1&0&1&0\\ 1&0&1&0&1\\ 0&1&0&1&0\\ 1&0&1&0&1 \end{pmatrix} \\ &= \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&1 \end{pmatrix}. \end{aligned}$$

The previous example has shown that converse implication of Theorem 2 does not hold. However, we are able to prove it if we assume additional structure.

Lemma 6. *Let $\mathcal{R}$ be an allegory, and let $(A \times B, \pi, \rho)$ be a categorical product in* MAP($\mathcal{R}$). *Furthermore, assume that there exists an $R \in \mathcal{R}$ that splits $\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile}$. Then $(A \times B, \pi, \rho)$ is a weak relational product.*

Proof. By Lemma 5 it remains to show the inclusion $\sqsubseteq$ of P3. Let $R : C \to A \times B$ be the splitting of $\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile}$, and define $\tilde{\pi} := R;\pi$ and $\tilde{\rho} := R;\rho$. We want to show that $(C, \tilde{\pi}, \tilde{\rho})$ is a weak relational product of A and B. Once verified Lemma 2 implies that $(C, \tilde{\pi}, \tilde{\rho})$ is another categorical product of A and B in MAP($\mathcal{R}$), and hence isomorphic to $A \times B$. It is easy to verify (cf. [1]) that the isomorphism is given by the two mapping $h : C \to A \times B$ and $k : A \times B \to C$ fulfilling $h;\pi = \tilde{\pi}$, $h;\rho = \tilde{\rho}$, $k;\tilde{\pi} = \pi$ and $k;\tilde{\rho} = \rho$. Theorem 2 also shows $k = \pi;\tilde{\pi}^{\smile} \sqcap \rho;\tilde{\rho}^{\smile}$. Furthermore, in [4] it was shown the inverse of an isomorphism in an allegory is its converse so that $h^{\smile} = k$ follows. We conclude

$$\begin{aligned}
\mathbb{I}_{A\times B} &= k;h && \text{pair of isomorphisms}\\
&= (\pi;\tilde{\pi}^{\smile} \sqcap \rho;\tilde{\rho}^{\smile});h\\
&= (\pi;\pi^{\smile};h^{\smile} \sqcap \rho;\rho^{\smile};h^{\smile});h\\
&= (\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});h^{\smile};h && \text{Lemma 2(1)}\\
&= (\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});k;h\\
&= \pi;\pi^{\smile} \sqcap \rho;\rho^{\smile}
\end{aligned}$$

In order to show that $(C, \tilde{\pi}, \tilde{\rho})$ is a weak relational product we derive Axiom P1 from

$$\begin{aligned}
\tilde{\pi}^{\smile};\tilde{\pi} &= \pi^{\smile};R^{\smile};R;\pi\\
&= \pi^{\smile};(\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});\pi\\
&= \mathbb{I}_A \sqcap \pi^{\smile};\rho;\rho^{\smile};\pi && \text{Lemma 2(2)}\\
&\sqsubseteq \mathbb{I}_A,
\end{aligned}$$

and Axiom P2 analogously. We get the totality of $\tilde{\pi}$ from $\mathbb{I}_C = R;R^{\smile} \sqsubseteq R;\pi;\pi^{\smile};R^{\smile} = \tilde{\pi};\tilde{\pi}^{\smile}$ and for $\tilde{\rho}$ analogously. Together with

$$\begin{aligned}
&\tilde{\pi};\tilde{\pi}^{\smile} \sqcap \tilde{\rho};\tilde{\rho}^{\smile}\\
&= R;\pi;\pi^{\smile};R^{\smile} \sqcap R;\rho;\rho^{\smile};R^{\smile}\\
&\sqsubseteq R;(\pi;\pi^{\smile};R^{\smile} \sqcap R^{\smile};R;\rho;\rho^{\smile};R^{\smile})\\
&\sqsubseteq R;(\pi;\pi^{\smile} \sqcap R^{\smile};R;\rho;\rho^{\smile};R^{\smile};R);R^{\smile}\\
&\sqsubseteq R;(\pi;\pi^{\smile} \sqcap (\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});\rho;\rho^{\smile};(\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile}));R^{\smile}\\
&= R;(\pi;\pi^{\smile} \sqcap (\pi;\pi^{\smile};\rho \sqcap \rho);(\rho^{\smile};\pi;\pi^{\smile} \sqcap \rho^{\smile}));R^{\smile} && \text{Lemma 2(2)}\\
&\sqsubseteq R;(\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});R^{\smile}\\
&= R;R^{\smile};R;R^{\smile}\\
&= \mathbb{I}_C
\end{aligned}$$

we have shown Axiom P3. Last but not least, the computation

$$\begin{aligned}
\tilde{\pi}^{\smile};\tilde{\rho} &= \pi^{\smile};R^{\smile};R;\rho\\
&= \pi^{\smile};(\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile});\rho\\
&= \pi^{\smile};\rho \sqcap \pi^{\smile};\rho && \text{Lemma 2(2)}\\
&= \pi^{\smile};\rho
\end{aligned}$$

implies Axiom P4. □

If MAP($\mathcal{R}$) has products but lacks the required splitting, one still can split the idempotents in the extension Split($\mathcal{R}$) providing any splitting. However, this will enlarge the category of maps and will destroy the product structure.

Example. Consider the allegory of the previous example, and assume we embed this allegory into an allegory $\mathcal{R}'$ so that $R : C \to B$ splits $p_1; p_1^\smile \sqcap p_2; p_2^\smile$. Then the triple (B, p_1, p_2) is not longer a categorical product of A, which follows from the previous lemma, of course. In the standard (minimal) embedding there are not enough maps from C to B so that B is still a categorical product. We want to prove this for all possible embeddings of this concrete example.

Notice that we have $(p_1; p_1^\smile \sqcap p_2; p_2^\smile); p_1 = p_1$ and $(p_1; p_1^\smile \sqcap p_2; p_2^\smile); p_2 = p_2$ in $\mathcal{R}$. Now, we compute

$$\begin{aligned}
(R; p_1)^\smile; R; p_1 &= p_1^\smile; R^\smile; R; p_1 \\
&= p_1^\smile; (p_1; p_1^\smile \sqcap p_2; p_2^\smile); p_1 \\
&= p_1^\smile; p_1 && \text{see above} \\
&\sqsubseteq \mathbb{I}_A, \\
R; p_1; (R; p_1)^\smile &= R; p_1; p_1^\smile; R^\smile \\
&\sqsupseteq R; R^\smile \\
&= \mathbb{I}_C
\end{aligned}$$

so that $R; p_1$ (and analogously $R; p_2$) is a map. Now, assume that there is a map $h : C \to B$ with $h; p_1 = R; p_2$ and $h; p_2 = R; p_1$ (R is not univalent!). Since $R^\smile$ is a map we conclude that $R^\smile; h$ is a map, and we have $R^\smile; h; p_1 = R^\smile; R; p_2 = (p_1; p_1^\smile \sqcap p_2; p_2^\smile); p_2 = p_2$. Analogously, we get $R^\smile; h; p_2 = p_1$. The map g_3 also satisfies $g_3; p_1 = p_2$ and $g_3; p_2 = p_1$. Assuming that B is still a categorical product (in $\mathcal{R}'$), we conclude $g_3 = R^\smile; h$. Finally, we obtain

$$g_3 = R^\smile; h = R^\smile; R; R^\smile; h = (p_1; p_1^\smile \sqcap p_2; p_2^\smile); R^\smile; h = (p_1; p_1^\smile \sqcap p_2; p_2^\smile); g_3,$$

which is not true in $\mathcal{R}$.

3.2. Equational theory

In this section we want to show that for certain allegories the weak relational product can be defined by equations. This seems to be of particular interest because the allegories considered to construct weak relational products (cf. next section) are of that kind.

Lemma 7. *Let $\mathcal{R}$ be a division allegory in which all partial identities split. Then $(A \times B, \pi, \rho)$ is a weak relational product iff the Axioms* P1–P3 *and*

$$\text{(P4u)} \qquad \mathrm{unp}(R)^\smile; \mathrm{unp}(S) \sqsubseteq \pi^\smile; \rho$$

for all relations $R : C \to A$ and $S : C \to B$ hold.

Proof. The implication $\Leftarrow$ is trivial since $\mathrm{unp}(f) = f$ for all mappings by Lemma 3 (3). For the converse implication assume P1–P4 and let $R : C \to A$ and $S : C \to B$ be arbitrary relations. Now, let $i := \mathbb{I}_C \sqcap \mathrm{unp}(R); \mathrm{unp}(R)^\smile \sqcap \mathrm{unp}(S); \mathrm{unp}(S)^\smile$ and $s : D \to C$ be its splitting. Then the relation $s; \mathrm{unp}(R)$ is univalent since s and R are. Furthermore, this relation is total because $\mathbb{I}_D = s; s^\smile; s; s^\smile = s; i; s^\smile \sqsubseteq s\mathrm{unp}(R); \mathrm{unp}(R)^\smile; s^\smile$. Analogously, we get that $s; \mathrm{unp}(S)$ is a mapping. Notice, that we have $Q = (\mathbb{I}_A \sqcap Q; Q^\smile); Q$ for arbitrary relations $Q : A \to B$ and $i; j = i \sqcap j$ for partial identities $i, j : A \to A$. Proofs of those properties can be found in [4,9,10]. We conclude

$$
\begin{aligned}
&\mathrm{unp}(R)^{\smile}; \mathrm{unp}(S)\\
&\quad = \mathrm{unp}(R)^{\smile}; (\mathbb{I}_D \sqcap \mathrm{unp}(R); \mathrm{unp}(R)^{\smile}); (\mathbb{I}_D \sqcap \mathrm{unp}(S); \mathrm{unp}(S)^{\smile}); \mathrm{unp}(S)\\
&\quad = \mathrm{unp}(R)^{\smile}; (\mathbb{I}_D \sqcap \mathrm{unp}(R); \mathrm{unp}(R)^{\smile} \sqcap \mathbb{I}_D \sqcap \mathrm{unp}(S); \mathrm{unp}(S)^{\smile}); \mathrm{unp}(S)\\
&\quad = \mathrm{unp}(R)^{\smile}; i; \mathrm{unp}(S)\\
&\quad = \mathrm{unp}(R)^{\smile}; s^{\smile}; s; \mathrm{unp}(S)\\
&\quad = (s; \mathrm{unp}(R))^{\smile}; s; \mathrm{unp}(S)\\
&\quad \sqsubseteq \pi^{\smile}; \rho.
\end{aligned}
$$

This completes the proof. □

With following example we want to show that (P4) cannot be replaced by a generalized version of (P4u), i.e., by the property

(P4u) $\quad R^{\smile}; S \sqsubseteq \pi^{\smile}; \rho \quad$ for all univalent relations R, S.

Example. Again, we are going to construct the example by taking a closure of certain sets of relations. Consider the sets $U := \{u, v, w\}$ and $V := \{(u, u), (u, v), (v, u), (v, v), w\}$. Now, choose the following three relations (we use the same matrix notation as before):

$$
\underset{p_1}{\begin{pmatrix} 1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}}
\quad
\underset{p_2}{\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}}
\quad
\underset{R}{\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix}}
$$

$\mathcal{R}$ is obtained by taking the closure of those relations. This time we have the following cardinalities:

$$|\mathcal{R}[V, V]| = 79, \quad |\mathcal{R}[V, U]| = 29, \quad |\mathcal{R}[U, U]| = 13.$$

In this allegory V is a weak relational product of U with itself. Furthermore, p_2;R is univalent and we have

$$
p_1^{\smile}; (p_2; R) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix} \not\sqsubseteq \begin{pmatrix} 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} = p_1^{\smile}; p_2.
$$

3.3. Unsharpness property

We want to start with an example indicating that unsharpness is a common property for weak relational products.

Example. We want to show that unsharpness may hold for a weak relational product, i.e. there are relations Q, R, S and T with $(Q; \pi^{\smile} \sqcap R; \rho^{\smile}); (\pi; S \sqcap \rho; T) \neq Q; S \sqcap R; T$. Our example will show that even under the additional assumption of totality of the relations involved unsharpness is possible. Suppose $\mathcal{R}$ is a weak pairing allegory with a greatest element in $\mathcal{R}[A, B]$ and $\pi^{\smile}; \rho \neq \top\!\!\top_{AB}$. Then we have

$$
\begin{aligned}
(\mathbb{I}_A; \pi^{\smile} \sqcap \top\!\!\top_{AB}; \rho^{\smile}); (\pi; \top\!\!\top_{AB} \sqcap \rho; \mathbb{I}_B) &= (\pi^{\smile} \sqcap \top\!\!\top_{A(A\times B)}); (\top\!\!\top_{(A\times B)B} \sqcap \rho)\\
&= \pi^{\smile}; \rho\\
&\neq \top\!\!\top_{AB}\\
&= \mathbb{I}_A; \top\!\!\top_{AB} \sqcap \top\!\!\top_{AB}; \mathbb{I}_B.
\end{aligned}
$$

Notice that $\mathcal{R}$ is a weak pairing allegory, i.e., has weak relational products for every pair of objects. A similar situation for relational products is not possible.

The previous example indicates that unsharpness is very common for weak relational products. In fact, sharpness cannot hold for weak (but not strong) relational products.

Theorem 3. *Let $\mathcal{R}$ be an allegory with greatest elements, and $(A \times B, \pi, \rho)$ a weak relational product of A and B. If*

$$(Q; \pi^{\smile} \sqcap R; \rho^{\smile}); (\pi; S \sqcap \rho; T) = Q; S \sqcap R; T$$

for all relations $Q : C \to A$, $R : C \to B$, $S : A \to D$ and $T : B \to D$, then $A \times B$ is a relational product, i.e., $\pi^{\smile}; \rho = \top\!\!\top_{AB}$.

Proof. We have

$$\top\!\!\top_{A\times B\ C} \sqsubseteq \pi; \pi^{\smile}; \top\!\!\top_{A\times B\ C} \sqsubseteq \pi; \top\!\!\top_{AC}$$

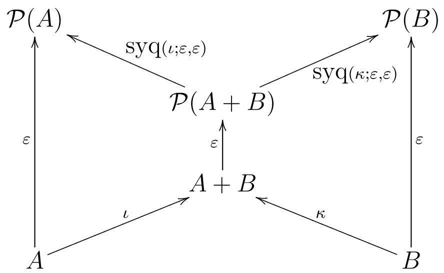
for all object C so that we conclude $\rho = \pi; \top\!\!\top_{AB} \sqcap \rho; \mathbb{I}_B$. Analogously, we get $\pi = \pi; \mathbb{I}_A \sqcap \rho; \top\!\!\top_{BA}$. This implies

$$\begin{aligned}\pi^{\smile}; \rho &= (\pi; \mathbb{I}_A \sqcap \rho; \top\!\!\top_{BA})^{\smile}; (\pi; \top\!\!\top_{AB} \sqcap \rho; \mathbb{I}_B)\\ &= (\mathbb{I}_A; \pi^{\smile} \sqcap \top\!\!\top_{AB}; \rho^{\smile}); (\pi; \top\!\!\top_{AB} \sqcap \rho; \mathbb{I}_B)\\ &= \mathbb{I}_A; \top\!\!\top_{AB} \sqcap \top\!\!\top_{AB}; \mathbb{I}_B\\ &= \top\!\!\top_{AB},\end{aligned}$$

where the third = follows from the sharpness property. □

3.4. Creating weak relational products and representability

The main proposition of this section (Corollary 1) states that any (small) allegory can be embedded into a weak pairing allegory. This theorem is based on the fact that the cartesian product of two power sets can be constructed by the power set of the disjoint union of the sets. An abstract version of this proposition is given in the next lemma and summarized by the following diagram.



Notice, that the constructed weak relational product is not necessarily a relational product [10].

Lemma 8. *Let $\mathcal{R}$ be an allegory, A and B objects of $\mathcal{R}$ so that the relational sum $(A + B, \iota_{AB}, \kappa_{AB})$ and the relational powers $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A + B)$ exist. Then $(\mathcal{P}(A + B), \mathrm{syq}(\iota_{AB}; \varepsilon_{A+B}, \varepsilon_A), \mathrm{syq}(\kappa_{AB}; \varepsilon_{A+B}, \varepsilon_B))$ is a weak relational product of $\mathcal{P}(A)$ and $\mathcal{P}(B)$.*

Proof. Axiom P1 follows immediately from

$$\begin{aligned}\pi^{\smile}; \pi &= \mathrm{syq}(\iota; \varepsilon, \varepsilon)^{\smile}; \mathrm{syq}(\iota; \varepsilon, \varepsilon) &&\\ &= \mathrm{syq}(\varepsilon, \iota; \varepsilon); \mathrm{syq}(\iota; \varepsilon, \varepsilon) && \text{Lemma 1(3)}\\ &\sqsubseteq \mathrm{syq}(\varepsilon, \varepsilon) && \text{Lemma 1(4)}\\ &= \mathbb{I}_{\mathcal{P}(A)}.\end{aligned}$$

Axiom P2 is shown analogously. Since $\mathrm{syq}(\iota;\varepsilon,\varepsilon)$ is total by definition we get

$$\begin{aligned}\pi;\pi^{\smile} &= \mathrm{syq}(\iota;\varepsilon,\varepsilon);\mathrm{syq}(\iota;\varepsilon,\varepsilon)^{\smile}\\ &= \mathrm{syq}(\iota;\varepsilon,\varepsilon);\mathrm{syq}(\varepsilon,\iota;\varepsilon) && \text{Lemma 1(3)}\\ &= \mathrm{syq}(\iota;\varepsilon,\iota;\varepsilon) && \text{Lemma 1(5)}\end{aligned}$$

and $\rho;\rho^{\smile} = \mathrm{syq}(\kappa;\varepsilon,\kappa;\varepsilon)$ analogously. Furthermore, we have

$$\begin{aligned}&\varepsilon;((\iota;\varepsilon)\backslash(\iota;\varepsilon)\sqcap(\kappa;\varepsilon)\backslash(\kappa;\varepsilon))\\ &\quad= (\iota^{\smile};\iota\sqcup\kappa^{\smile};\kappa);\varepsilon;((\iota;\varepsilon)\backslash(\iota;\varepsilon)\sqcap(\kappa;\varepsilon)\backslash(\kappa;\varepsilon))\\ &\quad\sqsubseteq \iota^{\smile};\iota;\varepsilon;(\iota;\varepsilon)\backslash(\iota;\varepsilon)\sqcup\kappa^{\smile};\kappa;\varepsilon;(\kappa;\varepsilon)\backslash(\kappa;\varepsilon)\\ &\quad\sqsubseteq \iota^{\smile};\iota;\varepsilon\sqcup\kappa^{\smile};\kappa;\varepsilon && \text{Lemma 1(1)}\\ &\quad= (\iota^{\smile};\iota\sqcup\kappa^{\smile};\kappa);\varepsilon\\ &\quad= \varepsilon\end{aligned}$$

so that $(\iota;\varepsilon)\backslash(\iota;\varepsilon)\sqcap(\kappa;\varepsilon)\backslash(\kappa;\varepsilon)\sqsubseteq\varepsilon\backslash\varepsilon$ follows. Again, the similar inclusion $(\iota;\varepsilon)^{\smile}/(\iota;\varepsilon)^{\smile}\sqcap(\kappa;\varepsilon)^{\smile}/(\kappa;\varepsilon)^{\smile}\sqsubseteq\varepsilon^{\smile}/\varepsilon^{\smile}$ is shown analogously. Together, we conclude

$$\begin{aligned}\pi;\pi^{\smile}\sqcap\rho;\rho^{\smile} &= \mathrm{syq}(\iota;\varepsilon,\iota;\varepsilon)\sqcap\mathrm{syq}(\kappa;\varepsilon,\kappa;\varepsilon)\\ &= (\iota;\varepsilon)\backslash(\iota;\varepsilon)\sqcap(\iota;\varepsilon)^{\smile}/(\iota;\varepsilon)^{\smile}\sqcap(\kappa;\varepsilon)\backslash(\kappa;\varepsilon)\sqcap(\kappa;\varepsilon)^{\smile}/(\kappa;\varepsilon)^{\smile}\\ &= \varepsilon\backslash\varepsilon\sqcap\varepsilon^{\smile}/\varepsilon^{\smile}\\ &= \mathrm{syq}(\varepsilon,\varepsilon)\\ &= \mathbb{I}_{\mathcal{P})(A+B)}.\end{aligned}$$

In order to prove P4 let $f: C\to A$ and $g: C\to B$ be mappings. The relation $\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon)$ is a mapping by definition, and we have

$$\begin{aligned}&\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);\pi\\ &\quad= \mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);\mathrm{syq}(\iota;\varepsilon,\varepsilon)\\ &\quad= \mathrm{syq}(\iota;\varepsilon;\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon)^{\smile},\varepsilon) && \text{Lemma 1(2)}\\ &\quad= \mathrm{syq}(\iota;\varepsilon;\mathrm{syq}(\varepsilon,\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile}),\varepsilon) && \text{Lemma 1(3)}\\ &\quad= \mathrm{syq}(\iota;(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile}),\varepsilon) && \text{Lemma 1(6)}\\ &\quad= \mathrm{syq}(\varepsilon;f^{\smile},\varepsilon)\\ &\quad= f;\mathrm{syq}(\varepsilon,\varepsilon) && \text{Lemma 1(2)}\\ &\quad= f.\end{aligned}$$

$\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);\rho = g$ is shown analogously. We conclude

$$\begin{aligned}f^{\smile};g &= f^{\smile};\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);\rho\\ &= f^{\smile};\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);(\pi;\pi^{\smile}\sqcap\rho;\rho^{\smile});\rho\\ &\sqsubseteq f^{\smile};\mathrm{syq}(\iota^{\smile};\varepsilon;f^{\smile}\sqcup\kappa^{\smile};\varepsilon;g^{\smile},\varepsilon);\pi;\pi^{\smile};\rho\\ &= f^{\smile};f;\pi^{\smile};\rho\\ &\sqsubseteq \pi^{\smile};\rho.\end{aligned}$$

This completes the proof. □

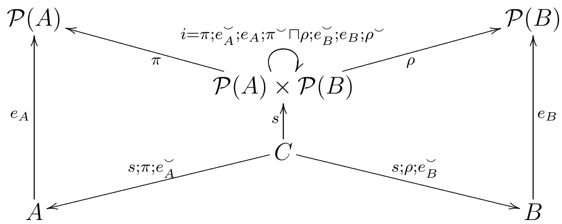
If the allegory provides splitting of partial identities, the construction described above can be distributed to any pair of objects.

Theorem 4. *Any systemic complete allegory is a weak pairing allegory.*

Proof. Let $\mathcal{R}$ be a systemic complete allegory, and let A and B be objects of $\mathcal{R}$. By Lemma 8 there is a weak relational product $(\mathcal{P}(A) \times \mathcal{P}(B), \pi, \rho)$ of $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Let be $i := \pi; e_A^{\smile}; e_A; \pi^{\smile} \sqcap \rho; e_B^{\smile}; e_B; \rho^{\smile}$. The relation i is a partial identity, which is shown as follows

$$\begin{aligned} i &= \pi; e_A^{\smile}; e_A; \pi^{\smile} \sqcap \rho; e_B^{\smile}; e_B; \rho^{\smile} \\ &\sqsubseteq \pi; \pi^{\smile} \sqcap \rho; \rho^{\smile} && e_A \text{ and } e_B \text{ are univalent} \\ &= \mathbb{I}_{\mathcal{P}(A)\times\mathcal{P}(B)}. && \text{Axiom P3} \end{aligned}$$

Since $\mathcal{R}$ is systemic complete there is an object C and a relation $s : C \to \mathcal{P}(A) \times \mathcal{P}(B)$ that splits i. Notice, that s is an injective mapping since i is a partial identity. We want to show that C together with the relations $\tilde{\pi} := s; \pi; e_A^{\smile}$ and $\tilde{\rho} := s; \rho; e_B^{\smile}$ is a weak relational product of A and B.



Axiom P1 is shown by

$$\begin{aligned} \tilde{\pi}^{\smile}; \tilde{\pi} &= (s; \pi; e_A^{\smile})^{\smile}; s; \pi; e_A^{\smile} \\ &= e_A; \pi^{\smile}; s^{\smile}; s; \pi; e_A^{\smile} \\ &= e_A; \pi^{\smile}; \pi; e_A^{\smile} && s \text{ is total and injective} \\ &\sqsubseteq e_A; e_A^{\smile} && \text{Axiom P1} \\ &= \mathbb{I}_A, && e_A \text{ is total and injective} \end{aligned}$$

and Axiom P2 follows analogously. The computation

$$\begin{aligned} \tilde{\pi}; \tilde{\pi}^{\smile} \sqcap \tilde{\rho}; \tilde{\rho}^{\smile} &= s; \pi; e_A^{\smile}; e_A; \pi^{\smile}; s^{\smile} \sqcap s; \rho; e_B^{\smile}; e_B; \rho^{\smile}; s^{\smile} \\ &= s; (\pi; e_A^{\smile}; e_A; \pi^{\smile} \sqcap \rho; e_B^{\smile}; e_B; \rho^{\smile}); s^{\smile} && \text{Lemma 2(1)} \\ &= s; i; s^{\smile} \\ &= s; s^{\smile}; s; s^{\smile} \\ &= \mathbb{I}_C \end{aligned}$$

verifies Axiom P3. In order to prove Axiom P4 we first observe

$$\begin{aligned}
\tilde{\pi}^{\smile}; \tilde{\rho} &= e_A; \pi^{\smile}; s^{\smile}; s; \rho; e_B^{\smile} \\
&= e_A; \pi^{\smile}; i; \rho; e_B^{\smile} && s \text{ splits } i \\
&= e_A; \pi^{\smile}; (\pi; e_A^{\smile}; e_A; \pi^{\smile} \sqcap \rho; e_B^{\smile}; e_B; \rho^{\smile}); \rho; e_B^{\smile} \\
&= e_A; (e_A^{\smile}; e_A; \pi^{\smile}; \rho \sqcap \pi^{\smile}; \rho; e_B^{\smile}; e_B); e_B^{\smile} && \text{Lemma 2(2)} \\
&= e_A; e_A^{\smile}; e_A; \pi^{\smile}; \rho; e_B^{\smile} \sqcap e_A; \pi^{\smile}; \rho; e_B^{\smile}; e_B; e_B^{\smile} && \text{Lemma 2(2)} \\
&= e_A; \pi^{\smile}; \rho; e_B^{\smile} \sqcap e_A; \pi^{\smile}; \rho; e_B^{\smile} && e_A, e_B \text{ total and injective} \\
&= e_A; \pi^{\smile}; \rho; e_B^{\smile}.
\end{aligned}$$

Now, let $f : D \to A$ and $g : D \to B$ be mappings. Then $f; e_A$ and $g; e_B$ are also mappings from C to $\mathcal{P}(A)$ and $\mathcal{P}(B)$, respectively. This implies $e_A^{\smile}; f^{\smile}; g; e_B = (f; e_A)^{\smile}; g; e_B \sqsubseteq \pi^{\smile}; \rho$. We conclude

$$\begin{aligned}
f^{\smile}; g &= e_A; e_A^{\smile}; f^{\smile}; g; e_B; e_B^{\smile} && e_A, e_B \text{ total and injective} \\
&\sqsubseteq e_A; \pi^{\smile}; \rho; e_B^{\smile} \\
&= \tilde{\pi}^{\smile}; \tilde{\rho},
\end{aligned}$$

which finally verifies Axiom P4. □

Since the systemic completion of a small allegory is systemic complete ([4] 2.221 and 2.434) we have shown the main result of this section.

Corollary 1. *Any small allegory may be faithfully represented in a weak pairing allegory.*

This corollary also shows that there are indeed weak pairing allegories in which the weak relational product is not always a relational product. For example, consider the allegory induced by the non-representable McKenzie relation algebra. According to Corollary 1 this allegory can be embedded into a weak pairing allegory. This allegory can not have all relational products since then it would be representable [4], which is a contradiction.

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References

[1] A. Asperti, G. Longo, Categories, Types and Structures, The MIT Press, Cambridge, Massachusetts, London, England, 1991.
[2] R. Berghammer, A. Haeberer, G. Schmidt, P.A.S. Veloso, Comparing two different approaches to products in abstract relation algebra, Algebraic Methodology and Software Technology, Proceedings of the 3rd International Conference on Algebraic Methodology and Software Technology (AMAST'93), Springer, 1994, pp. 167–176.
[3] J. Desharnais, Monomorphic characterization of *n*-ary direct products, Inform. Sci. 119 (3–4) (1999) 275–288.
[4] P. Freyd, A. Scedrov, Categories, Allegories, North-Holland, 1990.
[5] H. Furusawa, W. Kahl, A study on symmetric quotients, Technical Report 1998-06, University of the Federal Armed Forces Munich, 1998.
[6] R.D. Maddux, On the derivation of identities involving projection functions, in: Csirmaz, Gabbay, de Rijke (Eds.), Logic Colloquium'92, Center for the Study of Language and Information Publications, Stanford 1995, pp. 145–163.
[7] J.P. Olivier, D. Serrato, Catégories de Dedekind, Morphismes dans les Catégories de Schröder, C.R. Acad. Sci. Paris 290 (1980) 939–941.
[8] J.P. Olivier, D. Serrato, Squares and rectangles in relational categories – three cases: semilattice, distributive lattice and boolean non-unitary, Fuzzy Sets and Systems 72 (1995) 167–178.
[9] G. Schmidt, T. Ströhlein, Relationen und Graphen, Springer, 1989, English version: Relations and Graphs, Discrete Mathematics for Computer Scientists, EATCS Monographs on Theoret. Comput. Sci., Springer, 1993.

[10] M. Winter, Strukturtheorie heterogener Relationenalgebren mit Anwendung auf Nichtdetermismus in Programmiersprachen, Dissertationsverlag NG Kopierladen GmbH, München, 1998.
[11] M. Winter, Weak relational products, Relations and Kleene algebra in computer science, LNCS 4136 (2006) 417–431.
[12] H. Zierer, Relation algebraic domain constructions, TCS 87 (1991) 163–188.