Covers and Envelopes over gr-Gorenstein Rings

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In this paper we study the existence of Gorenstein injective envelopes and
Gorenstein projective and flat covers in the category of graded modules and we
relate them with the corresponding envelopes and covers in the category of
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INTRODUCTION

Graded commutative rings are a classical topic in commutative algebra.
In recent years there has been considerable interest in various aspects of
graded noncommutative rings (cf. [17]). Certain results on fixed subrings of
a ring with automorphisms have been shown to be easily obtained from the
induced graded structure. On the other hand, the so-called strongly graded
rings were investigated by Dade and the results were applied to the study
of representations of a group $H$ related to a normal subgroup $N$ of $H$.
Moreover, graded noncommutative rings appear in a natural way in the so
called noncommutative algebraic geometry.

The homological theory of graded rings is very important because of its
applications in algebraic geometry (cf. [13] and its references). Iwanga–Gorenstein rings are defined in [14] as noetherian rings on both
sides and with finite self-injective dimension on both sides too (in the case
of commutative rings, this definition of a Gorenstein ring coincides with a
Gorenstein ring of finite Krull dimension, originally defined by Bass).
Their importance is due to the fact that a relative homological theory can
be developed for them (cf. [7, 9]).

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The corresponding theory for graded rings was developed in [1, 2], where Gorenstein gr-injective, gr-projective, and gr-flat modules were introduced. Now, in this paper we study the existence of envelopes and covers by these classes of graded modules. This is secured in the case of gr-Gorenstein rings (cf. [1]). But we are also interested in another problem: to relate envelopes and covers in the category of graded modules, whose existence is studied in this paper and envelopes and covers in the category of modules obtained in [10, 12, 11].

In the first section we recall some basic notions in graded rings and modules and other results about gr-Gorenstein rings which are necessary.

In Section 2 we obtain the existence of Gorenstein gr-injective envelopes in the case of gr-Gorenstein rings, Theorem 2.4. The relation between the envelopes in the category of modules and graded modules is given in Theorem 2.9, where it is said that if $R$ is a graded by a finite group $G$, with $|G|$ invertible in $R$ and $R$ is gr-Gorenstein, then any graded $R$-module $M$ has a Gorenstein gr-injective preenvelope in the category of graded modules if and only if $M$ has a Gorenstein injective preenvelope in the category of modules.

In Section 3 we treat Gorenstein gr-projective covers. In [3] Auslander and Buchweitz studied the existence of Gorenstein projective precovers. We obtain the existence of this kind of covers in the context of commutative gr-Gorenstein rings which are sums of gr-local complete graded rings. This is the case of Theorem 3.9. The relation is given in Theorem 3.12.

In the last section we show that not every graded module over a gr-Gorenstein ring has a Gorenstein gr-projective cover. Therefore it is interesting to study the existence of Gorenstein gr-flat covers. This is done in Theorem 4.11.

1. PRELIMINARIES

All rings considered are associative with identity element and the (left or right) $R$-modules are unital. By $R$-$\text{Mod}$ we will denote the Grothendieck category of all the left $R$-modules. Let $G$ be a multiplicative group with neutral element $\epsilon$. A graded ring $R$ is a ring with identity $1$, together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma \tau}$ for all $\sigma, \tau \in G$. Thus $R_\sigma$ is a subring of $R$, $1 \in R_\epsilon$, and for every $\sigma \in G$, $R_\sigma$ is an $R_\sigma$-bimodule. $R$ is said to be strongly graded if $R_\sigma R_\tau = R_{\sigma \tau}$. A left graded $R$-module is a left $R$-module $M$ endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each $M_\sigma$ is a subgroup of the additive group of $M$ such that $R_\sigma M_\tau \subseteq M_{\sigma \tau}$ for all
For $M$ and $N$ graded left $R$-modules, we put
\[ \text{Hom}_{R,gr}(M, N) = \{ f : M \rightarrow N \mid f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \ \forall \sigma \in G \}. \]

$\text{Hom}_{R,gr}(M, N)$ is the group of all morphisms from $M$ to $N$ in the category $R$-gr of all graded left $R$-modules ($\text{gr}-R$ will denote the category of all graded right $R$-modules). It is well known that $R$-gr is a Grothendieck category (see [17]). An $R$-linear map $f : M \rightarrow N$ is said to be a graded morphism of degree $\tau$, $\tau \in G$, if $f(M_\sigma) \subseteq M_\tau$ for all $\sigma \in G$. Graded morphisms of degree $\tau$ build an additive subgroup $\text{HOM}_{R,gr}(M, N)$ of $R_{\tau}$.

If $f : M \rightarrow N$ is $R$-linear then $Ff : FM \rightarrow FN$ is a graded morphism given by $Ff(x) = f(x)$ for each $r \in R_{\tau}$. Separable functors and separability of $U$ and $F$ are studied in [18].

The injective objects of $R$-gr will be called gr-injective modules. Projective objects of $R$-gr are nothing but graded objects which are projective (as ungraded objects). We will denote the gr-injective dimension of a graded module $M$ by $\text{gr-id}(M)$, and $\text{pd}(M)$ will denote the projective dimension of $M$.

From [1] we recall that $R$ is gr-n-Gorenstein ring if $R$ is left and right gr-noetherian and if it has finite self gr-injective dimension on either side less than or equal to $n$. We will say that $R$ is gr-Gorenstein if it is gr-n-Gorenstein for some $n$. It is clear that any Gorenstein ring is gr-Gorenstein. Other examples of gr-Gorenstein rings are the group rings of a Gorenstein ring and $K[x, x^{-1}]$, where $K$ is a field. If $R$ is a graded ring by $G$ where $G$ is a polycyclic-by-finite group then $R$ is gr-Gorenstein if and only if it is Gorenstein. Therefore when $R$ is strongly graded $R$ is Gorenstein if and only if $R_e$ is Gorenstein (which generalizes [8, Theorem 3.5]).

**Theorem 1.1** [1, Theorem 2.8]. Let $R$ be a gr-n-Gorenstein ring and $M$ a graded left $R$-module. Then the following statements are equivalent:

(i) $\text{pd}(M) < \infty$ 
(ii) $\text{pd}(M) \leq n$ 
(iii) $\text{gr-id}(M) < \infty$ 
(iv) $\text{gr-id}(M) \leq n$. 

**COVERS OVER gr-GORENSTEIN RINGS**
By the above, for a gr-$n$-Gorenstein ring, the class of all graded left $R$-modules which have finite projective dimension and the class of all graded left $R$-modules which have finite gr-injective dimension are the same class. We will use $\mathcal{F}$ to denote this class. It is easy to check that $\mathcal{F}$ is closed under extensions, graded direct sums and summands, graded direct products, and graded direct limits.

Let $C$ be a Grothendieck category and let $\mathcal{F}$ be a class of objects in $C$ which is closed under extensions, direct summands, and isomorphisms. For an object $M$ in $C$, we recall the definition of $\mathcal{F}$-preenvelope and $\mathcal{F}$-envelope.

**Definition 1.1.** Let $\Phi : M \to F$ be a morphism with $F \in \mathcal{F}$. The pair $(\Phi, F)$ is called an $\mathcal{F}$-envelope of $M$ if the following conditions are satisfied:

(i) $\text{Hom}_C(F, F') \to \text{Hom}_C(M, F') \to 0$ is exact for any $F' \in \mathcal{F}$.

(ii) For any endomorphism $f$ of $F$ with $\Phi = f\Phi$, $f$ must be an automorphism of $F$.

In case that $(\Phi, F)$ verifies (i), we say that it is an $\mathcal{F}$-preenvelope.

Dually, we have the notions of $\mathcal{F}$-precovers and $\mathcal{F}$-covers.

**2. Gorenstein gr-Injective Envelopes**

Before starting the study of Gorenstein gr-injective envelopes let us recall from [1] that $M$ is said to Gorenstein gr-injective if it satisfies the following

**Theorem 2.1.** Let $R$ be a graded gr-Gorenstein ring and $M \in R$-gr. The following are equivalent for $M$:

(i) $\text{Ext}^1_R(M, L) = 0$ for any $L \in \mathcal{F}$.

(ii) There is an exact sequence in $R$-gr

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of gr-injective modules with $M = \text{Ker}(E^0 \to E^1)$ such that $\text{Hom}_{R}$-$\text{gr}(-, E)$ leaves the sequence exact for any gr-injective module $E$.

(iii) There is an exact sequence

$$E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$$

with $E_i$'s gr-injective.
**Proposition 2.2.** Let $\mathcal{L}$ be a class of graded left $R$-modules closed under extensions and let $0 \to M \to K \to L \to 0$ be an exact sequence in $R$-$\text{gr}$ with $L \in \mathcal{L}$ verifying

(i) for any exact sequence in $R$-$\text{gr}$ $0 \to M \to K \to L \to 0$ with $L \in \mathcal{L}$ there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \phi \\
0 & \to & M \\
\end{array}
\begin{array}{ccc}
& \to & K \\
& \downarrow & \downarrow \phi \\
& \to & L \\
\end{array}
\]

(ii) if the two rows are the same exact sequence with $\phi \in \text{Hom}_{R}$-$\text{gr}(L, L)$ then $\phi$ is an automorphism of $L$.

Then $\text{Ext}^1_{R}$-$\text{gr}(\bar{L}, K) = 0$ for all $\bar{L} \in \mathcal{L}$.

**Proof.** This is analogous to the ungraded case [10, Proposition 5.2].

**Proposition 2.3.** Let $R$ be a graded ring and $\mathcal{L}$ a class of graded left $R$-modules which is closed under direct limits and direct summands. If for some graded left $R$-module $M$ there exists an exact sequence in $R$-$\text{gr}$ $0 \to M \to N \to L \to 0$ with $L \in \mathcal{L}$, verifying the first condition of the above proposition, then there is an exact sequence in $R$-$\text{gr}$ $0 \to M \to \bar{N} \to \bar{L} \to 0$ verifying the two conditions of Proposition 2.2.

**Proof.** The proof is analogous to the ungraded case, but we include it for completeness.

Let $\mathcal{D}$ be the category whose objects are the exact sequences in $R$-$\text{gr}$, $\varepsilon : 0 \to M \to N \to L \to 0$, with $L \in \mathcal{L}$. A morphism from $\varepsilon \in \mathcal{D}$ and $\varepsilon$ will be given by a commutative diagram:

\[
\begin{array}{ccc}
\varepsilon : 0 & \to & M \\
\downarrow & & \downarrow \\
\varepsilon : 0 & \to & M \\
\end{array}
\begin{array}{ccc}
\to & \to & \to \\
\to & \to & \to \\
\end{array}
\begin{array}{ccc}
\varepsilon : 0 & \to & N \\
\downarrow & & \downarrow \\
\varepsilon : 0 & \to & L \\
\end{array}
\]

We note that $\varepsilon$ satisfies the first condition of the Proposition 2.2 if and only if $\text{Hom}_{\mathcal{D}}(\bar{L}, \varepsilon) \neq 0$ for all objects $\bar{L}$ in $\mathcal{D}$. Also note that $\mathcal{D}$ has direct limits since $\mathcal{L}$ is closed under direct limits (as we noted in the first section) and since the direct limit functor is exact.

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ whose objects are the $\varepsilon$ such that $\text{Hom}_{\mathcal{D}}(\bar{L}, \varepsilon) \neq 0$ for all objects $\bar{L}$ in $\mathcal{D}$. Then $\mathcal{C}$ may not have direct limits, but every directed system in $\mathcal{C}$ admits a map into some object of $\mathcal{C}$ (via its direct limit which is in $\mathcal{D}$).
Now suppose \( \varepsilon : 0 \to M \to N \to L \to 0 \in \mathcal{C} \) and that we have a morphism

\[
\varepsilon : 0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0
\]

such that \( \text{Ker}(g \circ g) = \text{Ker}(g) \), \( \text{Ker}(h \circ h) = \text{Ker}(h) \), \( \text{Im}(g \circ g) = \text{Im}(g) \), \( \text{Im}(h \circ h) = \text{Im}(h) \). Then \( \text{Im}(h) \) is a direct summand of \( L \) and so, by hypothesis, \( L \in \mathcal{C} \). Also the sequence \( 0 \to M \to \text{Im}(g) \to \text{Im}(h) \to 0 \) is exact and it is in \( \mathcal{C} \). Now, we can follow an analogous reasoning to [19, Theorem 2.2.4] to say that there is an exact sequence which verifies the two conditions of Proposition 2.2.

**Theorem 2.4.** If \( R \) is a gr-n-Gorenstein ring, then any graded left \( R \)-module has a Gorenstein gr-injective envelope.

**Proof.** Let \( M \) be a graded left \( R \)-module. By the proof of [1, Proposition 3.7], there exists an exact sequence in \( R \)-gr, \( 0 \to M \to K \to L \to 0 \) where \( K \) is Gorenstein gr-injective and \( L \in \mathcal{C} \). Since \( \text{Ext}^1_{R\text{-gr}}(L, K) = 0 \) for all \( L \in \mathcal{C} \), if \( 0 \to M \to K \to L \to 0 \) is exact in \( R \)-gr with \( L \in \mathcal{C} \), then we have that

\[
\text{Hom}_{R\text{-gr}}(K, K) \to \text{Hom}_{R\text{-gr}}(M, K) \to \text{Ext}^1_{R\text{-gr}}(L, K) = 0
\]

is exact and therefore we get a commutative diagram

\[
\begin{array}{ccc}
M & \longrightarrow & K \\
\downarrow & & \downarrow \\
M & \longrightarrow & K
\end{array}
\]

which shows that \( 0 \to M \to K \to L \to 0 \) verifies the first condition of Proposition 2.2. As we noted in Section 1, \( \mathcal{C} \) is closed under direct limits. Now, by Proposition 2.3, there is an exact sequence in \( R \)-gr \( 0 \to M \to K \to L \to 0 \) with \( L \in \mathcal{C} \) satisfying the two conditions of Proposition 2.2, and so, we get that \( K \) is Gorenstein gr-injective.

Now let \( K \) be a Gorenstein gr-injective module. Then we have that

\[
\text{Hom}_{R\text{-gr}}(K, K) \to \text{Hom}_{R\text{-gr}}(M, K) \to \text{Ext}^1_{R\text{-gr}}(L, K) = 0
\]

is exact, which shows that \( M \to K \) is a Gorenstein gr-injective preenve-
lope. Since $0 \to M \to K \to L \to 0$ satisfies the two conditions of Proposition 2.2, if we consider $K \to K$, we get a commutative diagram

$$
\begin{array}{c}
0 \to M \to K \to L \to 0 \\
\downarrow \downarrow f \downarrow \downarrow \\
0 \to M \to K \to L \to 0
\end{array}
$$

where $f$ is an automorphism, and therefore $K \to K$ is an automorphism, so we can conclude that $M \to K$ is a Gorenstein gr-injective envelope. □

So far we have proved the existence of Gorenstein gr-injective envelopes for any graded left module over a gr-n-Gorenstein ring. Now we are interested in relating the Gorenstein gr-injective envelope of a graded module and the Gorenstein injective envelope of its corresponding module (forgetting the graded structure), whose existence is proved in [10], but first, we need some general results.

Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories. We recall that if $(F, H): \mathcal{A} \to \mathcal{B}$ is an adjoint situation between the functors $F$ and $H$, we denote by $\varepsilon: FH \to 1_{\mathcal{A}}$ and $\eta: 1_{\mathcal{B}} \to HF$ the counit and the unit of the adjoint situation. Now let $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$ two full subcategories verifying $F(\mathcal{C}) \subseteq \mathcal{D}$ and $H(\mathcal{D}) \subseteq \mathcal{C}$.

**Proposition 2.5.** If $A \in \mathcal{A}$ has a $\mathcal{C}$-preenvelope $\Phi: A \to B$, then $F(\Phi): F(A) \to F(B)$ is a $\mathcal{D}$-preenvelope of $F(A)$.

**Proof.** Let $f: F(A) \to D$ a morphism in $\mathcal{B}$ with $D \in \mathcal{D}$. Since $\Phi: A \to B$ is a $\mathcal{C}$-preenvelope, there exists $g: B \to H(D)$ such that the following diagram is commutative:

$$
\begin{array}{c}
A \xrightarrow{\Phi} B \\
\downarrow H(f) \downarrow \downarrow \eta_A \\
H(D) \xrightarrow{\eta_B}
\end{array}
$$

If we apply $F$ to the above diagram we get that $F(g \circ \Phi) = F(H(f) \circ \eta_A) = f$, i.e., the following diagram is commutative,

$$
\begin{array}{c}
F(A) \xrightarrow{F(\Phi)} F(B) \\
\downarrow F(f) \downarrow \downarrow F(g)
\end{array}
$$

which shows that $F(\Phi): F(A) \to F(B)$ is a $\mathcal{D}$-preenvelope. □

**Proposition 2.6.** In the preceding adjoint situation, let us assume that the unit of the adjoint situation verifies that $\eta_Z: Z \to HF(Z)$ is an split monomorphism for all $Z \in \mathcal{C}$. If $Y \in \mathcal{A}$ is such that $F(Y) \in \mathcal{B}$ has a
\( \mathcal{D} \)-preenvelope \( \Phi : F(Y) \to X \), then \( Y \to \eta_Y H(\Phi) \to H(X) \) is a \( \mathcal{D} \)-preenvelope of \( Y \).

**Proof.** Let \( E \in \mathcal{D} \) and \( f : Y \to E \). Applying \( F \) we get the following commutative diagram:

\[
\begin{array}{ccc}
F(Y) & \xrightarrow{\Phi} & X \\
\downarrow{\Phi(f)} & & \downarrow{g} \\
F(E) & \xrightarrow{\eta_Y \Phi} & H(Y)
\end{array}
\]

Now, if we apply \( H \) to the above triangle, we get the following commutative diagram:

\[
\begin{array}{ccc}
HF(Y) & \xrightarrow{\eta_Y} & Y \\
\downarrow{H(\Phi)} & & \downarrow{f} \\
H(X) & \xrightarrow{HF(f)} & HF(E)
\end{array}
\]

From the above diagram we get that \( \varepsilon_E \circ H(g) \circ H(\Phi) \circ \eta_Y = \varepsilon_E \circ HF(f) \circ \eta_Y = \varepsilon_E \circ f = f \) which gives the desired result.

**Proposition 2.7.** Let \( R \) be a gr-n-Gorenstein ring graded by a finite group \( G \). If \( \Phi : M \to E \) is a Gorenstein injective envelope in \( R\text{-mod} \), with \( M, E \in R\text{-gr} \) and \( \Phi \) is a graded morphism, then \( \Phi : M \to E \) is a Gorenstein gr-injective envelope in \( R\text{-gr} \).

**Proof.** By [1, Theorem 3.10], the class of graded Gorenstein injective modules coincides with the class of Gorenstein gr-injective modules. Let \( E' \) be a graded Gorenstein injective module and \( g : M \to E' \) a graded morphism. Then there is a morphism in \( R\text{-mod} \) \( h : E \to E' \) such that \( h \circ \Phi = g \). By [17, Lemma I.2.1] there exists a graded morphism \( h' : E \to E' \) such that \( h' \circ \Phi = g \), which shows that \( \Phi : M \to E \) is a Gorenstein gr-injective preenvelope. Now, let \( f : E \to E \) be a graded morphism such that \( f \circ \Phi = \Phi \). Again by [17, Lemma I.2.1] we get that \( f \) is an automorphism in \( R\text{-gr} \), and therefore, \( \Phi : M \to E \) is a Gorenstein gr-injective envelope.

Now, we are in a good situation to solve the problem that we proposed in the Introduction: given \( M \in R\text{-gr} \), construct a Gorenstein gr-injective envelope via a known object in \( R\text{-mod} \), and conversely, given \( M \in R\text{-mod} \), construct a Gorenstein injective envelope via a known object in \( R\text{-gr} \) using the adjoint situation \((U, F)\) between the categories \( R\text{-gr} \) and \( R\text{-mod} \), where \( U \) is the forgetful functor and \( F \) is the right adjoint functor defined in Section 1.

**Proposition 2.8.** (i) Suppose that \( R \) is a gr-n-Gorenstein ring, graded by a finite group \( G \). If \( \Phi : M \to E \) is a Gorenstein gr-injective preenvelope in
If \( U(M) \xrightarrow{\Phi} E \) is a Gorenstein injective preenvelope in \( R\text{-mod} \), then

\[
M \xrightarrow{\eta} FU(M) \xrightarrow{F(\Phi)} F(E)
\]

is a Gorenstein \( gr \)-injective preenvelope, where \( \eta \) is the unit of the adjoint situation \((U, F)\).

(iii) If \( M \in R\text{-gr} \), then \( M \) has a Gorenstein \( gr \)-injective preenvelope if and only if \( U(M) \) has a Gorenstein injective preenvelope.

**Proof.** (i) This is a consequence of [1, Theorem 3.10] and Proposition 2.5.

(ii) Since \( U \) is separable [18, Proposition 2.4], by [4, Proposition 5(b)] we are in the conditions of Proposition 2.6, so we obtain the desired result.

(iii) This is a consequence of (i) and (ii).  

**Theorem 2.9.** Let \( R \) be a graded ring by a finite group \( G \) with \( |G| \) invertible in \( R \). Let us also suppose that \( R \) is \( gr \)-n-Gorenstein. Then, any graded \( R\text{-module} \) has a Gorenstein \( gr \)-injective preenvelope in \( R\text{-gr} \) if and only if any \( R\text{-module} \) has a Gorenstein injective preenvelope in \( R\text{-mod} \).

**Proof.** Let \( M \in R\text{-gr} \) and suppose that any \( R\text{-module} \) has a Gorenstein injective preenvelope in \( R\text{-mod} \). Let \( U(M) \xrightarrow{\Phi} E \) be a Gorenstein injective preenvelope in \( R\text{-mod} \). Then, by Proposition 2.8(ii), \( M \xrightarrow{\eta} FU(M) \xrightarrow{F(\Phi)} F(E) \) is a Gorenstein \( gr \)-injective preenvelope in \( R\text{-gr} \).

Conversely, let \( M \in R\text{-mod} \) and assume that any graded \( R\text{-module} \) has a Gorenstein \( gr \)-injective preenvelope in \( R\text{-gr} \). Let \( F(M) \xrightarrow{\Phi} E \) be a Gorenstein \( gr \)-injective preenvelope. By Proposition 2.8(i), \( U(F(M)) \xrightarrow{U(\Phi)} U(E) \) is a Gorenstein injective envelope. Now, let \( E' \) be a Gorenstein injective \( R\text{-module} \) and \( M \xrightarrow{f} E \) a morphism. Then, we get the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\varepsilon_M} & UF(M) \xrightarrow{U(\Phi)} U(E) \\
| & f & | \\
| & \varepsilon_M & |
\end{array}
\]

where \( \varepsilon_M \) is the morphism which appears in [4, Proposition 5(a)] due to the separability of \( F \) (see [18, Corollary 3.7]), and the existence of \( g \) is due to the fact that \( UF(M) \xrightarrow{U(\Phi)} U(E) \) is a Gorenstein injective preenvelope.  

Remark 1. Although the last result is not true for envelopes, it is convenient to note its importance since an envelope can be obtained from a preenvelope by using the argument in Theorem 2.4.

From the above remark and from Theorem 2.9 we get the following

Corollary 2.10. Let $R$ be a strongly graded ring by a finite group $G$ with $|G|$ invertible in $R$. If $R$ is Gorenstein then any left $R$-module has a Gorenstein injective envelope in $R$-mod $\iff$ any left $R_e$-module has a Gorenstein injective envelope in $R_e$-mod.

3. GORENSTEIN gr-PROJECTIVE COVERS

As for Gorenstein gr-injective modules, we can give the following theorem which characterizes Gorenstein gr-projective modules.

Theorem 3.1. Let $R$ be a gr-Gorenstein ring and $M \in R$-gr. The following are equivalent for $M$:

(i) $\text{Ext}_{R, gr}^1(M, L) = 0$ for any $L \in \mathcal{D}$.

(ii) There is an exact sequence in $R$-gr

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules with $M = \text{Ker}(P^0 \to P^1)$ such that $\text{Hom}_{R, gr}(P, -)$ leaves the sequence exact for any projective graded $R$-module $P$.

(iii) There is an exact sequence in $R$-gr

$$0 \to M \to P^0 \to P^1 \to \cdots \to P^{n-1}$$

with $P^i$'s projective.

In this section all graded modules considered are finitely generated. In this case a Gorenstein gr-projective module $M$ can be also characterized by $\text{EXT}_{R, gr}^i(M, R) = 0 \forall i \geq 1$.

Our aim is to prove the existence of Gorenstein gr-projective covers in some particular cases. First, we will prove the existence of Gorenstein gr-projective precovers in the case of a gr-Gorenstein ring.

Following the notation of [3], let $R$ be a gr-$n$-Gorenstein ring and $\mathbf{X}$ the category of finitely generated graded $R$-modules such that $\text{Ext}_{R, gr}^i(M, R) = 0 \forall i > 0$. $\mathbf{X}$ is closed under direct summands and if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact in $\mathbf{X}$ and when $M_1$ and $M_3$ or $M_2$ and $M_3$ are in $\mathbf{X}$, then the third one is also in $\mathbf{X}$. 
Let $\Omega$ be the subcategory of finitely generated projective graded $R$-
modules. By definition, $\Omega$ is a subcategory of $X$ and it is closed under
direct summands. Let us show that $\Omega$ is a cogenerator for $X$, i.e., any
$M \in X$ is embedded in $N \in \Omega$ such that the cokernel of this morphism is
also in $X$.

Let $\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a gr-projective resolution
of $M$ in $X$. If for any $M \in R$-gr $M^*$ denotes $\text{HOM}_R(M, R)$, we have that
the complex $0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \cdots$ is acyclic. Let us prove the
following

**Lemma 3.2.** With the above notation, for any $M$ in $X$ we have

(i) $\forall j \geq 0$, $K_j = \text{Ker}(d_j^\bullet)$ satisfy $\text{EXT}^i_R(K_j, R) = 0 \forall i > 0$.

(ii) $M \rightarrow M^*$ is an isomorphism of graded modules.

(iii) If $0 \rightarrow L \rightarrow Q \rightarrow M^* \rightarrow 0$ is exact in $R$-gr with $Q$ finitely
generated projective, then $L^*$ satisfies that $\text{EXT}^i_R(L^*, R) = 0 \forall i > 0$.

**Proof.** (i) The $P_i^*$ are finitely generated projective graded modules.
Then these modules satisfy $\text{EXT}^i_R(P_i^*, R) = 0 \forall i > 0$. So $\forall m \geq 0$ we get the natural isomorphisms

$$\text{EXT}^i_R(K_{j-m}, R) \rightarrow \text{EXT}^{i+m}_R(K_j, R) \quad \forall i > 0.$$ 

Since by hypothesis $\text{EXT}^k_R(-, R) = 0$ for $k > n$, it is enough to take
$m \geq n$ to conclude that $\text{EXT}^i_R(K_j, R) = 0 \forall i > 0, \forall j \geq 0$.

(ii) Let $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be exact, with $P_i$ finitely
generated projective. If we apply the functor $\text{HOM}_R(-, R)$, we get $0 \rightarrow M^* \rightarrow
P_0^* \rightarrow P_1^* \rightarrow \cdots$. Let $N = \text{Coker}(P_0^* \rightarrow P_1^*)$. So

$$\text{HOM}_R(N, R) = \text{Ker}(\text{HOM}_R(P_1^*, R) \rightarrow \text{HOM}_R(P_2^*, R))$$

$$= \text{Ker}(P_1 \rightarrow P_2)$$

since $P = \text{HOM}_R(P^*, R)$ when $P$ is finitely generated projective.
Therefore, we get an exact sequence

$$\text{HOM}_R(N, R) \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$ 

In the exact sequence

$$0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow N \rightarrow 0$$

let $B = \text{Coker}(M^* \rightarrow P_0^*) = P_0^*/\text{im}(M^* \rightarrow P_0^*) = P_0^*/\text{Ker}(P_0^* \rightarrow P_1^*) \cong
\text{im}(P_0^* \rightarrow P_1^*) = \text{Ker}(P_1^* \rightarrow N)$.
Let us consider now the exact sequences
\[ 0 \to B \to P_1^* \to N \to 0 \]
and
\[ 0 \to M^* \to P_0^* \to B \to 0 \]
which give us the exact sequences
\[ 0 \to N^* \to P_1^{**} \to B^* \to \text{EXT}^1_R(N, R) \to 0 \]
and
\[ 0 \to B^* \to P_0^{**} \to M^{**} \to \text{EXT}^1_R(B, R) \to 0 \]
since \( P_1^* \), \( P_0^* \) are projective. In this way we obtain the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & N^* & \to & P_1^{**} & \to & B^* & \to & \text{EXT}^1_R(N, R) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N^* & \to & P_0^{**} & \to & P_1^{**} & \to & M & \to & 0 \\
| & & | & & | & & | & & | & & | \\
| & & | & & | & & \downarrow & & \downarrow & & \downarrow \\
\text{EXT}^1_R(B, R) & \to & & & & & & & & & & 0 \\
\end{array}
\]

Now, \( \text{EXT}^1_R(B, R) \cong \text{EXT}^2_R(N, R) \) because \( N \) has a projective resolution in \( R\text{-gr} \cdots \to P_1^* \to P_0^* \to 0 \) such that \( B = \text{Ker}(P_1^* \to N) \). The morphism \( \text{EXT}^1_R(N, R) \to M \) is obtained by the universal property of the kernel and it also occurs for \( M \to M^{**} \), so we get an exact sequence
\[ 0 \to \text{EXT}^1_R(N, R) \to M \to M^{**} \to \text{EXT}^2_R(N, R) \to 0, \]
where \( N = \text{Coker}(d_2^N) = \text{Im}(d_1^N) \), so, from (i) in this lemma, we have that \( \text{EXT}^1_R(N, R) = \text{EXT}^2_R(N, R) = 0 \), obtaining the desired isomorphism.

(iii) \( M^* = K_0 \). From (i), we get that the sequence
\[ 0 \to M^{**} \to Q^* \to L^* \to 0 \]
is exact. From (ii), \(M^{**} \cong M\) and since \(M \in X\) and \(Q^*\) is projective, we have that \(\text{Ext}_R^i(L^*, R) = 0\) \(\forall i > 1\). We have to show now that \(\text{Ext}_R^1(L^*, R) = 0\) or, equivalently, that the dual sequence

\[0 \to L^{**} \to Q^{**} \xrightarrow{p^{**}} M^{***} \to 0\]

is again exact, which is clear, because \(Q^* = Q^{***}\), \(M^* = M^{***}\), and \(p^{**} = p\).

Now, combining (ii) and (iii) in the preceding lemma, we get that any \(M\) in \(X\) is embedded in a finitely generated projective graded module \(Q^*\), and that the cokernel of this morphism is isomorphic to \(L^*\), which is also in \(X\).

The following theorem is a particular case of [3, Theorem 1.1].

**Theorem 3.3.** Let \(R\) be a gr-Gorenstein ring. Then for any finitely generated graded \(R\)-module \(N\) satisfying \(\text{Ext}_R^i(N, R) = 0\) for all big enough \(i\), there exist \(Y\) and \(Y^N\) finitely generated of finite projective dimension in \(R\)-gr and \(X\) and \(X^N\) finitely generated in \(R\)-gr such that

\[\text{Ext}^i_{R\text{-gr}}(X^N, R) = 0\quad\text{and}\quad\text{Ext}^i_{R\text{-gr}}(X^N, R) = 0\quad\forall i > 0\]

and the exact sequences in \(R\)-gr

\[0 \to Y_N \to X^N \to N \to 0\]

and

\[0 \to N \to Y^N \to X^N \to 0.\]

**Proof.** This is a consequence of the preceding lemma.

**Theorem 3.4.** Let \(R\) be a gr-Gorenstein ring. Then any finitely generated \(M \in R\text{-gr}\) has a Gorenstein gr-projective precover.

**Proof.** From the preceding theorem we get an exact sequence in \(R\)-gr

\[0 \to L \to C \to M \to 0\]

where \(C\) is Gorenstein gr-projective and \(\text{pd}(L) < \infty\). Now if \(C\) is another Gorenstein gr-projective \(R\)-module and if we apply \(\text{Hom}_{R\text{-gr}}(C', -)\) to this exact sequence we get

\[0 \to \text{Hom}_{R\text{-gr}}(C', L) \to \text{Hom}_{R\text{-gr}}(C', C) \to \text{Hom}_{R\text{-gr}}(C', M) \to 0\]

since \(\text{Ext}^1_{R\text{-gr}}(C', L) = 0\), which shows that \(C \to M\) is a Gorenstein gr-projective precover.

To show the existence of Gorenstein gr-projective covers we first need some known results. The following proposition can be seen as a graded version of the Artin–Rees property for noetherian rings.
**Proposition 3.5.** Let $R$ be a left gr-noetherian ring and $I$ a graded ideal generated by a central system composed by homogeneous elements. Then for any finitely generated graded left $R$-module $M$, any graded submodule $N$ of $M$ and $s \in \mathbb{N}$ there exists $t \in \mathbb{N}$ such that $IM \cap N \subset IN$.

**Proof.** Let $M, N,$ and $s$ be as in our hypothesis and let $M'$ be a maximal graded submodule of $M$ such that $M' \cap N = IM$. This election makes that $M/M'$ is a gr-essential extension of $N/I'M$. Let us show that there is some $t \in \mathbb{N}$ such that $I(M/M') = 0$, which gives us that $IM \cap N \subset IN$. It is enough to prove it for $s = 1$, since in other cases we can apply the same reasoning to $IN, \ldots, I^{s-1}N$.

From now on we will denote $M/M'$ by $M$ and $N/IN$ will be $N$. We have that $M$ is a gr-essential extension of $N$ and that $I$ annihilates $N$. Let $I = Rc_1 + \cdots + Rc_n,$ where \{c_1, \ldots, c_n\} is a central system composed of homogeneous elements and let us make induction over $n$ to show that if $I$ annihilates $N$, then some power of $I$ will make the same with $M$. Let $\mu : M \to M$ be defined by $\mu(m) = c_1m \forall m \in M$. $\mu$ is a graded morphism of degree $gr(c_1)$, so $\text{Ker}(\mu)$ is a graded submodule of $M$. Since $M$ is gr-noetherian there is $r \in \mathbb{N}$ such that $\text{Ker}(\mu^r) = \text{Ker}(\mu^{r+1}) \forall k \in \mathbb{N}$, i.e., $\text{Im}(\mu^r) \cap \text{Ker}(\mu^r) = 0$. Since $IN = 0$, $c_1N = 0$ which implies that $N$ is contained in $\text{Ker}(\mu)$. Therefore $N \cap \text{Im}(\mu^r) \subset \text{Im}(\mu^r) \cap \text{Ker}(\mu^r) = 0$ and since $N$ is gr-essential in $M$, then $\text{Im}(\mu^r) = 0$. Let $t$ be the least natural number such that $\mu^t = 0$. We have a injective morphism $M/\text{Ker}(\mu) \to \text{Ker}(\mu^{-1})$. It will be enough to show that $\text{Ker}(\mu)$ and $\text{Ker}(\mu^{-1})$ are annihilated by large powers of $I$. Since $c_1$ is central $\text{Ker}(\mu)$ is an $R/Rc_1$-module, $N$ is annihilated by $I/Rc_1$, so by the induction hypothesis $\text{Ker}(\mu)$ is annihilated by some power of $I/Rc_1$, since $N$ is gr-essential in $\text{Ker}(\mu)$, and therefore $\text{Ker}(\mu)$ is annihilated as an $R$-module by $I^h$ for some $h \in \mathbb{N}$. Now if we consider the morphism $\text{Ker}(\mu^{-1})/\text{Ker}(\mu) \to \text{Ker}(\mu^t)$, we have that $I^h\text{Ker}(\mu^{-1})/\text{Ker}(\mu) = 0$ which implies that $I^h\text{Ker}(\mu^{t-1}) \subset \text{Ker}(\mu)$, so $I^hI^h\text{Ker}(\mu^{t-1}) \subset I^h\text{Ker}(\mu) = 0$.

From now on, we will suppose that the ring $R$ is commutative. Now, by using the preceding result and an analogous reasoning in the ungraded case [5, Lemma 7.15], we can get the following

**Proposition 3.6.** Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated graded $R$-modules over a gr-noetherian ring $R$ and let $I$ be a graded ideal of $R$. Then the sequence of the $I$-adic completed graded $R$-modules $0 \to (M')^\wedge \to (M)^\wedge \to (M'')^\wedge \to 0$ is exact.

**Proposition 3.7.** If $R$ is a gr-noetherian ring and $M$ is a finitely generated graded $R$-module, then $R^\wedge \otimes_R M \cong M^\wedge$.
Proof. It is clear that if \( I \) is a graded ideal, then the \( I \)-adic gr-completion commutes with the direct sum. Let \( F \) be a gr-free \( R \)-module. Hence \( F \cong \bigoplus_{\sigma \in S} R(\sigma) \), with \( S \subseteq G \). Then

\[
\hat{R}^g \otimes_R F \cong \bigoplus_{\sigma \in S} \left( \bigoplus_{\sigma \in S} R(\sigma) \otimes_R \hat{R}(\sigma) \right)
\]

\[
\cong \bigoplus_{\sigma \in S} \hat{R}^g(\sigma) \cong \left( \bigoplus_{\sigma \in S} (R(\sigma))^{\hat{g}} \right) \cong F^{\hat{g}}.
\]

Now, let \( M \) be a finitely generated graded \( R \)-module. Then we have the exact sequence \( 0 \to N \to F \to M \to 0 \) exact in \( R \)-gr with \( F \) gr-free and \( N \) finitely generated, which gives a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & N^{\hat{g}} & \rightarrow & F^{\hat{g}} & \rightarrow & M^{\hat{g}} & \rightarrow & 0 \\
\gamma & \downarrow & \beta & \downarrow & \alpha & & \\
0 & \rightarrow & \hat{N} & \rightarrow & \hat{F} & \rightarrow & \hat{M} & \rightarrow & 0
\end{array}
\]

where the upper row is exact. Following an analogous reasoning as in the proof of [5, Lemma 7.15] we get that \( \delta \) is an epimorphism, and since \( \beta \) is an isomorphism, we have that \( \delta \) is an epimorphism. Since \( N \) is finitely generated, \( \gamma \) is also an epimorphism and the lower row is exact from the above proposition. Now by chasing the diagram we obtain that \( \alpha \) is a monomorphism. 

In order to introduce our following result, we recall that a graded ring is gr-local if it has only a maximal graded ideal.

**Proposition 3.8.** Let \( R \) be a commutative gr-noetherian graded by a finite group ring. If \( R \) is a finite sum of gr-local complete graded rings (respect the \( \mathfrak{M} \)-adic topology, where \( \mathfrak{M} \) is its maximal graded ideal), then \( \lim_{\leftarrow} M_n = 0 \) for any projective system of finitely generated graded \( R \)-modules \( (M_n) \) and \( \forall i \geq 1 \) (\( \lim_{\leftarrow} \) denotes the right derived functors of the inverse limit functor).

Proof. It is enough to consider the case where \( R \) is a gr-local complete graded ring. By the preceding proposition, every finitely generated graded \( R \)-module is complete. Let us prove that \( M/\mathfrak{M}^nM \) has finite length \( \forall n \geq 1 \). Let

\[
0 = \mathfrak{M}^nM \subseteq \mathfrak{M}^{n-1}M \subseteq \ldots \subseteq \mathfrak{M}^1M \subseteq \mathfrak{M}^0M \subseteq \mathfrak{M}^0M.
\]

But \( (\mathfrak{M}^{-1}M/\mathfrak{M}^nM)/(\mathfrak{M}^0M/\mathfrak{M}^nM) \) is finitely generated over the gr-residue field of \( R \) and so has finite length. Therefore, \( M \) is pseudocompact in its \( \mathfrak{M} \)-adic topology. Since, any morphism between finitely generated
graded $R$-modules is continuous, by [16, Proposition 2], we obtain that 
\[ \lim_{i \to 0} M_i = 0 \forall i > 0. \]

Now, we can develop a dual reasoning as in the previous section and 
obtain the duals of Propositions 2.2 and 2.3.

**Theorem 3.9.** If $R$ is a commutative gr-Gorenstein graded by a finite 
group ring which is a sum of gr-local complete graded rings, then any finitely 
generated graded left $R$-module has a Gorenstein gr-projective cover.

**Proof.** Let $M \in R$-gr. By Theorem 3.3 there exists an exact sequence 
$0 \to L \to K \to M \to 0$ with $K$ Gorenstein gr-projective and $pd(L) < \infty$. If $0 \to \overline{L} \to \overline{K} \to M \to 0$ is exact with $pd(\overline{L}) < \infty$, then we get that 
\[ \text{Hom}_{R\text{-gr}}(K, \overline{K}) \to \text{Hom}_{R\text{-gr}}(K, M) \to \text{Hom}_{R\text{-gr}}(K, \overline{L}) = 0 \]
is exact, so we can obtain a commutative diagram 
\[
\begin{array}{ccc}
K & \to & M \\
\downarrow & & \downarrow \\
\overline{K} & \to & M
\end{array}
\]
By the preceding proposition the projective limit of exact sequences 
$0 \to L_i \to K_i \to M \to 0$ with the modules finitely generated is exact.

Let us show that $\lim L_i$ has finite graded dimensions. It is enough to 
prove 
\[ \text{Ext}_{R\text{-gr}}^{n+1}(A, \lim L_i) = 0 \]
for any finitely generated graded $R$-module $A$, but note that in this case, 
$\text{Ext} = \text{Ext}$ since $A$ is finitely generated. So following the same reasoning 
as in [16, Proposition 5] we get the desired result.

Now we can apply the duals of Propositions 2.2 and 2.3 to obtain the 
existence of Gorenstein gr-projective covers.

Now we are interested again in relating the graded and the ungraded 
cases. The following three results have dual proofs to Propositions 2.7 and 
2.8 and Theorem 2.9.

**Proposition 3.10.** Let $R$ be graded by a finite group $G$ gr-Gorenstein 
gr-ring. If $\Phi : C \to M$ is a Gorenstein projective cover in $R$-mod, with $C, M \in 
R$-gr and $\Phi$ a graded morphism, then $\Phi : C \to M$ is a Gorenstein gr-projective 
cover.

**Proposition 3.11.** Let $R$ be a graded ring. Then

(i) If $\Phi : C \to M$ is a Gorenstein projective precover of $M \in R$-mod, 
then $F(\Phi) : F(C) \to F(M)$ is a Gorenstein gr-projective cover of $F(M) \in R$- 
gr.
(ii) If $F$ is separable and $C^F$ is a Gorenstein gr-projective precover, then $U(C^F) \xrightarrow{\epsilon_M} UF(M) \xrightarrow{\mu_M} M$ is a Gorenstein projective precover.

(iii) If $F$ is separable, then $M \in R\text{-mod}$ has a Gorenstein projective precover if and only if $F(M)$ has a Gorenstein projective precover.

**Theorem 3.12.** Let $R$ be a graded ring by a finite group $G$ with $|G|$ invertible in $R$. Let us also suppose that $R$ is gr-$n$-Gorenstein. Then, any graded $R$-module has a Gorenstein gr-projective precover in $R\text{-gr}$ if and only if any $R$-module has a Gorenstein projective precover in $R\text{-mod}$.

**Remark 2.** We note that as in Section 2 the existence of Gorenstein projective precovers in the categories $R\text{-gr}$ and $R\text{-mod}$ can be also related with the category $R_{\text{gr}}\text{-mod}$ when the ring satisfies conditions of the theorem above and it is strongly graded.

We also note that we can obtain Gorenstein gr-projective covers from precovers in the cases studied in this section.

### 4. GORENSTEIN gr-FLAT COVERS

In the section above we have studied the existence of Gorenstein gr-projective covers. However, we cannot claim they exist in general for any graded module. For example, let $R$ be a local ring with finite global dimension. We can consider the group ring $R[G]$. Then $R[G]$ is a gr-Gorenstein gr-local ring and Gorenstein gr-projective modules are just graded projective modules, and therefore Gorenstein gr-projective covers are just graded projective covers. Although every finitely generated graded left $R[G]$-module has a (Gorenstein) graded projective cover, this is not true for any graded left $R[G]$-module, because the existence of graded projective covers for any graded left $R[G]$-module is equivalent to the existence of projective covers for any left $R$-module.

Our aim in this section is to show that if $R$ is a graded gr-Gorenstein ring, then any graded left $R$-module has a Gorenstein gr-flat cover. The process of obtaining Gorenstein gr-flat covers is basically the same as in [11]. Before studying the existence of Gorenstein gr-flat covers, let us introduce some results about Gorenstein gr-flat modules and graded modules which are needed.

Let $\mathbb{Q}$ and $\mathbb{Z}$ be the rational numbers field and the integer numbers ring respectively and consider $\mathbb{Q}/\mathbb{Z}$ the quotient ring. It can be graded with the trivial graduation, i.e., $(\mathbb{Q}/\mathbb{Z})_e = 0$ $\forall e \in G$, $e \neq e$, and $(\mathbb{Q}/\mathbb{Z})_e = \mathbb{Q}/\mathbb{Z}$. Then we have the following

**Definition 4.1.** Let $M = \bigoplus_{e \in G} M_e$ be a graded left $R$-module. We define the graded right $R$-module $M^+ = \text{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. 


Remark 3. Note that each homogeneous component of $M^+$ is

$$(\text{HOM}_z(M, \mathbb{Q}/\mathbb{Z}))_\sigma = \{ f : M \to \mathbb{Q}/\mathbb{Z} : f(M_\mu) = 0 \ \forall \beta \in G, \ \beta \neq \sigma^{-1} \}. $$

Therefore $M^+$ can be written as $M^+ = \bigoplus_{\sigma \in G} \text{HOM}_z(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

It is easy to see that if $M \in R\text{-gr}$, then $M^+ \in \text{gr-R}$.

The following lemmas can be proved immediately as in the case of the character module of a left $R$-module.

**Lemma 4.1.** $0 \to N \to M \to P \to 0$ is a short exact sequence in $R\text{-gr}$ if and only if $0 \to P^+ \to M^+ \to N^+ \to 0$ is a short exact sequence in $\text{gr-R}$.

**Lemma 4.2.** Let $M \in R\text{-gr}$. $M$ is pure in $M^{++}$.

We recall from [2] the following definition.

**Definition 4.2.** Let $M$ be a graded left $R$-module. $M$ is called Gorenstein gr-flat if and only if there is an exact sequence in $R\text{-gr}$

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat left $R$-modules such that remains exact when the functor $E \otimes_R -$ is applied for any gr-injective right $R$-module $E$ and $M = \ker(F^0 \to F^1)$.

By using an analogous reasoning as in [7, Theorem 5.1] it is proved that if $R$ is gr-coherent then any $M \in R\text{-gr}$ has a graded flat preenvelope. Then we can define right derived functors of $\otimes_R$ which we will denote by $\text{Tor}_i$.

**Theorem 4.3 [2, Theorem 2.10].** Let $R$ be a gr-Gorenstein ring. The following statements are equivalent for any graded left $R$-module $M$:

(i) $M$ is Gorenstein gr-flat.

(ii) There is an exact sequence in $R\text{-gr}$

$$0 \to M \to F^0 \to F^1 \to \cdots \to F^n$$

with each $F^i$ flat.

(iii) If $N$ is a finitely generated graded $R$-module, then any homomorphism $N \to M$ can be factored $N \to C \to M$ where $C$ is a finitely generated Gorenstein gr-projective module.

(iv) $M \cong \lim_{\to} C_i$ for some direct system $((C_i), (f_i))$, with $C_i$ a finitely generated Gorenstein gr-projective module for all $i$.

(v) $\text{Tor}_i(E, M) = 0$, $\forall i \geq 1$ and for all gr-injective right $R$-module $E$.

(vi) $\text{Tor}_i(E, M) = 0$ for $1 \leq i \leq n$ and for all gr-injective right $R$-module $E$. 


(vii) $M^+$ is Gorenstein gr-injective.

(viii) $\text{Tor}_i(L, M) = 0$ for all $L$ such that gr-id$(L) < \infty$.

**Corollary 4.4.** Let $R$ be a gr-Gorenstein ring. Then the direct limit of Gorenstein gr-flat modules is Gorenstein gr-flat. Graded products and graded direct sums of Gorenstein gr-flat modules are also Gorenstein gr-flat.

**Proof.** Since $\text{Tor}_i(\cdot, M)$ commutes with direct limits and direct sums, apply (v) of the above theorem to obtain the desired result. Graded products of Gorenstein gr-flat modules are also Gorenstein gr-flat from (ii).

**Proposition 4.5.** Let $R$ be a gr-Gorenstein ring. If $M$ is a Gorenstein gr-injective right $R$-module, then $M^+$ is Gorenstein gr-flat.

**Proof.** If $M$ is Gorenstein gr-injective, then there is an exact sequence in gr-$R$ by gr-injective modules

$$
\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots
$$

such that $M = \text{Ker}(E^0 \to E^1)$. Then

$$
\cdots \to (E^1)^+ \to (E^0)^+ \to (E_0)^+ \to (E_1)^+ \to \cdots
$$

is an exact sequence in $R$-gr of flat modules, where $M^+ = \text{Ker}(E_0^+ \to E_1^+)$. We use induction to show that the sequence remains exact when $N \otimes_R -$ is applied, for any graded right $R$-module $N$, with finite flat dimension. Now, if $E$ is a gr-injective module, then by [1, Theorem 2.8], fd$(E)$ is finite, so $M^+$ is Gorenstein gr-flat.

The following two lemmas have analogous proofs to [11, Lemmas 3.2, 3.3].

**Lemma 4.6.** Let

$$
0 \to K \to F \to M \to 0
$$

and

$$
0 \to L \to G \to M \to 0
$$

be exact where $F \to M$ and $G \to M$ are Gorenstein gr-flat precovers of $M$. Then $G \otimes K \cong F \otimes L$.

**Lemma 4.7.** Suppose $0 \to A \to B \to C \to 0$ is exact and that both $A$ and $C$ have Gorenstein gr-flat covers. Also suppose that $\text{Ext}^1_{R, gr}(H, A) = 0$ for
any Gorenstein gr-flat module $A$. Then there is a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & K \longrightarrow N \longrightarrow L \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & F \longrightarrow D \longrightarrow G \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & A \longrightarrow B \longrightarrow C \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

such that $D = F \oplus G$ and such that $F \to A$ and $G \to C$ are Gorenstein gr-flat covers. Moreover, $K$ and $L$ verify $\text{Ext}^1_{R \text{-gr}}(H, K) = 0$, $\text{Ext}^1_{R \text{-gr}}(H, L) = 0$ for any Gorenstein gr-flat module $H$ and hence so does $N$.

**Definition 4.3.** A graded left $R$-module $Q$ is called pure gr-injective if for every pure sequence in $R$-gr $0 \to L \to M \to N \to 0$ and every graded homomorphism $\phi : L \to Q$, there exists $\psi : M \to Q$ such that $\psi \alpha = \phi$.

**Lemma 4.8.** (i) $K^+$ is pure gr-injective for any $K \in R$-gr.

(ii) $Q$ is pure gr-injective if and only if $Q$ is a direct summand of $K^+$ for some $K \in R$-gr.

**Proof.** (i) Let $0 \to L \to M \to N \to 0$ be a pure sequence in $R$-gr. Then

\[
0 \to L \otimes_R K \to M \otimes_R K \to N \otimes_R K \to 0
\]

is exact for any $K \in R$-gr. From Lemma 4.1 we get that

\[
0 \to (N \otimes_R K)^+ \to (M \otimes_R K)^+ \to (L \otimes_R K)^+ \to 0
\]

is exact, and therefore, by [17, Proposition 1.2.14],

\[
0 \to \text{Hom}_R(N, K^+) \to \text{Hom}_R(M, K^+) \to \text{Hom}_R(L, K^+) \to 0
\]

is exact, which shows that $K^+$ is pure gr-injective.

(ii) Assume that $Q$ is pure gr-injective. By Lemma 4.2

\[
0 \to Q \xrightarrow{f} Q^+ \to \text{Coker}(f) \to 0
\]

is pure in $R$-gr, we get that this sequence splits, which shows that $Q$ is a direct summand of $Q^+$. 

Conversely, if $K^+ = Q \oplus H$ and $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is pure in $\text{gr}-R$ and we consider $\phi : L \to Q$, there exists $g : M \to K^+$, since $K^+$ is pure $\text{gr}$-injective, such that $g \alpha = f \phi$. Let us define $\psi : M \to Q$ as $\psi = g \pi$, where $\pi$ verifies $f \pi = \text{Id}_Q$.

**Lemma 4.9.** If $R$ is gr-$n$-Gorenstein and $M$ is a pure gr-injective graded left $R$-module, then we have an exact sequence $0 \to K \to F \to M \to 0$ such that both $K$ and $F$ are pure gr-injective and $F \to M$ is a Gorenstein gr-flat cover of $M$.

**Proof.** Let us show that any pure gr-injective module $M$ has a Gorenstein gr-flat cover which is also pure gr-injective. Since $M$ is pure gr-injective, by Lemma 4.8, it is a direct summand of $M^{++}$. Then we have a split exact sequence

$$0 \to N \to M^{++} \to M \to 0.$$  

For $M^+$, there is an exact sequence by the dual of [3, Theorem 1.1] and there is an exact sequence in gr-$R$

$$0 \to M^+ \to H \to L \to 0$$

such that $H$ is Gorenstein gr-injective and $L$ has finite gr-injective dimension. Since $L^+$ has finite flat dimension, $L^+$ has finite gr-injective dimension. By Proposition 4.5, $H^+$ is Gorenstein gr-flat. Now we can get

$$0 \to L^+ \to H^+ \to M^{++} \to 0.$$  

By Theorem 4.3(viii) we obtain, for any Gorenstein gr-flat module $F$, that $\text{Ext}^1_R(F, L) \cong (\text{Tor}_1(L, F))^+ = 0$. This means that $H^+$ is a Gorenstein gr-flat precover of $M^{++}$ and therefore it is so for $M$, since $M$ is a direct summand of $M^{++}$. Now, if $F \to M$ is a Gorenstein gr-flat cover of $M$, $F$ is a direct summand of $H^+$ and hence, $F$ is pure gr-injective. $K$ is also pure gr-injective by following the same reasoning as in [11, Lemma 3.6].

The following lemma has an analogous proof to the ungraded case [11, Lemma 3.8].

**Lemma 4.10.** Let $R$ be a gr-$n$-Gorenstein ring and let $M$ have finite gr-injective dimension. Then there is an exact sequence in $R$-gr

$$0 \to M \to P \to L \to 0$$

such that $\text{Ext}^1_{R, \text{gr}}(P, H) = 0$ for any Gorenstein gr-flat module $H$ and where $L$ is Gorenstein gr-flat.

Now we can state the main result of this section.

**Theorem 4.11.** Let $R$ be a gr-$n$-Gorenstein ring. Then every left $R$-module has a Gorenstein gr-flat cover.
**Proof.** Before starting the proof, let us remark that Theorem 3.3 can be proved for any graded left $R$-module by using the same reasoning as in the ungraded case [11, Theorem 2.2]. By this theorem, for any $M \in R$-gr there is an exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow M \rightarrow 0$$

such that $L$ has finite dimensions and $C$ is Gorenstein gr-projective. From Theorem 4.3, we know that any Gorenstein gr-projective module is Gorenstein gr-flat. Then by the preceding lemma, there exists an exact sequence

$$0 \rightarrow L \rightarrow P \rightarrow N \rightarrow 0$$

such that $\text{Ext}^{1}_{gr}(P, H) = 0$ for any Gorenstein gr-flat module $H$ and $N$ is Gorenstein gr-flat. Now, if $D$ is the pushout of $L \rightarrow C$ and $L \rightarrow P$ we get that $D \rightarrow M$ is a Gorenstein gr-flat precover. Then, since by Corollary 4.4 the direct limit of Gorenstein gr-flat modules is Gorenstein gr-flat, apply an analogous reasoning as in [6, Theorem 2.1].

**Remark 4.** We can rewrite the results concerning the relation between the graded and the ungraded cases obtained in Section 3 in terms of Gorenstein gr-flat covers.

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**REFERENCES**